for each t with $\delta_{n+1} \le |t - t_0| \le \delta_n$. Finally, define $x(t_0) = t_0 x_0$. We show that $x'(t_0) = x_0$. If $\delta_{n+1} \le |\Delta t| \le \delta_n$, then

$$\left\| \frac{x(t_0 + \Delta t) - x(t_0)}{\Delta t} - x_0 \right\| = \left\| \frac{x(t_0 + \Delta t) - (t_0 + \Delta t)x_0}{\Delta t} \right\| < \frac{1}{n}. \ \blacksquare$$

PROBLEM. Let X be an F-space with trivial dual. Does every continuous function from $[a, b]$ into X have a primitive? What happens for $X = L_p$ with $0 \le p < 1$?

Addendum (January 1994). Recently Professor N. J. Kalton sent me his short preprint "The existence of primitives for continuous functions in quasi-Banach space" which contains an affirmative answer to the Problem in the setting of quasi-Banach spaces.

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A recurrence theorem for square-integrable martingales

by

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Abstract. Let $(M_n)_{n \ge 0}$ be a zero-mean martingale with canonical filtration $(\mathcal{F}_n)_{n \ge 0}$ and stochastically L_2-bounded increments $Y_1, Y_2, \ldots$, which means that

$$P(|Y_n| > t \mid \mathcal{F}_{n-1}) \le 1 - H(t) \quad \text{a.s. for all } n \ge 1, \ t > 0$$

and some square-integrable distribution H on $[0, \infty)$. Let $V^2 = \sum_{n \ge 1} E(Y_n^2 \mid \mathcal{F}_{n-1})$. It is the main result of this paper that each such martingale is a.s. convergent on $\{V < \infty\}$ and recurrent on $\{V = \infty\}$, i.e. $P(M_n \in [-c, c] \text{ i.o.} \mid V = \infty) = 1$ for some $c > 0$. This generalizes a recent result by Durrett, Kesten and Lawler [4] who consider the case of only finitely many square-integrable increment distributions. As an application of our recurrence theorem, we obtain an extension of Blackwell's renewal theorem to a fairly general class of processes with independent increments and linear positive drift function.

1. Introduction. Let $(S_n)_{n \ge 0}$ be a random walk with i.i.d. zero-mean, non-vanishing increments $X_1, X_2, \ldots$ Then $(S_n)_{n \ge 0}$ is recurrent with recurrence set $\Re = \mathbb{R}$ in case of non-arithmetic increments, and $\Re = d\mathbb{Z}$ if $X_1, X_2, \ldots$ are d-arithmetic for some $d > 0$. In any case

$$(1.1) \qquad\qquad P(|S_n| \le c \text{ i.o.}) = 1$$

for all $c > 0$. Dispensing with the stationarity assumption on $X_1, X_2, \ldots$, (1.1) need no longer be true. Durrett, Kesten and Lawler [4] give an example of a random walk $(S_n)_{n \ge 0}$ which converges a.s. to ∞, even though its increments are independent and drawn from a set of merely two zero-mean distributions. On the other hand, they also show that (1.1) holds true for sufficiently large c provided that $X_1, X_2, \ldots$ are independent and drawn from a finite set of distributions with mean 0 and finite, positive variances. In fact, their result is even stated for so-called controlled random walks, that is, general martingales with square-integrable conditional increment distributions drawn from a finite set. Although their proof uses the finiteness of the latter

set, there is no strong intuitive evidence for this being indispensable as long as the conditional variances remain bounded. It is the major contribution of this article to support the previous conjecture by extending the result to a rather general class of martingales with square-integrable increments, to be defined further below. While Durrett, Kesten and Lawler's work was initiated by related questions on p-type branching processes, formulated by A. Spataru in a letter to F. Spitzer (see [4]), our interest grew out of renewal theoretic investigations on a certain class of linear submartingales in [1]–[3]. In fact, the proof of Blackwell's renewal theorem for linearly growing random walks with independent, but non-stationary increments naturally leads to the question of recurrence of a martingale of the form $M_n = S_n - S'_n$, $n \geq 0$, if a coupling argument is used. Here $(S_n)_{n\geq 0}$ is the random walk of interest and $(S'_n - S'_0)_{n\geq 0}$ forms an independent copy of $(S_n - S_0)_{n\geq 0}$. The delay S'_0 is chosen in a suitable manner, as usual for a coupling proof. A more detailed account is given in Section 2 where we derive an extension of Blackwell's renewal theorem to a fairly broad class of linear random walks with independent increments.

Before we formulate our results, we give some definitions and notations. A sequence $(Y_n)_{n\geq 0}$, adapted to a filtration $(\mathcal{F}_n)_{n\geq 0}$, is called *stochastically bounded* (with respect to $(\mathcal{F}_n)_{n\geq 0}$) if there are integrable distributions F, G such that for all $n \geq 1$ and $t \in \mathbb{R}$,

$$(1.2) \qquad G(t) \ \leq \ P(Y_n \leq t \mid \mathcal{F}_{n-1}) \ \leq \ F(t) \quad \text{a.s.}$$

where the convention of identifying a distribution with its distribution function has been used, i.e. $F(t) = F((-\infty, t])$, etc. F and G are called a *minorant* and a *majorant* for $(Y_n)_{n\geq 1}$, resp. If F and G have both finite variances, $(Y_n)_{n\geq 1}$ is called *stochastically L_2-bounded*. It is easily seen that the latter assumption is slightly stronger than conditional uniform integrability of $Y_1^2, Y_2^2, \ldots$ with respect to $(\mathcal{F}_n)_{n\geq 0}$ and slightly weaker than the frequently encountered conditional Lyapunov-type condition

$$(1.3) \qquad \sup_{n\geq 1} \|E(|Y_n|^{2+\delta} \mid \mathcal{F}_{n-1})\|_\infty < \infty \quad \text{for some } \delta > 0,$$

where $\|\cdot\|_\infty$ denotes the usual L_∞-norm on the underlying probability space. In particular, it implies uniform boundedness of the conditional variances of $Y_1, Y_2, \ldots$ Note also that the condition for stochastic L_2-boundedness in the Abstract is indeed equivalent to the one chosen here.

Now let $(M_n)_{n\geq 0}$ be a zero-mean square-integrable martingale with increments $Y_1, Y_2, \ldots$, canonical filtration $(\mathcal{F}_n)_{n\geq 0}$ and

$$(1.4) \quad V_n^2 = \sum_{j=1}^n E(Y_j^2 \mid \mathcal{F}_{n-1}) \quad \text{and} \quad V^2 = \lim_{n\to\infty} V_n^2 = \sum_{j\geq 1} E(Y_j^2 \mid \mathcal{F}_{n-1}).$$

For $n \geq 0$ let further

$$(1.5) \qquad \widehat{M}_n = M_{\nu_n}\mathbf{1}(\nu_n < \infty) + \sum_{j=1}^n W_{n+1-j}\mathbf{1}(\nu_{j-1} < \infty = \nu_j),$$

where

$$(1.6) \quad \nu_0 = 0 \quad \text{and} \quad \nu_n = \inf\{j \geq 1 : V_{\nu_{n-1}+j}^2 - V_{\nu_{n-1}}^2 \geq 1\} \quad \text{for } n \geq 1$$

and $(W_n)_{n\geq 0}$ defines a random walk with i.i.d. standard normal increments and independent of $(M_n)_{n\geq 0}$. It is easily verified that $(\widehat{M}_n)_{n\geq 0}$ forms again a zero-mean square-integrable martingale, whose increments are denoted by $\widehat{Y}_1, \widehat{Y}_2, \ldots$ and its canonical filtration by $(\widehat{\mathcal{F}}_n)_{n\geq 0}$. The introduction of $(W_n)_{n\geq 0}$ is for purely technical reasons which become apparent in the proof of Theorem 1 in Section 3.

THEOREM 1. *Let $(M_n)_{n\geq 0}$ be a zero-mean square-integrable martingale such that $\widehat{Y}_1, \widehat{Y}_2, \ldots$ are stochastically L_2-bounded (with respect to $(\widehat{\mathcal{F}}_n)_{n\geq 0}$). Then, for some $c > 0$,*

$$(1.7) \qquad P(M_n \text{ converges or } M_n \in [-c, c] \ i.o.) = 1.$$

More precisely, M_n converges a.s. on $\{V < \infty\}$ and

$$(1.8) \qquad P(M_n \in [-c, c] \ i.o. \mid V = \infty) = 1 \quad \text{for some } c > 0.$$

Observe that stochastic L_2-boundedness of $\widehat{Y}_1, \widehat{Y}_2, \ldots$ could have been claimed also only for $\widehat{Y}_1\mathbf{1}(\nu_1 < \infty), \widehat{Y}_2\mathbf{1}(\nu_2 < \infty), \ldots$ because outside the restricting events the $\widehat{Y}_n$ are obviously just chi-square variables of degree 1 and thus clearly stochastically L_2-bounded.

At first sight (1.8) does not exclude the possibility of M_n being convergent on $\{V = \infty\}$. However, the following corollary sets us right.

COROLLARY 1. *In the situation of Theorem 1, on $\{V = \infty\}$,*

$$(1.9) \qquad \liminf_{n\to\infty} M_n = -\infty \quad \text{and} \quad \limsup_{n\to\infty} M_n = \infty \quad \text{a.s.}$$

This is an extension of the Chung–Fuchs–Ornstein Theorem and has been known for martingales $(M_n)_{n\geq 0}$ which satisfy $E \sup_{n\geq 1} |Y_n| < \infty$, where (1.9) holds true on the event $\{M_n \text{ diverges}\}$, see [6], p. 32. Corollary 1 shows that under the assumptions of Theorem 1 the latter event a.s. coincides with $\{V = \infty\}$.

Our second corollary deals with an interesting subclass of martingales for which Theorem 1 holds true, namely those being very close to random walks with i.i.d. zero-mean increments with positive variance. It contains, in particular, all martingales with conditional increment distributions drawn from a finite set of distributions with positive variance. In the latter case, Corollary 2 has been obtained by Durrett, Kesten and Lawler [4], Theorem 1.

COROLLARY 2. *Let* $(M_n)_{n \geq 0}$ *be a zero-mean square-integrable martingale with stochastically L_2-bounded $Y_1, Y_2, \ldots$ which further satisfy*

$$(1.10) \qquad V_n^2 - V_{n-1}^2 = E(Y_n^2 \mid \mathcal{F}_{n-1}) \geq \sigma_*^2 \quad a.s.$$

for all $n \geq n_0 \geq 1$ and some $\sigma_*^2 > 0$ not depending on n. Then, for some $c > 0$,

$$(1.11) \qquad P(M_n \in [-c, c] \ i.o.) = 1$$

and (1.9) *holds true.*

A combination of Theorem 1 with a geometric trial argument will prove

COROLLARY 3. *In the situation of Theorem 1, define*

$$(1.12) \quad \begin{aligned} \alpha_k^\pm(t) &= \liminf_{n \to \infty} \max_{1 \leq j \leq k} \{ \| P(0 < (M_{n+j} - M_n)^\pm < t \mid \mathcal{F}_n) \|_\infty \}, \\ \beta_k^\pm(t) &= \liminf_{n \to \infty} \max_{1 \leq j \leq k} \{ \| P((M_{n+j} - M_n)^\pm = t \mid \mathcal{F}_n) \|_\infty \} \end{aligned}$$

for $t > 0$ and $k \geq 1$. Suppose $V = \infty$ a.s. and, for some $\varepsilon > 0$,

$$(1.13) \qquad \sup_{k \geq 1} \alpha_k^-(\delta) \vee \alpha_k^+(\delta) > 0 \quad \text{for all } \delta \in (0, \varepsilon).$$

Then $(M_n)_{n \geq 0}$ is recurrent on $\mathbb{R}$, i.e. $P(M_n \in [x - \varepsilon, x + \varepsilon] \ i.o.) = 1$ for all $x \in \mathbb{R}$ and $\varepsilon > 0$. If $(M_n)_{n \geq 0}$ is concentrated on a lattice $d\mathbb{Z}$, $d > 0$, and if instead of (1.13),

$$(1.14) \qquad \sup_{k \geq 1} \beta_k^-(d) \vee \beta_k^+(d) > 0$$

holds true, then $(M_n)_{n \geq 0}$ is recurrent on $d\mathbb{Z}$, i.e. $P(M_n = kd \ i.o.) = 1$ for each $k \in \mathbb{Z}$.

Dispensing with conditions like (1.13) or (1.14), it is easy to provide an example of a recurrent martingale which never visits a sufficiently small neighborhood of 0.

A number of papers have dealt with related recurrence problems, the earliest one we know of being by Lamperti [10]. Kemperman [7], Rogozin and Foss [11] and Lalley [9] consider so-called oscillating random walks, a subclass of random walks with finitely many increment distributions. The latter are studied by Durrett, Kesten and Lawler [4], as already mentioned, and in a recent companion paper by Kesten and Lawler [8].

The proofs of the previous results are presented in Section 3. The next section is devoted to an extension of Blackwell's renewal theorem to certain random walks with independent increments by combining Corollary 3 with a result from [2].

2. An extension of Blackwell's renewal theorem.

Suppose $(S_n)_{n \geq 0}$ is a random walk with $S_0 = 0$ and *stochastically stable* (s.s.) increments $X_1, X_2, \ldots$ with mean $\theta \in (0, \infty)$, which means that $X_1, X_2, \ldots$ are stochastically bounded and satisfy the mean stability condition

$$(2.1) \qquad \lim_{k \to \infty} \sup_{n \geq 0} \| k^{-1} E(S_{n+k} - S_n \mid \mathcal{G}_n) - \theta \|_\infty = 0,$$

where $\mathcal{G}_n = \sigma(S_0, \ldots, S_n)$. These random walks have been shown in [2] to satisfy the following Blackwell-type renewal theorem (see Proposition 5.1): If $U = \sum_{n \geq 0} P(S_n \in \cdot)$ denotes the renewal measure of $(S_n)_{n \geq 0}$ and ℓ_0 Lebesgue measure on $\mathbb{R}$, then for each $\varepsilon > 0$, there is a delay distribution H such that

$$(2.2) \qquad (\theta + \varepsilon)^{-1} \ell_0(I) \leq \liminf_{t \to \infty} H * U(t + I)$$
$$\leq \limsup_{t \to \infty} H * U(t + I) \leq (\theta - \varepsilon)^{-1} \ell_0(I).$$

It has further been shown in [2], and also in [3], that even

$$(2.3) \qquad \lim_{t \to \infty} U(t + I) = \theta^{-1} \ell_0(I)$$

holds true for certain subclasses of random walks, the intrinsic feature of which is that they "contain" a random walk with i.i.d. increments ([2]) or with independent increments sharing an appropriate distributional regularity property ([3]). As for the former ones, (2.3) follows by employing a coupling argument based upon the recurrence of a zero-mean random walk with i.i.d. increments. With the results in Section 1 of this paper, we can proceed in a similar manner to obtain (2.3) for a fairly general class of random walks with independent increments to be introduced below. Their obvious property must be that their symmetrizations form *recurrent* martingales on $\mathbb{R}$ making for a successful coupling.

Given a random walk $(S_n)_{n \geq 0}$ with canonical filtration $\mathcal{G}_n$ and increments $X_1, X_2, \ldots$, we call $(x_n)_{n \geq 1}$ a *sequence of uniform points of increase* for $(X_n)_{n \geq 1}$ if there is a positive function f, defined on any interval $(0, \varepsilon)$, such that for all $\delta \in (0, \varepsilon)$ and $n \geq n(\delta)$,

$$(2.4) \quad P(0 < (X_n - x_n)^- < \delta \mid \mathcal{G}_{n-1})$$
$$\vee P(0 < (X_n - x_n)^+ < \delta \mid \mathcal{G}_{n-1}) \geq f(\delta) \quad \text{a.s.}$$

As one can easily see, $(S_n)_{n \geq 0}$ is non-arithmetic under (2.4), i.e.

$$P(S_n - S_0 \in d\mathbb{Z} \text{ for all } n \geq 0) < 1 \quad \text{for all } d > 0.$$

It is further easily verified that each stochastically bounded sequence $(X_n)_{n \geq 1}$ with $P(X_n \in \cdot \mid \mathcal{G}_{n-1})$ being continuous a.s. for all $n \geq 1$ possesses a sequence of uniform points of increase. Combining (2.2) with Corollary 3, we can prove

THEOREM 2. *Let $(S_n)_{n \geq 0}$ be a random walk with $S_0 = 0$, renewal measure U and independent, s.s. increments $X_1, X_2, \ldots$ with positive mean θ.*

Suppose further that $(X_n)_{n\geq 1}$ possesses a sequence $(x_n)_{n\geq 1}$ of uniform points of increase. Then (2.3) holds true.

Proof. Suppose first $(X_n)_{n\geq 1}$ to be even stochastically L_2-bounded, thus in particular uniformly square-integrable. Let $(X'_n)_{n\geq 1}$ be an independent copy of $(X_n)_{n\geq 1}$ with associated random walk $(S'_n)_{n\geq 0}$. Let further S'_0 be independent of $(X_n, X'_n)_{n\geq 1}$ and with distribution H. It follows that $H * U$ is the renewal measure of $(S'_n)_{n\geq 0}$. Given an arbitrary $\varepsilon > 0$, we choose H such that (2.2) holds true. (2.4) and a simple estimation imply

$$(2.5) \quad 2P(0 < (X_n - X'_n)^{\pm} < 2\delta) = P(0 < |X_n - X'_n| < 2\delta)$$
$$\geq P(0 < |X_n - x_n| < \delta)P(0 < |X'_n - x_n| < \delta) \geq f(\delta)^2 \quad \text{a.s.}$$

for each sufficiently small $\delta > 0$, which shows that $M_n = S_n - S'_n$, $n \geq 0$, satisfies condition (1.13) of Corollary 3. Furthermore, $\lim_{\delta\to 0} f(\delta) = 0$, so that

$$E(X_n - X'_n)^2 \geq \int_0^{2\delta} 2tP(t < |X_n - X'_n| < 2\delta)\,dt \geq \int_0^{2\delta} 2t(f(\delta)^2 - f(t)^2)\,dt > 0$$

for all $n \geq n_0(\delta)$. This shows that $(M_n)_{n\geq 0}$ also satisfies the conditions of Corollary 2 and we thus infer that it is recurrent on $\mathbb{R}$. As a consequence, the ε-coupling time

$$T = \inf\{n \geq 0 : |S_n - S'_n| < \varepsilon\}$$

is a.s. finite, and the associated coupling process

$$(2.6) \quad \widehat{S}_n = S_n \mathbf{1}(T \geq n) + (S'_n - S'_T + S_T)\mathbf{1}(T < n), \quad n \geq 0,$$

defines a copy of $(S_n)_{n\geq 0}$ and thus has the same renewal measure U. (2.3) is now derived by a standard estimation of $U(t+I) - H*U(t+I)$, the details of which may e.g. be found in [2].

If $(X_n)_{n\geq 1}$ is only stochastically (L_1-)bounded, then note first that $(x_n)_{n\geq 1}$ must be bounded because of a simple tightness argument. As a consequence, we can choose X'_n to be an independent copy of X_n, given $|X_n| \leq 1 + \sup_{j\geq 1} x_j$, and equal to X_n otherwise. We leave it to the reader to verify that the above coupling argument still applies because $M_n = S_n - S'_n$, $n \geq 0$, is a martingale with now even bounded increments, and the coupling process $(\widehat{S}_n)_{n\geq 0}$ in (2.6) is still a copy of $(S_n)_{n\geq 0}$.

Remarks. (a) It is not difficult to formulate an arithmetic version of condition (2.4) and then of Theorem 2 also. However, in contrast to the arithmetic case this does not lead to an improvement over the results in [2] and [3], and we therefore omit further details.

(b) It is natural to ask why $X_1, X_2, \ldots$ have to be independent in the above theorem, because $M_n = S_n - S'_n$, $n \geq 0$, is also recurrent without this assumption. Unfortunately, the coupling process $(\widehat{S}_n)_{n\geq 0}$ in (2.6) then

needs no longer be a copy of $(S_n)_{n\geq 0}$, which is essential for the coupling argument.

3. Proofs of the results in Section 1. In the following, we assume that the martingale $(M_n)_{n\geq 0}$ under consideration is defined on a measurable space $(\Omega, \mathcal{A})$ which is large enough to carry a family of probability measures P_ψ such that $P_\psi((M_n)_{n\geq 0} \in \cdot) = \psi$ for $\psi \in \Psi$, the set of all martingal distributions "starting at 0", i.e. $P_\psi(M_0 = 0) = 1$. We denote by $\Psi_{F,G}$ the subset of those ψ such that, under P_ψ and with respect to its canonical filtration $(\mathcal{F}_n)_{n\geq 0}$, $(M_n)_{n\geq 0}$ has stochastically L_2-bounded increments $Y_1, Y_2, \ldots$ with minorant F and majorant G which are thus assumed to be square-integrable. As a further subclass we consider $\Psi_{F,G}^+$ containing those ψ such that $Y_1, Y_2, \ldots$ additionally satisfy (1.10) with $\sigma_* = 1$ under P_ψ, i.e.

$$(3.1) \quad E_\psi(Y_n^2 \mid \mathcal{F}_{n-1}) \geq 1 \quad \text{a.s. for all } n \geq 1.$$

Let us also define

$$\psi_n(x, \cdot) = P_\psi((M_{n+k} - M_n)_{k\geq 0} \in \cdot \mid (M_0, \ldots, M_n) = x),$$
$$Q_{n,k}^\psi(x, \cdot) = P_\psi(M_{n+k} - M_n \in \cdot \mid (M_0, \ldots, M_n) = x)$$

for $\psi \in \Psi$, $n \geq 0$, $k \geq 1$ and $x \in \mathbb{R}^{n+1}$, and finally

$$\mathcal{D}_{F,G} := \{Q_{n,1}^\psi(x, \cdot) : n \geq 0, x \in \mathbb{R}^{n+1} \text{ and } \psi \in \Psi_{F,G}^+\}.$$

We first prove

LEMMA 1. *For all square-integrable* F, G *with* $G \leq F$, *there is some* $\varepsilon > 0$ *such that*

$$(3.2) \quad \inf_{Q \in \mathcal{D}_{F,G}} Q((\varepsilon, \infty)) > 0 \quad \text{and} \quad \inf_{Q \in \mathcal{D}_{F,G}} Q((-\infty, -\varepsilon)) > 0.$$

Proof. Let $\mathcal{D} = \mathcal{D}_{F,G}$, $a < 1$ and suppose there is a sequence $Q_n \in \mathcal{D}$, $n \geq 1$, such that $Q_n([-a, a]) \to 1$ as $n \to \infty$. Since $\mathcal{D}$ is relatively compact, we may assume that Q_n converges weakly to some Q. Consequently, by stochastic L_2-boundedness

$$\lim_{n\to\infty} \int x^2 Q_n(dx) = \int x^2 Q(dx) \leq a^2 < 1,$$

which contradicts (3.1). We have thus shown $\inf_{Q\in\mathcal{D}} Q([-\varepsilon, \varepsilon]^c) > 0$. A similar argument utilizing $\int x\, Q(dx) = 0$ for all $Q \in \mathcal{D}$ also proves (3.2). We omit further details.

For $b \in \mathbb{R}$, we next define the first passage times

$$\tau^+(b) = \inf\{n \geq 0 : M_n > b\}, \quad \tau^-(b) = \inf\{n \geq 0 : M_n < -b\}$$

and

$$\tau(b) = \inf\{n \geq 0 : |M_n| > b\}.$$

LEMMA 2. *For all square-integrable* F, G *with* $G \leq F$, *there is some positive constant* C *such that*

$$(3.3) \qquad \sup_{\psi \in \Psi_{F,G}^+} E_\psi \tau(b) \leq Cb^2.$$

Moreover,

$$(3.4) \qquad \lim_{b \to \infty} \sup_{\psi \in \Psi_{F,G}^+} |P_\psi(M_{\tau(b)} > b) - 1/2| = 0.$$

P r o o f. Without loss of generality we may assume $\varepsilon = 1$ in (3.2). Then for all $b \in \mathbb{N}$,

$$\eta_b := \sup_{\psi \in \Psi_{F,G}^+} \sup_{n \geq 0} \sup_{x \in \mathbb{R}^{n+1}} Q_{n,2b}^\psi(x, [-2b, 2b]) < 1,$$

which implies after successive conditioning

$$P_\psi(\tau(b) > 2nb) \leq P_\psi(|M_{2bj} - M_{2b(j-1)}| \leq 2b \text{ for } 1 \leq j \leq n) \leq \eta_b^k$$

and therefore

$$(3.5) \qquad P_\psi(\tau(b) < \infty) = 1 \quad \text{and} \quad E_\psi \tau(b)^p < \infty$$

for all $\psi \in \Psi_{F,G}^+$, $b > 0$ and $p > 0$. For the proof of (3.3), we employ a similar argument to that in Gundy and Siegmund [5]. For $m, n \geq 1$ put $Y_n(m) = Y_n 1(|Y_n| \geq m)$. By stochastic L_2-boundedness of $Y_1, Y_2, \dots$ we can choose m so large that, given any $\varepsilon \in (0, 1)$,

$$(3.6) \qquad \sup_{\psi \in \Psi_{F,G}^+} \sup_{n \geq 1} \|E_\psi(Y_n(m)^2 \mid \mathcal{F}_{n-1})\|_\infty < \varepsilon^2.$$

With the help of the optional sampling theorem, (3.1), (3.6) and the inequality $M_{\tau(b)}^2 \leq (b + |Y_{\tau(b)}|)^2$, we now obtain for all $\psi \in \Psi_{F,G}^+$,

$$
\begin{aligned}
E_\psi \tau(b) &\leq E_\psi \Big(\sum_{j=1}^{\tau(b)} E_\psi(Y_j^2 \mid \mathcal{F}_{j-1}) \Big) = E_\psi M_{\tau(b)}^2 \\
&\leq E_\psi (b + |Y_{\tau(b)}|)^2 \leq E_\psi(b + m + |Y_{\tau(b)}(m)|)^2 \\
&\leq (b + m)^2 + 2(b + m)\{E_\psi Y_{\tau(b)}(m)^2\}^{1/2} + E_\psi Y_{\tau(b)}(m)^2 \\
&\leq (b + m)^2 + 2(b + m)\Big\{E_\psi\Big(\sum_{j=1}^{\tau(b)} Y_j(m)^2\Big)\Big\}^{1/2} + E_\psi\Big(\sum_{j=1}^{\tau(b)} Y_j(m)^2\Big) \\
&\leq (b + m)^2 + 2\varepsilon(b + m)\{E_\psi \tau(b)\}^{1/2} + \varepsilon^2 E_\psi \tau(b),
\end{aligned}
$$

hence

$$\Big(\frac{b + m}{\{E_\psi \tau(b)\}^{1/2}} + \varepsilon \sigma_* \Big)^2 \geq 1,$$

and the latter inequality obviously implies (3.3) for some $C > 0$.

In order to prove (3.4), let $R_b = M_{\tau(b)} - b$, $R_{-b} = -b - M_{\tau(b)}$ and $A_b = \{R_b > 0\}$ for $b \geq 0$. Note that $A_b^c = \{R_{-b} > 0\}$ and that

$$R_b \mathbf{1}(A_b) + R_{-b} \mathbf{1}(A_b^c) \leq |Y_{\tau(b)}|.$$

Another appeal to the optional sampling theorem combined with the previous inequality and (3.3) then gives for all $\psi \in \Psi_{F,G}^+$,

$$
\begin{aligned}
0 = E_\psi M_{\tau(b)} &= b(2P_\psi(A_b) - 1) + \int_{A_b} R_b \, dP_\psi - \int_{A_b^c} R_{-b} \, dP_\psi \\
&\leq b(2P_\psi(A_b) - 1) + m + \{E_\psi Y_{\tau(b)}(m)^2\}^{1/2} \\
&\leq b(2P_\psi(A_b) - 1) + m + \varepsilon\{E_\psi \tau(b)\}^{1/2} \\
&\leq b(2P_\psi(A_b) - 1) + m + \varepsilon C^{1/2} b
\end{aligned}
$$

and similarly

$$0 \geq b(2P_\psi(A_b) - 1) - m - \varepsilon C^{1/2} b.$$

Both inequalities together show

$$|P_\psi(A_b) - 1/2| \leq m/b + \varepsilon C^{1/2}.$$

But $\varepsilon > 0$ can be made arbitrarily small by choosing m sufficiently large, so that

$$\lim_{b \to \infty} \sup_{\psi \in \Psi_{F,G}^+} |P_\psi(A_b) - 1/2| = 0,$$

which is the same as (3.4).

LEMMA 3. *In the situation of Lemma 2,* $\tau^+(b)$ *is* P_ψ-*a.s. finite for all* $b \geq 0$ *and* $\psi \in \Psi_{F,G}^+$. *Furthermore, there exists* $c > 0$ *such that*

$$(3.7) \qquad \kappa := \inf_{\psi \in \Psi_{F,G}^+} \inf_{b \geq 0} P_\psi(\tau^+(b) \leq c(b+1)^2) > 0.$$

P r o o f. Choose a so large that, by (3.4), $P_\psi(M_{\tau(b)} < -b) \leq 2/3$ for all $b \geq a$ and $\psi \in \Psi_{F,G}^+$. Define

$$\tau_\xi(b) = \inf\{n \geq 1 : |M_{\xi+n} - M_\xi| > b\}$$

for $b \geq 0$ and each a.s. finite stopping time ξ with respect to $(\mathcal{F}_n)_{n \geq 0}$, and then recursively

$$\xi_1 = \tau(b) \quad \text{and} \quad \xi_n = \xi_{n-1} + \tau_{\xi_{n-1}}(|M_{\xi_{n-1}}|) \quad \text{for } n \geq 2.$$

As one can easily see, $\{\tau^+(b) > \xi_n\} \subset A_n$, where

$$A_n := \{M_{\xi_1} < -b, \ M_{\xi_2} - M_{\xi_1} < -|M_{\xi_1}|, \dots, M_{\xi_n} - M_{\xi_{n-1}} < -|M_{\xi_{n-1}}|\}$$

for each $n \geq 1$, so that for each $\psi \in \Psi_{F,G}^+$,

$$P_\psi(\tau^+(b) > \xi_n) \leq \int_{A_{n-1}} P_{\psi_{\xi_{n-1}}((M_0,\dots,M_{\xi_{n-1}}),\cdot)}(M_{\tau(|M_{\xi_{n-1}}|)} < -|M_{\xi_{n-1}}|)\, dP_\psi$$

$$\leq \frac{2}{3} P_\psi(A_{n-1}) \leq \dots \text{(inductively)} \leq \left(\frac{2}{3}\right)^n,$$

thus $P_\psi(\tau^+(b) < \infty) = 1$, where $\psi_{\xi_{n-1}}((M_0,\dots,M_{\xi_{n-1}}),\cdot) \in \Psi_{F,G}^+$ P_ψ-a.s. should be observed.

As for (3.7), we note that for all $b \geq a$, with a as chosen at the beginning of the proof, and for C as in (3.3),

$$\inf_{\psi \in \Psi_{F,G}^+} P_\psi(\tau^+(b) \leq 4Cb^2) \geq \inf_{\psi \in \Psi_{F,G}^+} P_\psi(\tau(b) \leq 4Cb^2,\ M_{\tau(b)} > b)$$

$$\geq \inf_{\psi \in \Psi_{F,G}^+} P_\psi(M_{\tau(b)} > b) - \sup_{\psi \in \Psi_{F,G}^+} P_\psi(\tau(b) > 4Cb^2)$$

$$\geq \frac{1}{3} - \sup_{\psi \in \Psi_{F,G}^+} \frac{E_\psi \tau(b)}{4Cb^2} \geq \frac{1}{3} - \frac{1}{4} > 0.$$

The desired conclusion follows by choosing $c \geq 4C$ large enough so that $P_\psi(\tau^+(b) \leq c(b+1)^2)$ also has a uniform positive lower bound for all $\psi \in \Psi_{F,G}^+$ and $0 \leq b \leq a$.

Next, we define for $b \geq 0$ and $\psi \in \Psi_{F,G}^+$,

$$H_\psi(b,\cdot) = E_\psi\left(\sum_{n=0}^{\tau^+(b)-1} \mathbf{1}(M_n \in \cdot)\right) = \sum_{n \geq 0} P_\psi(\tau^+(b) > n,\ M_n \in \cdot).$$

LEMMA 4. *In the situation of Lemma 2, there exists some $c > 0$ such that for all $x \geq 0$,*

(3.8) $$\sup_{\psi \in \Psi_{F,G}^+} \sup_{b \geq 0} H_\psi(b, (b-x, b]) \leq c(x+1)^2.$$

Proof. Let $R_b^+ = M_{\tau^+(b)} - b$ for $b \geq 0$. We obtain for all $\psi \in \Psi_{F,G}^+$ and $b \geq 0$,

$$H_\psi(b, (b-x, b])$$

$$= \int_{\{R_{b-x}^+ \leq x\}} E_{\psi_\tau}\left(\sum_{n=0}^{\tau^+(x-R_{b-x}^+)-1} \mathbf{1}(M_n \in (-R_{b-x}^+, x - R_{b-x}^+])\right) dP_\psi$$

$$[\text{where } \psi_\tau = \psi_\tau((M_0,\dots,M_\tau),\cdot), \text{ and } \tau = \tau^+(b-x)]$$

$$= \int_{\{R_{b-x}^+ \leq x\}} H_{\psi_\tau}(x - R_{b-x}^+, (-R_{b-x}^+, x - R_{b-x}^+])\, dP_\psi$$

$$\leq \sup_{\phi \in \Psi_{F,G}^+} \sup_{y \leq x} H_\phi(y, [y-x, y]).$$

A similar calculation with c, κ as in Lemma 3 and with N denoting the integer part of $c(y+1)^2$ leads to

$$H_\psi(y, (y-x, y])$$

$$\leq c(y+1)^2 + E_\psi\left(\sum_{n=N+1}^{\tau^+(y)-1} \mathbf{1}(M_n \in (y-x, y], \tau^+(y) > N)\right)$$

$$= c(y+1)^2$$

$$+ \int_{\{\tau^+(y) > N\}} H_{\psi_N((M_0,\dots,M_N),\cdot)}(y - M_N, (y - M_N - x, y - M_N])\, dP_\psi$$

$$\leq c(x+1)^2 + (1-\kappa)\sup_{\phi \in \Psi_{F,G}^+} \sup_{b \geq 0} H_\phi(b, (b-x, b])$$

for all $\psi \in \Psi_{F,G}^+$ and $0 \leq y \leq x$. By combining both previous inequalities and taking suprema we finally infer

$$\sup_{\psi \in \Psi_{F,G}^+} \sup_{b \geq 0} H_\psi(b, (b-x, b]) \leq c(1+x)^2 + (1-\kappa)\sup_{\psi \in \Psi_{F,G}^+} \sup_{b \geq 0} H_\psi(b, (b-x, b]),$$

and this obviously proves (3.8).

Needless to say, all previous results for $\tau^+(b)$ are also true for $\tau^-(b)$ as follows by switching to $(-M_n)_{n \geq 0}$. After these preparations we are now ready to prove the results in Section 1. We begin with the proof of Theorem 1 which can be combined with that of Corollary 2.

Proof of Theorem 1 and Corollary 2. Put $\sigma_n^2 = E(Y_n^2 \mid \mathcal{F}_{n-1})$ for $n \geq 1$. Then, for each $b > 0$, $M_n(b) = \sum_{j=1}^n Y_j \mathbf{1}(V_j^2 \leq b)$, $n \geq 0$, is a zero-mean square-integrable martingale which further satisfies

$$EM_n(b)^2 = \sum_{j=1}^n EY_j^2 \mathbf{1}(V_j^2 \leq b) = \sum_{j=1}^n \sigma_j^2 \mathbf{1}(V_j \leq b) \leq EV_n^2 \mathbf{1}(V_n^2 \leq b) \leq b$$

for all $n \geq 0$. Thus $M_n(b)$ converges a.s. for each $b > 0$ by the martingale convergence theorem, and this is equivalent to the a.s. convergence of M_n on $\{V < \infty\}$.

For the second part of the proof we readopt the notation which we have used for the lemmata above. So we denote by P_ψ (instead of P) the un-

derlying probability measure if $\psi \in \Psi$ is the given distribution of $(M_n)_{n\geq0}$. The resulting dependence of V_n and V on ψ, however, is suppressed in the following, which means that we keep writing V_n and V only. Suppose $P_\psi(V = \infty) > 0$ and consider the martingale $(\widehat{M}_n)_{n\geq0}$ as defined in (1.5); let $\widehat{\psi}$ denote its distribution under P_ψ. By assumption of Theorem 1, $\widehat{\psi} \in \Psi^+_{F,G}$ if F, G are a minorant and a majorant of $\widehat{Y}_1, \widehat{Y}_2, \ldots$, respectively. Validity of (3.1) for the latter sequence is evident from (1.6). Moreover, the quadratic variation sequence $\widehat{V}_n^2 := \sum_{j=1}^n E_\psi(\widehat{Y}_j^2 \mid \widehat{\mathcal{F}}_{j-1})$ converges a.s. to ∞ as $n \to \infty$, and finally

(3.9) $P_\psi(\widehat{M}_n \in [-c,c] \text{ i.o.}) = 1 \;\Rightarrow\; P_\psi(M_n \in [-c,c] \text{ i.o.} \mid V = \infty) = 1,$

where the latter assertion is the one still to be proved. We will therefore finish the proof of Theorem 1 by showing the first assertion of (3.9), and in order to save notation we replace $(\widehat{M}_n)_{n\geq0}$ by $(M_n)_{n\geq0}$ itself and assume that its distribution ψ is an element of $\Psi^+_{F,G}$, in particular $V = \infty$ P_ψ-a.s. This, however, means nothing but proving Corollary 2 next.

Let $\tau_\xi^{\pm}(b) = \inf\{n \geq 1 : \pm(M_{\xi+n} - M_\xi) > b\}$ for each a.s. finite stopping time ξ and define $\zeta_0 = \tau^-(0)$,

$$\xi_1 = \zeta_0 + \tau^+(-M_{\xi_0}), \quad \zeta_1 = \xi_1 + \tau^-_{\xi_1}(M_{\xi_1}),$$
$$\xi_2 = \zeta_1 + \tau^+_{\zeta_1}(-M_{\zeta_1}), \quad \zeta_2 = \xi_2 + \tau^-_{\xi_2}(M_{\xi_2}), \ldots,$$

which are easily seen to be the successive random times where M_n moves from $(-\infty, 0)$ to $(0, \infty)$ or vice versa. Since $Y_1, Y_2, \ldots$ are supposed to be stochastically L_2-bounded with minorant F and majorant G, we inductively infer from Lemma 3 that $\xi_n < \infty$ and $\zeta_n < \infty$ P_ψ-a.s. for all $n \geq 1$ where $\psi_\tau((M_0, \ldots, M_\tau), \cdot) \in \Psi^+_{F,G}$ P_ψ-a.s. for each a.s. finite stopping time τ is repeatedly used. Recalling the definitions of R_b^+ and $H_\psi(b, \cdot)$ from the proof of Lemma 4, we now infer that for all $n \geq 1$, $s > 0$ and $P_\psi((\zeta_{n-1}, M_0, \ldots, M_{\zeta_{n-1}}) \in \cdot)$-almost all $(k, x) \in \mathbb{N} \times \mathbb{R}^k$,

(3.10) $P_\psi(M_{\xi_n} > s \mid \zeta_{n-1} = k, (M_0, \ldots, M_k) = x) = P_{\psi_k(x,\cdot)}(R^+_{-x_k} > s)$

$$= \sum_{n\geq1} \int_{(-\infty,-x_k]} P_{\psi_k(x,\cdot)}(Y_n > s - x_k - y \mid M_{n-1} = y)$$
$$\times \, dP_{\psi_k(x,\cdot)}(M_{n-1} \in dy, \; \tau^+(-x_k) > n)$$
$$\leq \int_{(-\infty,-x_k]} (1 - G(s - x_k - y)) H_{\psi_k(x,\cdot)}(-x_k, dy)$$
$$= \int_{(s,\infty)} H_{\psi_k(x,\cdot)}(-x_k, (-x_k + s - t, -x_k]) \, G(dt)$$

$$\leq \int_{(s,\infty)} c(s - t + 1)^2 \, G(dt),$$

where $M_{\zeta_{n-1}} = x_k < 0$, stochastic L_2-boundedness of $Y_1, Y_2, \ldots, \psi_k(x, \cdot) \in \Psi^+_{F,G}$, a change of variables and Lemma 4 for the final inequality have been utilized. Since G is square-integrable, we thus obtain for sufficiently large s,

$$\sup_{n\geq1} \|P_\psi(M_{\xi_n} > s \mid \mathcal{F}_{\zeta_{n-1}})\|_\infty < 1/2$$

and then by successive conditioning $P_\psi(M_{\xi_n} \in [0, s] \text{ i.o.}) = 1$. A similar procedure yields $P_\psi(M_{\zeta_n} \in [-s, 0] \text{ i.o.}) = 1$, and both results together yield the asserted recurrence of $(M_n)_{n\geq0}$ provided $\psi \in \Psi^+_{F,G}$.

R e m a r k. Basically we have just shown that Durrett, Kesten and Lawler's [4] proof of the recurrence part of Theorem 1 in case of finitely many increment distributions runs through also if $Y_1, Y_2, \ldots$ are merely stochastically L_2-bounded with a uniform lower bound for the conditional variances. The major difference is that their proof of the crucial Lemma 3 (their Lemma (4.1)) is based on Skorokhod imbedding and makes explicit use of dealing only with finitely many increment distributions. Moreover, there seems to be a little gap in their argument corresponding to (3.10), because they use (3.8) of Lemma 4 as stated here, but prove it only with $\sup_{b\geq0}$ replaced by $\sup_{0\leq b\leq x}$.

P r o o f   o f   C o r o l l a r y 1. The same reasoning as in the previous proof shows that we may assume without loss of generality that $(M_n)_{n\geq0}$ has a distribution $\psi \in \psi^+_{F,G}$ with square-integrable F, G, so that in particular $P_\psi(V = \infty) = 1$. By Theorem 1, M_n then visits an interval $[-c,c]$ i.o. P_ψ-a.s., and we denote by $\xi_0 = 0, \xi_1, \xi_2, \ldots$ the successive visit times. By Lemma 1, there is some $\varepsilon > 0$ such that

$$\beta := \inf_{n\geq0, j\geq1} \|P_\psi(Y_{\xi_n+j} > \varepsilon \mid \mathcal{F}_{\xi_{n+j-1}})\|_\infty > 0.$$

Consequently, by successive conditioning for every $m \geq 1$,

$$P_\psi(M_{\xi_{mk}+m} - M_{\xi_{mk}} \leq m\varepsilon \text{ for } 0 \leq k < n) \leq (1 - \beta^m)^n \to 0$$

as $n \to \infty$, and therefore

$$P_\psi(M_{\xi_{mk}+m} > -c + m\varepsilon \text{ i.o.}) = 1$$

for $m \geq 1$. But the latter obviously implies $\limsup_{n\to\infty} M_n = \infty$ P_ψ-a.s. The other assertion of the corollary follows analogously when considering $(-M_n)_{n\geq0}$.

P r o o f   o f   C o r o l l a r y 3. We only give a sketch of the proof here from which the formal procedure can be easily deduced. Consider the non-arithmetic case, assume (1.13) and pick an arbitrarily small $\delta > 0$ and an

arbitrary $x \in \mathbb{R}$. Denote by $[-c, c]$ the recurrence interval of $(M_n)_{n \geq 0}$. It is then easily verified that there exists some $k = k(\delta)$ sufficiently large such that for each (stopping) time $\xi \geq k$ with $M_\xi \in [-c, c]$ there is a further one $\zeta \geq \xi$ such that $P(M_\zeta \in [x - \delta, x + \delta] \mid \mathcal{F}_\xi) \geq \gamma$ and $\gamma > 0$ only depends on δ, k and x. Thus we obtain the desired result by a geometric trial argument similar to the one used in the previous proof.

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On the invertibility of isometric semigroup representations

by

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Abstract. Let T be a representation of a suitable abelian semigroup S by isometries on a Banach space. We study the spectral conditions which will imply that $T(s)$ is invertible for each s in S. On the way we analyse the relationship between the spectrum of T, $\mathrm{Sp}(T, S)$, and its unitary spectrum $\mathrm{Sp}_u(T, S)$. For $S = \mathbb{Z}_+^n$ or $\mathbb{R}_+^n$, we establish connections with polynomial convexity.

1. Introduction. This paper deals chiefly with the question of invertibility of isometric representations of abelian semigroups. Quite apart from its intrinsic interest, the problem has a bearing on the study of the asymptotic behaviour of bounded semigroups of operators. Let S be a suitable subsemigroup of a locally compact, abelian group G, and T be a bounded representation of S on a complex Banach space X. Let $\mathrm{Sp}_u(T, S)$ be the set of all characters in the dual group Γ which are approximate eigenvalues for T, and $P\sigma_u(T^*)$ be the set of characters in Γ which are eigenvalues for T^*. If $\mathrm{Sp}_u(T)$ is countable and $P\sigma_u(T^*)$ is empty, then, for each x in X, $\|T(t)x\| \to 0$ as $t \to \infty$ through S. This was shown in [11] for norm-continuous representations of $\mathbb{R}_+$, in [7] for arbitrary representations of $\mathbb{R}_+$, in [3] (independently) for arbitrary representations of $\mathbb{Z}_+$ and $\mathbb{R}_+$, in [9] for norm-continuous representations of general semigroups, and in [4] for arbitrary (strongly continuous) representations. The arguments in [7], [9], and [4] all used a functional analytic construction to reduce the problem to the study of isometric semigroups.

If T is a representation of S by isometries and $\mathrm{Sp}_u(T, S)$ is countable, the question arises whether T is automatically invertible. For $S = \mathbb{R}_+$ this was an ingredient in the proof of [7], where a short argument using the Hille–Yosida Theorem showed that T is invertible whenever $\mathrm{Sp}_u(T, \mathbb{R}_+) \neq i\mathbb{R}$, and in the case when $S = \mathbb{Z}_+$ it is elementary that T is invertible whenever