arbitrary $x \in \mathbb{R}$. Denote by $[-c, c]$ the recurrence interval of $(M_n)_{n\geq 0}$. It is then easily verified that there exists some $k = k(\delta)$ sufficiently large such that for each (stopping) time $\xi \geq k$ with $M_\xi \in [-c, c]$ there is a further one $\zeta \geq \xi$ such that $P(M_\zeta \in [x-\delta, x+\delta] \mid \mathcal{F}_\xi) \geq \gamma$ and $\gamma > 0$ only depends on δ, k and x. Thus we obtain the desired result by a geometric trial argument similar to the one used in the previous proof.

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On the invertibility of isometric semigroup representations

by

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Abstract. Let T be a representation of a suitable abelian semigroup S by isometries on a Banach space. We study the spectral conditions which will imply that $T(s)$ is invertible for each s in S. On the way we analyse the relationship between the spectrum of T, $\mathrm{Sp}(T,S)$, and its unitary spectrum $\mathrm{Sp_u}(T,S)$. For $S = \mathbb{Z}_+^n$ or $\mathbb{R}_+^n$, we establish connections with polynomial convexity.

1. Introduction. This paper deals chiefly with the question of invertibility of isometric representations of abelian semigroups. Quite apart from its intrinsic interest, the problem has a bearing on the study of the asymptotic behaviour of bounded semigroups of operators. Let S be a suitable subsemigroup of a locally compact, abelian group G, and T be a bounded representation of S on a complex Banach space X. Let $\mathrm{Sp_u}(T,S)$ be the set of all characters in the dual group Γ which are approximate eigenvalues for T, and $P\sigma_{\mathrm{u}}(T^*)$ be the set of characters in Γ which are eigenvalues for T^*. If $\mathrm{Sp_u}(T)$ is countable and $P\sigma_{\mathrm{u}}(T^*)$ is empty, then, for each x in X, $\|T(t)x\| \to 0$ as $t \to \infty$ through S. This was shown in [11] for norm-continuous representations of $\mathbb{R}_+$, in [7] for arbitrary representations of $\mathbb{R}_+$, in [3] (independently) for arbitrary representations of $\mathbb{Z}_+$ and $\mathbb{R}_+$, in [9] for norm-continuous representations of general semigroups, and in [4] for arbitrary (strongly continuous) representations. The arguments in [7], [9], and [4] all used a functional analytic construction to reduce the problem to the study of isometric semigroups.

If T is a representation of S by isometries and $\mathrm{Sp_u}(T,S)$ is countable, the question arises whether T is automatically invertible. For $S = \mathbb{R}_+$ this was an ingredient in the proof of [7], where a short argument using the Hille–Yosida Theorem showed that T is invertible whenever $\mathrm{Sp_u}(T,\mathbb{R}_+) \neq i\mathbb{R}$, and in the case when $S = \mathbb{Z}_+$ it is elementary that T is invertible whenever

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$\mathrm{Sp_u}(T) \neq \mathbb{T}$. In [4], the question of invertibility for general S was avoided, but it will be shown by Theorem 5.1 that T is indeed invertible whenever $\mathrm{Sp_u}(T,S)$ is countable. This leads in turn (although the details are not given here) to a greatly simplified proof of the stability theorem mentioned above.

The spectrum $\mathrm{Sp}(T,S)$ of T was defined in [4] and identified with the Gelfand spectrum of a certain commutative Banach algebra $\mathcal{A}_T$, while $\mathrm{Sp_u}(T,S)$ was shown to contain the Shilov boundary of $\mathcal{A}_T$; here we show that in fact $\mathrm{Sp_u}(T,S)$ coincides with the Shilov boundary, and we deduce that $\mathrm{Sp}(T,S)$ is a certain hull of $\mathrm{Sp_u}(T,S)$. For $S=\mathbb{Z}_+^n$, $\mathrm{Sp}(T,\mathbb{Z}_+^n)$ is the polynomially convex hull in $\mathbb{C}^n$ of the subset $\mathrm{Sp_u}(T,\mathbb{Z}_+^n)$ of $\mathbb{T}^n$, while for $S=\mathbb{R}_+^n$ a similar result holds after applying a Möbius transform of $i\mathbb{R}^n$ into $\mathbb{T}^n$; these matters are dealt with in Sections 2 and 4. Using some facts about polynomial convexity of subsets of $\mathbb{T}^n$, we are able to deduce that $\mathrm{Sp}(T,S)=\mathrm{Sp_u}(T,S)$ under certain conditions. In all these cases, we are further able to deduce that T is invertible, and these results are presented in Section 5. For discrete semigroups, invertibility follows directly from the property that $\mathrm{Sp}(T,S)=\mathrm{Sp_u}(T,S)$, but we do not know whether the same holds for $S=\mathbb{R}_+^n$. In Section 3 it is shown that any obstruction to invertibility arises in translation semigroups on quotients of $L^1(S)$.

The results which we use concerning polynomial convexity are due to Stolzenberg and Alexander. We are very grateful to Professor Alexander for providing us with Proposition 4.5, which enabled us to improve our original version of Theorem 5.3.

2. The spectrum and S-hulls. We shall adopt the terminology and conventions of [4] with only minor changes. Thus, G shall denote a locally compact, abelian group with dual Γ, and S will be a measurable subsemigroup of G with non-empty interior S^0 in G which satisfies $S-S=G$. Here, G is assumed to be equipped with the Haar measure, and we consider S with the restriction of that measure. $L^1(S)$ shall be identified with a subspace of $L^1(G)$. The *dual* of S, S^*, is the space of all non-zero, continuous, bounded, complex homomorphisms of S, and by the *unitary part* of S^* we mean

$$S_{\mathrm{u}}^* = \{\chi\in S^* : |\chi(s)|=1 \text{ for all } s \text{ in } S\}.$$

We shall identify S_{u}^* with Γ in the obvious way. For f in $L^1(S)$ and χ in S^* let

$$\widehat{f}(\chi)=\int_S f(s)\chi(s)\,ds.$$

Finally, we assume that $\{\widehat{f} : f\in L^1(S)\}$ separates the points of S^* from each other and from zero.

For $S=\mathbb{Z}_+^n$ we identify $(\mathbb{Z}_+^n)^*$ with $\mathbb{D}^n$, where $\mathbb{Z}_+=\{0,1,2,\ldots\}$ and $\mathbb{D}=\{z\in\mathbb{C}: |z|\le 1\}$, by the relationship $\chi(m_1,\ldots,m_n)=z_1^{m_1}\ldots z_n^{m_n}$.

Similarly, for $S=\mathbb{R}_+^n$, we identify $(\mathbb{R}_+^n)^*$ with $\mathbb{C}_-^n$, where $\mathbb{R}_+=\{t\in\mathbb{R}: t\ge 0\}$ and $\mathbb{C}_-=\{z\in\mathbb{C}: \mathrm{Re}\, z\le 0\}$, by $\chi(t_1,\ldots,t_n)=\exp(t_1z_1+\ldots+t_nz_n)$.

If T is a bounded representation of S on a complex Banach space X we define the *spectrum of T with regard to S* to be

$$\mathrm{Sp}(T,S)=\{\chi\in S^* : |\widehat{f}(\chi)|\le \|\widehat{f}(T)\| \text{ for all } f \text{ in } L^1(S)\},$$

where $\widehat{f}(T):X\to X$ is defined by

$$\widehat{f}(T): x\mapsto \int_S f(s)T(s)x\,ds.$$

We shall denote the unitary part of the spectrum, $\mathrm{Sp}(T,S)\cap S_{\mathrm{u}}^*$, by $\mathrm{Sp_u}(T,S)$; $\mathrm{Sp}(T,S)$ is clearly a closed subset of S^*. If T is a C_0-semigroup, that is, if $S=\mathbb{R}_+$, then $\mathrm{Sp_u}(T,S)=\sigma(A)\cap i\mathbb{R}$ where A is the generator of T.

Suppose now that U is a bounded representation of G; then the definition of $\mathrm{Sp}(U,G)$ in the preceeding paragraph coincides with the standard definition of the Arveson spectrum [8] and of the finite L-spectrum of U [6]. In [4, Proposition 2.2] it was shown that $\mathrm{Sp_u}(U,S)$ is equal to the unitary approximate point spectrum $A\sigma_{\mathrm{u}}(U)$; that is, $\chi\in\mathrm{Sp}(U,S)$ if and only if there exists a net (x_α) of elements in X such that $\|x_\alpha\|=1$ and

$$\|U(s)x_\alpha-\chi(s)x_\alpha\|\to 0 \quad \text{uniformly on compact subsets of } S.$$

From the conditions imposed on S above it may easily be deduced that, if C is a compact subset of G, then there exists an s in S such that $s+C\subseteq S$. This fact may be used to prove the non-trivial half of the equality

$$\mathrm{Sp}(U,G)=\mathrm{Sp_u}(U,S), \tag{2.1}$$

where we are here identifying Γ and S_{u}^*. It also allows us to use the notation $A\sigma_{\mathrm{u}}(U)$ without ambiguity.

A representation T of S by isometries on a complex Banach space X is said to be *invertible* if the operator $T(s)$ is invertible for each s in S. In such a case T may clearly be extended to a group representation U by defining

$$U(t-s)=T(t)T(s)^{-1},\qquad t,s\in S.$$

PROPOSITION 2.1. *Let T be a representation of S by isometries on a Banach space X. There exist a Banach space X_d containing X (by isometric isomorphism), and a representation T_d of G by isometries on X_d such that $T_d(s)x=T(s)x$ $(x\in X, s\in S)$ and $\mathrm{Sp}(T_d,G)=\mathrm{Sp_u}(T,S)$.*

Proof. The existence of X_d and T_d is given in [5, Theorem 1]; moreover, we may assume that $\{T_d(t)x : t\in G, x\in X\}$ is dense in X_d. (Then (X_d,T_d) is essentially unique.)

If χ is in $\mathrm{Sp_u}(T,S)$, then χ is an approximate eigenvalue of T and so there exists a net (x_α) of norm one elements of X such that $\|T(s)x_\alpha-\chi(s)x_\alpha\|\to 0$

uniformly on compact subsets of S. Let C be a compact subset of G and let $s \in S$ be such that $s + C \subseteq S$; then

$$\begin{aligned}\|T_d(t)x_\alpha - \chi(t)x_\alpha\| &= \|T_d(s)(T_d(t)x_\alpha - \chi(t)x_\alpha)\| \\ &\le \|T_d(s+t)x_\alpha - \chi(s+t)x_\alpha\| \\ &\quad + \|\chi(s+t)x_\alpha - \chi(t)T_d(s)x_\alpha\| \\ &= \|T_d(s+t)x_\alpha - \chi(s+t)x_\alpha\| + \|\chi(s)x_\alpha - T_d(s)x_\alpha\| \\ &\to 0 \quad \text{uniformly for } t \text{ in } C.\end{aligned}$$

Hence χ is in $\mathrm{Sp}(T_d, G)$.

Conversely, let χ be in $\mathrm{Sp}(T_d, G)$; there is a net (y_α) in X_d of norm one elements such that $\|T_d(t)y_\alpha - \chi(t)y_\alpha\| \to 0$ uniformly on compact subsets of G. Using the density assumed in the first paragraph, we may arrange that $y_\alpha = T_d(-t_\alpha)x_\alpha$ for some t_α in S and x_α in X. For $s \in S$,

$$\begin{aligned}\|T(s)x_\alpha - \chi(s)x_\alpha\| &= \|T_d(s-t_\alpha)x_\alpha - \chi(s)T_d(-t_\alpha)x_\alpha\| \\ &= \|T_d(s)y_\alpha - \chi(s)y_\alpha\| \\ &\to 0 \quad \text{uniformly on compact subsets of } S.\end{aligned}$$

Thus $\chi \in \mathrm{Sp_u}(T, S)$.

The first statement of the following corollary was proved in [4, Corollary 3.3], but our proof is much more direct.

COROLLARY 2.2. *Let T be a representation of S by isometries on a Banach space X.*

(1) *If $X \ne \{0\}$, then $\mathrm{Sp_u}(T,S)$ is non-empty.*
(2) *T is norm-continuous if and only if $\mathrm{Sp_u}(T,S)$ is compact.*

Proof. Both statements follow from Proposition 2.1 and the corresponding results for group representations (see [8, Section 8.1]).

We now show that the invertibility theorems to be presented later in this paper are not vacuous, by proving that examples of representations exist with arbitrary closed unitary spectra.

Let E be a closed subset of Γ and define

$$J_E = \{f \in L^1(G) : \widehat{f}(\chi) = 0 \text{ for all } \chi \text{ in some open neighbourhood of } E\}^-,$$

so that J_E is a closed ideal in $L^1(G)$. Set $X = L^1(G)/J_E$ and define $U_E(s): X \to X$ for s in G by

$$U_E(s) : f(\cdot) + J_E \mapsto f(\cdot - s) + J_E \qquad (f \in L^1(G)).$$

It may easily be shown that U_E is a representation of G (by isometries) on X. If one writes $Y = \overline{L^1(S) + J_E}/J_E$ (regarding $L^1(S)$ as a subspace of $L^1(G)$) and $T_E(s) = U_E(s)|_Y$ for s in S, then T_E is a representation of S by isometries on Y. Let $\|\cdot\|_E$ denote the quotient norm of X and Y.

THEOREM 2.3. *In the notation above,* $\mathrm{Sp}(U_E, G) = \mathrm{Sp_u}(T_E, S) = E$.

Proof. By (2.1), $\mathrm{Sp_u}(U_E, S) = \mathrm{Sp}(U_E, G)$. If f is in $L^1(S)$, then $\|\widehat{f}(T_E)\| \le \|\widehat{f}(U_E)\|$ and hence $\mathrm{Sp_u}(T_E, S) \subseteq \mathrm{Sp_u}(U_E, S)$. It suffices to show therefore that (a) $E \subseteq \mathrm{Sp_u}(T_E, S)$, and (b) $\mathrm{Sp}(U_E, G) \subseteq E$.

(a) $E \subseteq \mathrm{Sp_u}(T_E, S)$: S satisfies the Følner condition; that is, there exists a net (Ω_α) of compact subsets of S with

$$|(\Omega_\alpha + s)\triangle\Omega_\alpha|/|\Omega_\alpha| \to 0$$

uniformly on compact subsets of S, where $|\cdot|$ denotes the Haar measure of G restricted to S. Suppose χ is in E and set

$$f_\alpha(s) = \frac{1}{|\Omega_\alpha|} 1_{\Omega_\alpha}(s)\chi(-s) \qquad \text{for } s \in S,$$

where 1_{Ω_α} is the characteristic function of Ω_α. We will show that $(f_\alpha + J_E)$ is a net of elements of norm one in Y such that

$$\|T_E(s)(f_\alpha + J_E) - \chi(s)(f_\alpha + J_E)\|_E \to 0 \tag{2.2}$$

uniformly on compact subsets of S; from these facts we may deduce the result.

For all α,

$$\|f_\alpha + J_E\|_E \le \|f_\alpha\|_1 = \int_G |f_\alpha(s)|\,ds = \int_{\Omega_\alpha} \frac{1}{|\Omega_\alpha|}\,ds = 1, \tag{2.3}$$

and for all g in J_E,

$$|(f_\alpha - g)^\wedge(\chi)| = |\widehat{f}_\alpha(\chi)| = \left|\int_G \frac{1}{|\Omega_\alpha|} 1_{\Omega_\alpha}(s)\chi(-s)\chi(s)\,ds\right| = 1. \tag{2.4}$$

Equation (2.4) then implies

$$1 \le \int_G |(f_\alpha - g)(s)\chi(s)|\,ds = \int_G |(f_\alpha - g)(s)|\,ds = \|f_\alpha - g\|_1,$$

which, together with (2.3), shows that $\|f_\alpha + J_E\|_E = 1$. To obtain (2.2) we note that

$$\begin{aligned}\|T_E(s)(f_\alpha + J_E) - \chi(s)(f_\alpha + J_E)\|_E &\le \|f_\alpha(\cdot - s) - \chi(s)f_\alpha(\cdot)\|_1 \\ &= \int_G \frac{|1_{\Omega_\alpha}(t-s)\chi(s-t) - 1_{\Omega_\alpha}(t)\chi(s-t)|}{|\Omega_\alpha|}\,dt \\ &= \int_{(\Omega_\alpha+s)\setminus\Omega_\alpha} \frac{1}{|\Omega_\alpha|}\,dt + \int_{\Omega_\alpha\setminus(\Omega_\alpha+s)} \frac{1}{|\Omega_\alpha|}\,dt \\ &= |(\Omega_\alpha + s)\triangle\Omega_\alpha|/|\Omega_\alpha| \to 0\end{aligned}$$

uniformly on compact subsets of S.

(b) $\mathrm{Sp}(U_E, G) \subseteq E$: If $f \in J_E$ and $g \in L^1(G)$, then

$$\widehat{f}(U_E)(g + J_E) = \int_G f(s)g(\cdot - s)\,ds + J_E = (f * g) + J_E.$$

It follows, J_E being an ideal, that $\widehat{f}(U_E)(g + J_E) = J_E$; in other words, $\widehat{f}(U_E) = 0$. The result now follows from the fact that for all χ not in E there exists an f in J_E such that $\widehat{f}(\chi) = 1$.

Let T be a bounded representation of S on X, and U be a bounded representation of G on X. Define

$$\mathcal{A}_T = \{\widehat{f}(T) : f \in L^1(S)\}^-$$

and

$$\mathcal{B}_U = \{\widehat{f}(U) : f \in L^1(G)\}^-,$$

where closures are taken in the norm topology of $\mathcal{B}(X)$.

The maximal ideal space of $\mathcal{A}_T$ may be identified with $\mathrm{Sp}(T, S)$, while that of $\mathcal{B}(U)$ may be identified with $\mathrm{Sp}(U, G)$, in the following way: Each χ in $\mathrm{Sp}(T, S)$ (respectively $\mathrm{Sp}(U, G)$) corresponds to an element ϕ_χ in $\mathcal{A}_T$ (resp. $\mathcal{B}_U$) where

$$\phi_\chi : \widehat{f}(T) \mapsto \widehat{f}(\chi), \qquad f \in L^1(S),$$

and similarly for $\mathcal{B}_U$. Moreover, each non-zero complex homomorphism of $\mathcal{A}_T$ (resp. $\mathcal{B}_U$) is of this form (see [4, Proposition 2.4]).

THEOREM 2.4. *The Shilov boundary of $\mathcal{A}_T$ contains $\mathrm{Sp_u}(T, S)$ and is contained in the approximate point spectrum $A\sigma(T)$. In particular, if T is a representation by isometries, then $A\sigma(T) = \mathrm{Sp_u}(T, S)$ and thus the Shilov boundary is equal to the unitary spectrum.*

Proof. By [4, Proposition 2.5], the Shilov boundary of $\mathcal{A}_T$, $\mathrm{Sh}(\mathcal{A}_T)$, is contained in the approximate point spectrum, so it remains to prove the inclusion $\mathrm{Sp_u}(T, S) \subseteq \mathrm{Sh}(\mathcal{A}_T)$.

Let χ be in $\mathrm{Sp_u}(T, S)$. Regarding χ as an element of Γ, and choosing any open neighbourhood V of 0 in Γ, we may find a function f in $L^1(G)$ such that $\widehat{f}(\chi) = 1$ and $\widehat{f} = 0$ outside $\chi + V$ (see [10, p. 49]). Since S is a subsemigroup with non-empty interior, there exists an $s \in S$ such that

$$\int_{G \setminus (S - s)} |f(t)|\,dt < 1/4.$$

Set $g_s(t) = f(t - s)$ for $t \in S$, so that $g_s \in L^1(S)$. For γ in Γ, we have

$$\begin{aligned} ||\widehat{g}_s(\gamma)| - |\widehat{f}(\gamma)|| &\le |\widehat{g}_s(\gamma) - \gamma(s)\widehat{f}(\gamma)| \\ &= \Big|\int_S f(t - s)\gamma(t)\,dt - \int_G f(t)\gamma(t + s)\,dt\Big| \\ &= \Big|\int_{G \setminus (S - s)} f(t)\gamma(t)\,dt\Big| \le \int_{G \setminus (S - s)} |f(t)|\,dt < 1/4, \end{aligned}$$

and thus the Gelfand transform of the element $\widehat{g}_s(T)$ of $\mathcal{A}_T$ peaks on $\chi + V$, which implies that χ is in the Shilov boundary of $\mathcal{A}_T$.

COROLLARY 2.5. *If T is a representation by isometries such that $\mathrm{Sp_u}(T, S)$ is countable, then $\mathrm{Sp}(T, S) = \mathrm{Sp_u}(T, S)$.*

Proof. If the Shilov boundary of any commutative Banach algebra is countable, then it is equal to the maximal ideal space [13, p. 55].

See Theorem 5.1 for a result closely related to Corollary 2.5.

S-hulls. For E a closed subset of S_u^* we define the *S-hull* of E as follows:

$$S\text{-hull}\,E = \{\chi \in S^* : |\widehat{f}(\chi)| \le \sup_{\gamma \in E} |\widehat{f}(\gamma)| \text{ for all } f \in L^1(S)\}.$$

If $E = S\text{-hull}\,E$, then we shall say that E is *S-convex*. Suppose T is an isometric representation of S with unitary spectrum E. If χ is in the S-hull of E, then $|\widehat{f}(\chi)| \le \sup_{\gamma \in E} |\widehat{f}(\gamma)| \le \|\widehat{f}(T)\|$ for each f in $L^1(S)$ by definition since E is contained in the spectrum; hence the S-hull of E is a subset of the spectrum.

Suppose instead that χ is in the spectrum of T; by Theorem 2.4 we know that E is the Shilov boundary of $\mathcal{A}_T$ and hence, for all f in $L^1(S)$, $|\widehat{f}(\chi)| \le \sup_{\gamma \in E} |\widehat{f}(\gamma)|$, which implies that χ is in $S\text{-hull}\,E$. We have thus proved the following:

PROPOSITION 2.6. *If T is a representation of S by isometries on X, then $\mathrm{Sp}(T, S) = S\text{-hull}\,\mathrm{Sp_u}(T, S)$. In particular, $\mathrm{Sp}(T, S) = \mathrm{Sp_u}(T, S)$ if and only if $\mathrm{Sp_u}(T, S)$ is S-convex.*

An important point here is that the S-hull of a set E depends only on S and E, so that if two isometric representations of S have identical unitary spectra, then their spectra agree entirely. It seems plausible that if $\mathrm{Sp}(T, S) = \mathrm{Sp_u}(T, S)$, then T is invertible. However, we are able to establish this only in certain cases and the question will be considered in the later sections of this paper.

If D is a dense subspace of $L^1(S)$, then it follows, since $\|\widehat{f}\|_\infty \le \|f\|_1$, that

$$S\text{-hull}\,E = \{\chi \in S^* : |\widehat{f}(\chi)| \le \sup_{\gamma \in E} |\widehat{f}(\gamma)| \text{ for all } f \in D\}.$$

If $S = \mathbb{Z}_+^n$ and D is the subset of $L^1(\mathbb{Z}_+^n)$ consisting of all sequences with only finitely many terms non-zero, then, identifying S^* with $\mathbb{D}^n$, the unit polydisc, we deduce that the S-hull of a set E is in fact its polynomially convex hull. Thus we see, when $S = \mathbb{Z}_+^n$, that the spectrum of an isometric representation is equal to the polynomially convex hull of its unitary spectrum and we shall, in this case, write P-hull for $\mathbb{Z}_+^n$-hull.

3. A characterisation theorem. In the remainder of this paper we will be trying to answer the question: Under what spectral conditions is an isometric semigroup representation invertible? T will be assumed to be a representation of S by isometries on X. First we show that any obstruction to this is already contained in the example of Section 2; we use the notation U_E and T_E as in Theorem 2.3.

THEOREM 3.1. *The following are equivalent:*

(1) *T_E is invertible*;

(2) $\overline{L^1(S) + J_E} = L^1(G)$;

(3) *Every representation of S by isometries on a Banach space with unitary spectrum E is invertible.*

Proof. Suppose (1) is true. Let f be in $L^1(G)$ and $\varepsilon > 0$. There exists an $s \in S$ that satisfies

$$\int_{G\setminus(S-s)} |f(t)|\,dt < \varepsilon/2;$$

if g denotes the restriction of f to $S-s$, this may be written as $\|g-f\|_1 < \varepsilon/2$. Because $g_s \in L^1(S)$, there exists by assumption an h in $L^1(S)$ such that

$$\|T_E(s)(h + J_E) - (g_s + J_E)\|_E < \varepsilon/2,$$

hence there is a k in J_E satisfying

$$\|h + k - g\|_1 = \|h_s + k_s - g_s\|_1 < \varepsilon/2.$$

It now follows trivially that $\|f - (h+k)\|_1 < \varepsilon$ and thus $f \in \overline{L^1(S) + J_E}$.

Suppose (2) is true. Let T be any representation of S by isometries on some Banach space X with unitary spectrum E. We aim to prove the following inequality: for f in $L^1(S)$,

$$\|\widehat{f}(T)\| \le \|f + J_E\|_E. \tag{3.1}$$

Once this is done the proof continues thus: let s_0 be an interior point of S and suppose $s \in S$, $x \in X$ with $\|x\| = 1$, and $\varepsilon > 0$. There exists a neighbourhood V of s_0, contained in S, with the property that

$$\|T(t)x - T(s_0)x\| < \varepsilon/2, \qquad t \in V.$$

Setting $f = (1/|V|)1_V$ we then have

$$\|\widehat{f}(T)x - T(s_0)x\| < \varepsilon/2.$$

$f(\cdot + s + s_0)$ is in $L^1(G)$ and so, by assumption, there exists a g in $L^1(S)$ with $\|g(\cdot) - f(\cdot + s + s_0) + J_E\|_E < \varepsilon/2$, but then, writing $y = \widehat{g}(T)x$, we have

$$\begin{aligned}\|T(s)y - x\| &= \|T(s+s_0)\widehat{g}(T)x - T(s_0)x\|\\ &\le \|T(s+s_0)\widehat{g}(T)x - \widehat{f}(T)x\| + \|\widehat{f}(T)x - T(s_0)x\|\\ &\le \|\widehat{g}_{s+s_0}(T)x - \widehat{f}(T)x\| + \varepsilon/2\\ &\le \|g(\cdot - s - s_0) - f(\cdot) + J_E\|_E + \varepsilon/2 < \varepsilon,\end{aligned}$$

so that x is in $T(s)[X]$; hence $T(s)[X] = X$ and $T(s)$ is invertible. It remains therefore to prove (3.1).

Let f be in $L^1(S)$ and g be in J_E, and let X_d and T_d be as in Proposition 2.1. Since $\mathrm{Sp}(T_d, G) = E$, it follows that $\widehat{g}(T_d) = 0$ by the theory of isometric group representations [8, Section 8.1]; hence

$$\begin{aligned}\|\widehat{f}(T)\| &= \sup_{\|x\|\le 1}\|\widehat{f}(T)x\| = \sup_{\|x\|\le 1}\|\widehat{f}(T_d)x\|\\ &\le \|\widehat{f}(T_d)\| = \|\widehat{f}(T_d) - \widehat{g}(T_d)\| \le \|f - g\|_1.\end{aligned}$$

Taking the infimum over all g in J_E, (3.1) is proved.

The final equivalence, (3) implies (1), is trivial.

LEMMA 3.2. *Suppose that 0 is in the closure of S^0 in G and that T is norm-continuous. Then for each s in S,*

$$\sigma(T(s)) \subseteq \{\chi(s) : \chi \in \mathrm{Sp}(T, S)\}.$$

Proof. The assumptions imply that $\mathcal{A}_T$ contains I_X and $T(s)$ for each s in S. For χ in $\mathrm{Sp}(T,S)$ we have $\phi_\chi(T(s)) = \chi(s)$ (where ϕ_χ is as in Section 2) and hence

$$\sigma_{\mathcal{A}_T}(T(s)) \subseteq \{\chi(s) : \chi \in \mathrm{Sp}(T, S)\}.$$

The result now follows since $\sigma(T(s)) = \sigma_{B(X)}(T(s)) \subseteq \sigma_{\mathcal{A}_T}(T(s))$.

THEOREM 3.3. *Let E be a closed subset of S_{u}^*.*

(1) *If E satisfies the conditions of Theorem 3.1, then E is S-convex;*

(2) *If E is S-convex and compact, then E satisfies the conditions of Theorem 3.1.*

Proof. (1) Suppose E satisfies the conditions of Theorem 3.1, so T_E is invertible and $T_E = U_E$. For any g in J_E, $\widehat{g}(U_E) = 0$ and hence

$$\begin{aligned}\mathcal{A}_{T_E} &= \{\widehat{f}(T_E) : f \in L^1(S)\}^- = \{\widehat{f}(U_E) : f \in L^1(S)\}^- \\ &= \{(f+g)^\wedge(U_E) : f \in L^1(S),\ g \in J_E\}^- \\ &= \{\widehat{h}(U_E) : h \in L^1(G)\}^- = \mathcal{B}_{U_E};\end{aligned}$$

the last but one step is justified by the fact that $L^1(S) + J_E$ is dense in $L^1(G)$ and because $\|\widehat{f}(U_E)\| \le \|f\|_1$ for all f in $L^1(G)$. The spectrum of T_E, S-hull E, is associated with the maximal ideal space of $\mathcal{A}_{T_E}$, which by the above is the same as that of $\mathcal{B}_{U_E}$. The latter is just E, however, so that S-hull $E = \mathrm{Sp}(T_E, S) = E$ and we are done.

(2) Now suppose E is compact and S-convex. By Corollary 2.2, any representation of S by isometries with compact unitary spectrum is norm-continuous. Proposition 2.6 and Lemma 3.2 then give that for each s in S^0, the spectrum of $T_E(s)$ is contained in the set $\{\chi(s) : \chi \in E\}$, but this set is contained in the circle $\{z \in \mathbb{C} : |z| = 1\}$ and hence $T_E(s)$ is invertible. The invertibility of all $T_E(s)$ follows from this.

COROLLARY 3.4. *Let E be a closed subset of the n-torus $\mathbb{T}^n$. E is polynomially convex if and only if $l^1(\mathbb{Z}^n_+) + J_E$ is dense in $l^1(\mathbb{Z}^n)$.*

Remark 3.5. It is not clear whether the conclusion of Theorem 3.3(2) is true in the case when E is not assumed to be compact. If $S = \mathbb{R}_+$, then by Remark 4.2 below, the closed S-convex subsets of $i\mathbb{R}$ are precisely the proper closed subsets. An isometric C_0-semigroup whose spectrum does not contain $i\mathbb{R}$ is invertible (again see Remark 4.2), and hence the full conclusion does hold when $S = \mathbb{R}_+$ even for non-compact E. We shall show in Section 5 that in the case $S = \mathbb{R}^n_+$ it is true provided that E is the union of its relatively open compact sets and note that it is true for all S if E is countable.

4. Some technicalities. Define $m : \widetilde{\mathbb{C}}^n \to \widetilde{\mathbb{C}}^n$, where $\widetilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, via

$$m : z_i \mapsto (z_i + 1)/(z_i - 1), \qquad i = 1, \ldots, n.$$

Note that m is a self-inverting transform.

LEMMA 4.1. *For $E \subseteq i\mathbb{R}^n$, $\mathbb{R}^n_+$-hull $E = \{z \in \mathbb{C}^n : m(z) \in P\text{-hull}\,\overline{m(E)}\}$.*

Proof. For ease of notation we shall also use m to denote the single transformation $z \mapsto (z+1)/(z-1)$. Suppose that $z \in \mathbb{C}^n_-$ is such that $m(z) \in P$-hull $\overline{m(E)}$. For f in $L^1(\mathbb{R}^n_+)$, let

$$g(w) = \begin{cases} \widehat{f}(m(w)), & w = (w_1, \ldots, w_n) \in \mathbb{D}^n,\ w_j \ne 1,\ j = 1, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

Since g is a function which is continuous on $\mathbb{D}^n$ and analytic in the interior, it may be approximated uniformly by polynomials p. From the definition of the polynomially convex hull we have $|p(m(z))| \le \sup_{w \in m(E)} |p(w)|$, and it follows that

$$|\widehat{f}(z)| \le \sup_{w \in m(E)} |\widehat{f}(m(w))| = \sup_{v \in E} |\widehat{f}(v)|.$$

Thus z is in $\mathbb{R}^n_+$-hull E and one inclusion is proved.

Let q be the polynomial $q(z) = (z_1 - 1) \ldots (z_n - 1)$. For $r = 0, 1, \ldots, n$ define

$$\mathcal{E}_r = \mathrm{span}\{q(z)(1 - \lambda_1 z_1)^{-1} \ldots (1 - \lambda_r z_r)^{-1} : |\lambda_i| < 1\}^-,$$

where the closure is taken in the topology of uniform convergence on $\mathbb{D}^n$. If $f(t) = \exp(\mu_1 t_1 + \ldots + \mu_n t_n)$, where $\mu = m(\lambda)$, then

$$\widehat{f}(m(z)) = (-\tfrac{1}{2})^n q(\lambda) q(z) (1 - \lambda_1 z_1)^{-1} \ldots (1 - \lambda_n z_n)^{-1}.$$

Hence, if $g \in \mathcal{E}_n$, then $g \circ m \in \{\widehat{f} : f \in L^1(\mathbb{R}^n_+)\}^-$, where the closure is taken in the topology of uniform convergence on $\mathbb{C}^n_-$.

We aim to prove the following:

INDUCTIVE HYPOTHESIS (I). *Suppose that $1 \le r \le n$, $1 \le k < \infty$, $p(z)$ is a polynomial depending only on the variables $z_1, \ldots z_{r-1}$. If $q(z)p(z) \in \mathcal{E}_{r-1}$ and $q(z)p(z)z_r^i \in \mathcal{E}_r$ $(0 \le i < k)$, then $q(z)p(z)z_r^k \in \mathcal{E}_r$.*

By assumption, $q(z)p(z) \in \mathcal{E}_{r-1}$. It follows from the definitions of $\mathcal{E}_{r-1}$ and $\mathcal{E}_r$ that $q(z)p(z)(1 - \lambda z_r)^{-1} \in \mathcal{E}_r$ whenever $|\lambda| < 1$. Also by assumption, $q(z)p(z)z_r^i \in \mathcal{E}_r$ for $0 \le i < k$, so

$$\sum_{i=k}^{\infty} q(z)p(z)\lambda^{i-k} z_r^i = \lambda^{-k} q(z)p(z)\Big((1 - \lambda z_r)^{-1} - \sum_{i=0}^{k-1} \lambda^i z_r^i\Big) \in \mathcal{E}_r$$

whenever $0 < |\lambda| < 1$. As $\lambda \to 0$, this function converges uniformly on $\mathbb{D}^n$ to $q(z)p(z)z_r^k$, which therefore belongs to $\mathcal{E}_r$, as required.

Since $q(z)$ is in $\mathcal{E}_0$, it follows from repeated applications of (I) that $q(z)p(z) \in \mathcal{E}_n$ for any polynomial p in $z_1, \ldots, z_n$, so $(q \circ m)(p \circ m) \in \{\widehat{f} : f \in L^1(\mathbb{R}^n_+)\}^-$.

Now suppose that z is in $\mathbb{R}^n_+$-hull E, so that $|\widehat{f}(z)| \le \sup_{v \in E} |\widehat{f}(v)|$ for all f in $L^1(\mathbb{R}^n_+)$. It follows from the preceding paragraph that

$$|q(m(z))p(m(z))| \le \sup_{w \in m(E)} |q(w)p(w)| \le 2^n \sup_{w \in m(E)} |p(w)|$$

for all polynomials p. Replacing $p(w)$ by $p(w)^k$ and taking kth roots, it follows that

$$|p(m(z))| \le \left(\frac{2^n}{|q(m(z))|}\right)^{1/k} \sup_{w \in m(E)} |p(w)|.$$

Letting k tend to infinity we deduce that $|p(m(z))| \le \sup_{w \in m(E)} |p(w)|$, and hence $m(z) \in P$-hull $\overline{m(E)}$.

Remark 4.2. A closed subset of the circle $\mathbb{T}$ is polynomially convex if and only if it is proper. The above lemma thus gives the following result: if T is an isometric C_0-semigroup, that is, a representation of $\mathbb{R}_+$ by isometries, then either $\mathrm{Sp}(T, \mathbb{R}_+) = \mathbb{C}_-$ or $\mathrm{Sp}(T, \mathbb{R}_+) \subseteq i\mathbb{R}$. This may be compared to the fact that if A is the generator of T, it is known that either $\sigma(A) = \mathbb{C}_-$ or T is invertible and $\sigma(A) \subseteq i\mathbb{R}$ (see [6, p. 39]); it is possible that $\sigma(A) = i\mathbb{R}$.

Remark 4.3. Suppose T is a bounded representation of $\mathbb{R}_+^n$, so that

$$T(t_1, \ldots, t_n) = T_1(t_1) \ldots T_n(t_n)$$

for commuting, bounded C_0-semigroups $T_1, \ldots, T_n$. Let A_j be the generator of T_j, and let $V_j = -(I + A_j)(I - A_j)^{-1}$. Although V_j may not be power bounded, $\sigma(V_j) = \{m(\lambda) : \lambda \in \sigma(A_j)\} \subseteq \mathbb{D}$. Define $\mathrm{Sp}(V)$ to be the set of z in $\mathbb{C}^n$ such that

$$|p(z)| \leq \|p(V)\|$$

for all polynomials p in n variables. Then $\mathrm{Sp}(V)$ is naturally identified with the Gelfand spectrum of the commutative Banach algebra generated by $I, V_1, \ldots, V_n$ in $\mathcal{B}(X)$. Let $\mathrm{Sp}'(V) = \{z \in \mathrm{Sp}(V) : z_j \neq 1\ (j = 1, \ldots, n)\}$.

One may now, by methods similar to those used in the proof of Lemma 4.1, prove the following result:

PROPOSITION 4.4. *If T is a bounded representation of $\mathbb{R}_+^n$ and V is defined as above, then $m(\mathrm{Sp}(T, S)) = \mathrm{Sp}'(V)$.*

A result in polynomial convexity. Let

$$Z_n = \{z \in \mathbb{D}^n : z_j = 1 \text{ for some } 1 \leq j \leq n\}.$$

The following result has been contributed to us by H. Alexander.

PROPOSITION 4.5. *If E is a compact set such that $Z_n \subseteq E \subseteq Z_n \cup \mathbb{T}^n$ and $E \setminus Z_n$ is the union of its compact, relatively open subsets, then E is polynomially convex.*

Proof. The proof is by induction on n. The case $n = 1$ is trivial.

Suppose that the result holds for subsets of $\mathbb{C}^{n-1}$. Let $\zeta = (\zeta_1, \ldots, \zeta_n) \in P\text{-hull}\,E$. We wish to show that $\zeta \in E$; there are three cases.

Case 1: $\zeta_j = 1$ for some j. Then $\zeta \in Z_n \subseteq E$.

Case 2: $|\zeta_1| = 1$, $\zeta_1 \neq 1$. Let $E_0 = \{w \in \mathbb{C}^{n-1} : (\zeta_1, w) \in E\}$. Since $\{z \in P\text{-hull}\,E : z_1 = \zeta_1\}$ is a peak set in $P\text{-hull}\,E$, $(\zeta_2, \ldots, \zeta_n) \in P\text{-hull}\,E_0$. However, $Z_{n-1} \subseteq E_0 \subseteq Z_{n-1} \cup \mathbb{T}^{n-1}$ and $E_0 \setminus Z_{n-1}$ is the union of its compact, relatively open, subsets, so the inductive hypothesis implies that $P\text{-hull}\,E_0 = E_0$. Thus $\zeta \in E$.

Case 3: $|\zeta_1| < 1$, $\zeta_j \neq 1$ $(j = 2, 3, \ldots, n)$. We shall show that this case contradicts the assumption that $\zeta \in P\text{-hull}\,E$.

Let q be the polynomial $q(z) = (z_1 - 1) \ldots (z_n - 1)$, so that $q(z) = 0$ if $z \in Z_n$, and $q(\zeta) \neq 0$. Let $F = \{z \in E : |q(z)| \geq \frac{1}{2}|q(\zeta)|\}$; then F is a compact subset of $E \setminus Z_n$. Now since $E \setminus Z_n$ is the union of its compact, relatively open, subsets, we can choose a compact, relatively open, subset E_1 of $E \setminus Z_n$ such that $F \subseteq E_1$. Setting $E_2 = E \setminus E_1$, we note that E_2 is also compact and open in E, and since $|q(z)| < \frac{1}{2}|q(\zeta)|$ for all z in E_2, we have $\zeta \notin P\text{-hull}\,E_2$.

Let V_1 and V_2 be disjoint open subsets of $\mathbb{C}^n$ such that

$$E_1 \subseteq V_1 \subseteq \{z \in \mathbb{C}^n : z_1 = re^{i\theta}, r > 0, 0 < \theta < 2\pi\} \quad \text{and} \quad E_2 \subseteq V_2.$$

Now, using Cases 1 and 2 above,

$$\bigcap_{0<r<1} \{z \in P\text{-hull}\,E : r \leq |z_1| \leq 1\} = \{z \in P\text{-hull}\,E : |z_1| = 1\}$$
$$= \{z \in E : |z_1| = 1\} \subseteq V_1 \cup V_2.$$

By compactness, we can choose $r < 1$ such that

$$\{z \in P\text{-hull}\,E : r \leq |z_1| \leq 1\} \subseteq V_1 \cup V_2.$$

Let

$$Q = \{z \in P\text{-hull}\,E : r \leq |z_1| \leq 1\} \cap V_1;$$

Q is compact and relatively open in $\{z \in P\text{-hull}\,E : r \leq |z_1| \leq 1\}$.

The argument is now completed by means of techniques originating in the work of Stolzenberg [12] (see also [2, p. 133]). Since $z \mapsto \log z_1$ is holomorphic in the neighbourhood V_1 of Q, it follows from the Local Maximum Modulus Principle [12, (1.6)] that

$$\partial_{\mathbb{C}}\{\log z_1 : z \in Q\} \subseteq \{\log z_1 : z \in (\partial_{P\text{-hull}\,E} Q) \cup (Q \cap E)\}.$$

Now, from the relations

$$\partial_{P\text{-hull}\,E} Q = \{z \in Q : |z_1| = 1 \text{ or } r\} \quad \text{and} \quad Q \cap E \subseteq E_1 \subseteq \{z : |z_1| = 1\},$$

it follows that

$$\partial_{\mathbb{C}}\{\log z_1 : z \in Q\} \subseteq \{w \in \mathbb{C} : \mathrm{Re}\, w = 0 \text{ or } \log r\}.$$

Since $\log z_1$ takes values in a horizontal strip, it follows that

$$\{\log z_1 : z \in Q\} \subseteq \{w \in \mathbb{C} : \mathrm{Re}\, w = 0 \text{ or } \log r\},$$

and hence $Q \subseteq \{z \in \mathbb{D}^n : |z_1| = 1 \text{ or } \log r\}$. It now follows that the compact set $\{z \in Q : |z_1| = 1\}$ is relatively open in $P\text{-hull}\,E$. Since $E \setminus \{z \in Q : |z_1| = 1\} = E_2$, it follows from the Shilov Idempotent Theorem that

$$P\text{-hull}\,E = \{z \in Q : |z_1| = 1\} \cup P\text{-hull}\,E_2.$$

Now we have the required contradiction since $\zeta \notin P\text{-hull}\,E_2$ and $|\zeta_1| < 1$.

COROLLARY 4.6. *Let E be a closed subset of $i\mathbb{R}^n$, and suppose that E is the union of its compact, relatively open subsets; then $\mathbb{R}_+^n\text{-hull}\,E = E$.*

Proof. Let m be as in Lemma 4.1. Then $m(E)$ is the union of its compact, relatively open, subsets, and $\overline{m(E)}$ is contained in $m(E) \cup Z_n$. By Proposition 4.5, P-hull $\overline{m(E)}$ is also contained in $m(E) \cup Z_n$ and the result follows from Lemma 4.1.

In the remainder of this section T is assumed to be isometric.

LEMMA 4.7. *If E is a compact, open subset of $\mathrm{Sp}(T,S)$ then there exists a closed subspace Y of X which is T-invariant and is such that $\mathrm{Sp}(T|_Y,S) = E$.*

Proof. We consider the commutative unital Banach algebra obtained by adjoining the identity operator I_X to $\mathcal{A}_T$: $\widehat{\mathcal{A}}_T = \mathcal{A}_T + \mathbb{C}I_X$. The maximal ideal space of this algebra is equal to that of $\mathcal{A}_T$ in the case when I_X is already in $\mathcal{A}_T$ (for example, when T is norm-continuous) and otherwise it is that of $\mathcal{A}_T$ with a point adjoined at infinity.

We need a unit to apply Shilov's Idempotent Theorem to obtain an idempotent P in $\widehat{\mathcal{A}}_T$ which satisfies $\phi_\chi(P) = 1$ for all $\chi \in E$ and $\phi_\chi(P) = 0$ for $\chi \notin E$. We define $Y = P[X]$ and claim that this is the required subspace.

If $f \in L^1(S)$ and $s \in S$, then $T(s)\widehat{f}(T) = \widehat{f_s}(T) = \widehat{f}(T)T(s)$; hence $T(s)P = PT(s)$ and Y is T-invariant. Let V denote the restriction of T to Y. Suppose f is in $L^1(S)$, $\lambda \in \mathbb{C}$ and $y \in Y$; then

$$(\widehat{f}(V) + \lambda I_X)y = (\widehat{f}(T) + \lambda I_X)Py = P(\widehat{f}(T) + \lambda I_X)y$$

and hence

$$\widehat{\mathcal{A}}_V = \{PB : B \in \widehat{\mathcal{A}}_T\}. \tag{4.1}$$

It follows from the definitions that $\mathrm{Sp}(V,S) \subseteq \mathrm{Sp}(T,S)$, so that the only possible characters on $\widehat{\mathcal{A}}_V$ are those of the form ϕ_χ, where χ is in $\mathrm{Sp}(T,S)$, and ϕ_∞, which maps $\widehat{f}(V) + \lambda I_X$ to λ. (4.1) then implies that the only characters are ϕ_χ for χ in E; in other words, $\mathrm{Sp}(V,S) = E$.

LEMMA 4.8. *If χ is an isolated point in the induced topology of $\mathrm{Sp}_{\mathrm{u}}(T,S)$, then χ is an eigenvalue of T.*

Proof. Since, by Theorem 2.4, $\mathrm{Sp}_{\mathrm{u}}(T,S)$ is equal to the Shilov boundary of $\mathcal{A}_T$, χ must be isolated in the Shilov boundary. However, any point isolated in the Shilov boundary must be isolated in the spectrum (see [13, p. 55]). It now follows from Lemma 4.7 that there is a (non-trivial) T-invariant subspace Y of X such that $T|_Y$ has spectrum $\{\chi\}$, from which we may deduce that χ is an eigenvalue of T (see [4, Proposition 4.1]).

A quotient construction. Let T be an isometric semigroup representation on X as above and define $L = \bigcap_{s\in S} T(s)[X]$; L is a closed, T-invariant subspace of X. Now let Y denote the quotient space X/L and V denote the induced isometric representation of S on Y. Clearly the spectrum of V is contained in that of T.

PROPOSITION 4.9. *In the notation above, the unitary spectrum of V contains no isolated points. Furthermore, if $S = \mathbb{R}_+^n$ for some n, then $\mathrm{Sp}(V,S)$ contains no non-empty, compact, open subsets.*

Proof. Suppose χ were an isolated point in $\mathrm{Sp}_{\mathrm{u}}(V,S)$; then by Lemma 4.8; χ would be an eigenvalue of V. Let $y \in Y$ be an associated eigenvector, so $y = x + L$ for some x in X, and let M be the sum of L and the linear span of x. It is easy to see that $T|_M$ would have to be invertible, but this would imply that $M \subseteq L$ and hence $x \in L$ and $y = 0$—a contradiction.

The proof for $\mathbb{R}_+^n$ works in a similar way, for suppose E were a compact open set in $\mathrm{Sp}(V,S)$; then by Lemma 4.7 there would exist a subspace Y of X such that the spectrum of $T|_Y$ were E. Then $T|_Y$ would be norm-continuous (see [5]) and therefore invertible. Now it is easy to obtain a contradiction as above.

5. Our main results.

Here we summarize the circumstances in which we are able to conclude that isometric representations are invertible.

THEOREM 5.1. *Let T be a representation of S by isometries on a Banach space X. If $\mathrm{Sp}_{\mathrm{u}}(T,S)$ is countable, then T is invertible.*

Proof. This follows easily from Proposition 4.9 and from the fact that a representation of S by isometries on a non-trivial Banach space has non-empty unitary spectrum by Corollary 2.2.

For multiparameter semigroups ($S = \mathbb{Z}_+^n$ or $S = \mathbb{R}_+^n$), we have established connections in Sections 2 and 4 between spectral properties and polynomial convexity of subsets of $\mathbb{T}^n$. The latter topic has been studied in [12] and [1], and leads to the following results.

THEOREM 5.2. *Let T be a representation of $\mathbb{Z}_+^n$ by isometries on a Banach space X; if $\mathrm{Sp}_{\mathrm{u}}(T,S)$ is polynomially convex, is contained in a Jordan arc or if it is totally disconnected, then T is invertible.*

Proof. The first part is merely a restatement of Theorem 3.3. The rest follow from the fact that if $\mathrm{Sp}_{\mathrm{u}}(T,S)$ is contained in a Jordan arc or if it is totally disconnected, then it is polynomially convex (see [12], [1]).

THEOREM 5.3. *Let T be a representation of $\mathbb{R}_+^n$ by isometries on a Banach space X. If $\mathrm{Sp}_{\mathrm{u}}(T,\mathbb{R}_+^n)$ is the union of its compact, relatively open subsets, then T is invertible.*

Proof. By Proposition 2.6 and Corollary 4.6,

$$\mathrm{Sp}(T,\mathbb{R}_+^n) = \mathbb{R}_+^n\text{-hull}\,\mathrm{Sp}_{\mathrm{u}}(T,\mathbb{R}_+^n) = \mathrm{Sp}_{\mathrm{u}}(T,\mathbb{R}_+^n).$$

Let Y and V be as in Proposition 4.9; then, since $\mathrm{Sp}(V,\mathbb{R}^n_+) \subseteq \mathrm{Sp}(T,\mathbb{R}^n_+)$, we see that $\mathrm{Sp}(V,\mathbb{R}^n_+)$ must be the union of its relatively open compact subsets. By Proposition 4.9, $\mathrm{Sp}(V,S)$ must be empty and hence $Y=\{0\}$ by Corollary 2.2, so that $X=\bigcap_{s\in S} T(s)[X]$. Thus each $T(s)$ is surjective and T is invertible.

The assumption in Theorem 5.3 that $\mathrm{Sp}_{\mathrm{u}}(T,\mathbb{R}^n_+)$ is the union of its compact, relatively open subsets is equivalent to the condition that the connected components of $\mathrm{Sp}_{\mathrm{u}}(T,\mathbb{R}^n_+)$ are bounded. (We are grateful to Robin Knight for showing us a proof of this.)

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Remarques sur la structure interne des composantes connexes semi-Fredholm

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Résumé. Soit $\mathcal{C}(X,Y)$ l'ensemble des opérateurs fermés à domaines denses dans l'espace de Banach X à valeurs dans l'espace de Banach Y, muni de la métrique du gap. Soit $F_n=\{T\in\mathcal{C}(X,Y) : T$ semi-Fredholm avec $\mathrm{ind}(T)=n\}$ et $C_{n,m}=\{T\in F_n : \alpha(T)=n+m\}$, où $\alpha(T)$ est la dimension du noyau de T. Nous montrons que $\bigcup_{m=0}^{j} C_{n,m}$ est un ouvert de F_n (et donc ouvert dans $\mathcal{C}(X,Y)$) et que $C_{n,m}$ est dense dans $\bigcup_{j\ge m} C_{n,j}$. Nous déduisons quelques résultats de densités. A la fin de se travail nous donnons un exemple d'espace de Banach X tel que, d'une part, F_n n'est pas connexe dans $B(X)$ et d'autre part, l'ensemble des opérateurs semi-Fredholm n'est pas dense dans $B(X)$, contrairement au cas Hilbertien.

Soient X et Y deux espaces de Banach et $\mathcal{C}(X,Y)$ l'ensemble des opérateurs fermés de domaines denses dans X et à valeurs dans Y. Pour $T\in\mathcal{C}(X,Y)$, notons $N(T)$ et $R(T)$ respectivement le noyau et l'image de T. Nous dirons que $T\in\mathcal{C}(X,Y)$ est *semi-Fredholm* (et nous notons $T\in S\Phi(X,Y)$) si $R(T)$ est fermé et $\min(\alpha(T),\beta(T))<\infty$, où $\alpha(T)=\dim N(T)$ et $\beta(T)=\operatorname{codim} R(T)$. Si $T\in S\Phi(X,Y)$, alors l'indice de T sera noté $\mathrm{ind}(T)=\alpha(T)-\beta(T)$.

Dans la suite, on utilisera les notions et les notations du [4, ch. IV].

THÉORÈME 1 [4, théorème 5.17, ch. IV]. *Soit $T,S\in\mathcal{C}(X,Y)$ et T semi-Fredholm. Alors $\exists\delta>0$ tel que si $\widehat{\delta}(S,T)<\delta$, alors* :

(1) *S est semi-Fredholm;*
(2) $\mathrm{ind}(T)=\mathrm{ind}(S)$;
(3) *$\alpha(S)\le\alpha(T)$ et $\beta(S)\le\beta(T)$,*

où $\widehat{\delta}(S,T)$ est le gap entre les graphes de S et de T (voir [4, ch. IV, §2]).

Remarque 1. Le théorème 1 montre que l'ensemble des opérateurs semi-Fredholm est un ouvert de $\mathcal{C}(X,Y)$ muni de la topologie du gap.

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