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On integrability in F-spaces

by

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Abstract. Some usual and unusual properties of the Riemann integral for functions $x : [a,b] \to X$ where X is an F-space are investigated. In particular, a continuous integrable l_p-valued function $(0 < p < 1)$ with non-differentiable integral function is constructed. For some class of quasi-Banach spaces X it is proved that the set of all X-valued functions with zero derivative is dense in the space of all continuous functions, and for any two continuous functions x and y there is a sequence of differentiable functions which tends to x uniformly and for which the sequence of derivatives tends to y uniformly. There is also constructed a differentiable function x with $x'(t_0) = x_0$ for given t_0 and x_0 and $x'(t) = 0$ for $t \neq t_0$.

Consider the classical definition of the Riemann integral in the setting of vector-valued functions $x : [a,b] \to X$ where X is an F-space (i.e. a complete metric linear space with an invariant metric). For a partition $T = \{t_k\}_{k=0}^n$ $(a = t_0 < t_1 < \ldots < t_n = b)$ of $[a,b]$ and a collection $\Lambda = \{\lambda_k\}_{k=1}^n$ $(\lambda_k \in [t_{k-1}, t_k])$ define the Riemann sum

$$\mathfrak{S}(T,\Lambda) = \sum_{k=1}^{n} x(\lambda_k)\Delta t_k, \qquad \Delta t_k = t_k - t_{k-1}.$$

A function x is said to be *integrable* on $[a,b]$ if $\mathfrak{S}(T,\Lambda)$ has a limit as $\max_k \Delta t_k \to 0$, i.e. if there exists an element $\int_a^b x(t)\,dt \in X$ such that for any $\varepsilon > 0$ there is a $\delta > 0$ such that for each partition T of $[a,b]$ with $\max_k \Delta t_k < \delta$ and each Λ,

$$\Big\|\mathfrak{S}(T,\Lambda) - \int\limits_a^b x(t)\,dt\Big\| < \varepsilon.$$

Some usual properties of the Riemann integral remains true: each integrable function is bounded and the integrability of x on both intervals $[a,b]$

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and $[b,c]$ implies its integrability on $[a,c]$ with

$$\int\limits_a^c x(t)\,dt = \int\limits_a^b x(t)\,dt + \int\limits_b^c x(t)\,dt.$$

Other properties are not so trivial or even false. For example, an F-space X is locally convex if and only if every continuous function $x:[a,b]\to X$ is integrable [4], [5, p. 121].

In Section 1 we prove some further properties of the Riemann integral.

Section 2 is devoted to the construction of an integrable continuous function $y:[0,1]\to l_p$ $(0<p<1)$ for which the function

$$x(t) = \int\limits_0^t y(s)\,ds$$

is not differentiable on the right at $t=0$.

The main unsolved question here is: does every continuous function $y:[a,b]\to X$ have a primitive? (of course, we assume that X is not locally convex). The result of Section 2 shows that the usual way of obtaining primitives fails even for integrable continuous functions. Another way of getting primitives is by passage to the limit. In Section 3 we show that this may also fail, by proving that for some class of F-spaces X and for any two continuous functions $x,y:[a,b]\to X$ there exists a sequence of differentiable functions $\{x_n\}_{n=1}^\infty$ such that x_n tends to x and x_n' tends to y uniformly on $[a,b]$. Finally, we show that there are differentiable functions with derivatives having points of discontinuity of the first kind.

We denote by $\|x\|$ the F-norm of X, i.e. $\|x\| = \varrho(x,0)$ where ϱ is the metric of X, and $\mathcal{L}(X)$ denotes the space of all continuous linear operators acting in X.

The author is grateful to L. V. Popova for her help in proving Theorem 2.1 and to Professor S. Rolewicz for valuable remarks.

1. Some connections with other properties

PROPOSITION 1.1. *Let X be an F-space and $x:[a,b]\to X$ be an integrable function on $[a,b]$. Then the set of all Riemann sums of x on $[a,b]$ is bounded in X (in particular, x is bounded).*

Proof. First we show that x is bounded. Supposing the contrary, let $\varepsilon_n \searrow 0$ and $s_n\in[a,b]$ be numbers such that $\|\varepsilon_n x(s_n)\| \ge \delta_0 > 0$ for each n. Choose $\delta>0$ so that for every partition of $[a,b]$ with diameter $<\delta$ any corresponding Riemann sum $\mathfrak{S}$ satisfies

$$\sup_{0<\varepsilon\le 1}\Big\|\varepsilon\Big(\mathfrak{S} - \int\limits_a^b x(t)\,dt\Big)\Big\| < \frac{\delta_0}{3}.$$

Let $m \ge (b-a)/\delta$. Decompose $[a,b]$ into intervals $\{I_k\}_{k=1}^m$ of length $(b-a)/m$. Let k_0 be an index such that there are infinitely many s_n's in I_{k_0}; say, $\{s_{i_n}\}_{n=1}^\infty \subset I_{k_0}$. Choose any $\xi_k\in I_k$ $(k=1,\ldots,m)$ and put

$$\mathfrak{S}_0 = \frac{b-a}{m}\sum_{k\ne k_0} x(\xi_k),\qquad \delta_n = \varepsilon_n\frac{m}{b-a},$$

$$\mathfrak{S}_{i_n} = \mathfrak{S}_0 + \frac{b-a}{m}x(s_{i_n}).$$

Then

$$\sup_{0<\varepsilon\le 1}\Big\|\varepsilon\Big(\mathfrak{S}_{i_n} - \int\limits_a^b x(t)\,dt\Big)\Big\| < \frac{\delta_0}{3}$$

for each n and hence

$$\|\varepsilon_{i_n}x(s_{i_n})\| = \Big\|\delta_{i_n}\frac{b-a}{m}x(s_{i_n})\Big\|$$
$$\le \|\delta_{i_n}\mathfrak{S}_0\| + \Big\|\delta_{i_n}\Big(\mathfrak{S}_{i_n} - \int\limits_a^b x(t)\,dt\Big)\Big\| + \Big\|\delta_{i_n}\int\limits_a^b x(t)\,dt\Big\|.$$

Since each of the terms on the right hand side can be made $<\delta_0/3$ for n large enough, the last inequality contradicts the assumption $\|\varepsilon_n x(s_n)\|\ge\delta_0$.

Thus, x is bounded. Let $\varepsilon_0>0$. It is not hard (using the integrability of x) to choose $\delta>0$ such that for each collection $\{I_k\}_{k=1}^m$ of subintervals of $[a,b]$ with disjoint interiors and with $\max_k\mu(I_k)<\delta$, and for any $\eta_k\in I_k$,

$$\sup_{0<\varepsilon\le 1}\Big\|\varepsilon\Big(\sum_{k=1}^m x(\eta_k)\mu(I_k) - \sum_{k=1}^m\int\limits_{I_k} x(t)\,dt\Big)\Big\| < \frac{\varepsilon_0}{2}.$$

Using boundedness of x, choose $\varepsilon_1\in(0,1]$ so that

$$\sup_{\substack{0<\varepsilon\le\varepsilon_1\\ 0\le t\le 1}}\|\varepsilon(b-a)x(t)\| < \frac{\varepsilon_0\delta}{2(b-a)}.$$

Let $\mathfrak{S}$ be an arbitrary Riemann sum constructed for some partition $a=t_0<\ldots<t_n=b$. Denote by n_0 the number of intervals $[t_{k-1},t_k]$ of length $\ge\delta$. Clearly, $n_0\le(b-a)/\delta$. Now denote by $\mathfrak{S}_0$ the part of $\mathfrak{S}$ which is obtained by summing over intervals of length $<\delta$. Then for $0<\varepsilon\le\varepsilon_1$ we have

$$\|\varepsilon\mathfrak{S}\| \le \|\varepsilon\mathfrak{S}_0\| + \sum_{\mu([t_{k-1},t_k])\ge\delta}\|\varepsilon(t_k-t_{k-1})x(\xi_k)\|$$
$$< \frac{\varepsilon_0}{2} + \sum_{\mu([t_{k-1},t_k])\ge\delta}\frac{\varepsilon_0\delta}{2(b-a)} = \frac{\varepsilon_0}{2} + n_0\frac{\varepsilon_0\delta}{2(b-a)} \le \varepsilon_0. \ \blacksquare$$

Now suppose that y is integrable on $[a,b]$ and $t_0 \in [a,b]$. If X is locally convex then one can show that

$$x(t) = \int_{t_0}^{t} y(s)\,ds \tag{1}$$

is a differentiable function at each point of continuity of y and the main formula of Integral Calculus is valid: $x' = y$. The situation changes when we pass to non-locally convex spaces. In general, we can only prove continuity of x.

PROPOSITION 1.2. *Let $y : [a,b] \to X$ be integrable on $[a,b]$ (where X is an F-space). Then the function x defined by* (1) *is uniformly continuous on $[a,b]$.*

Proof. By the Cantor theorem it is enough to prove the continuity of x at any point $t_1 \in [a,b]$. For given $\varepsilon > 0$ choose $\delta > 0$ so that for every partition $T = \{\tau_k\}_{k=0}^{n}$ of $[a,b]$, $a = \tau_0 < \ldots < \tau_n = b$, with $\operatorname{diam} T = \max_k(\tau_k - \tau_{k-1}) < \delta$ and for any points $\xi_k \in [\tau_{k-1}, \tau_k]$ ($\xi = \{\xi_k\}_{k=1}^{n}$) the corresponding Riemann sum $\mathfrak{S}(T,\xi)$ satisfies

$$\Big\|\mathfrak{S}(T,\xi) - \int_a^b y(t)\,dt\Big\| < \frac{\varepsilon}{4}.$$

Suppose that $t \in [a,b]$ and $0 < |t - t_1| < \delta$. Let $T_1 = \{t_1 = \tau_0 < \ldots < \tau_m = t\}$ be any partition of $[t_1, t]$ (of $[t, t_1]$ if $t < t_1$) and $\eta_i \in [\tau_{i-1}, \tau_i]$ for $i = 1, \ldots, m$ (set $\eta = \{\eta_i\}_{i=1}^{m}$). Let us supplement the collection $\{\tau_i\}_{i=1}^{m}$ with points from $[a,b] \setminus [t_1, t]$ so that the new partition T of $[a,b]$ satisfies $\operatorname{diam} T < \delta$. Denote by T' the partition of $[a,b]$ which is obtained from T by removing $\tau_1, \ldots, \tau_{m-1}$. Since $|t - t_1| < \delta$ and $\tau_i \in [t_1, t]$ and the ends $\tau_0 = t_1$, $\tau_m = t$ are still in T', we have $\operatorname{diam} T' < \delta$. Choose a collection of points ξ for T as follows. For the intervals $[\tau_{i-1}, \tau_i]$ retain the points η_i which have already been chosen and choose the left ends of the remaining intervals (right ends if $t < t_1$). For the partition T' denote by ξ' the collection of the left ends of intervals from T' (right ends if $t < t_1$). Then

$$\mathfrak{S}(T,\xi) - \mathfrak{S}(T',\xi') = \sum_{i=1}^{m} x(\eta_i)\Delta\tau_i - x(t_1)(t - t_1).$$

On the other hand, since $\operatorname{diam} T < \delta$ and $\operatorname{diam} T' < \delta$, we have

$$\|\mathfrak{S}(T,\xi) - \mathfrak{S}(T',\xi')\| \le \Big\|\mathfrak{S}(T,\xi) - \int_a^b y(t)\,dt\Big\| + \Big\|\int_a^b y(t)\,dt - \mathfrak{S}(T',\xi')\Big\| < 2\frac{\varepsilon}{4} = \frac{\varepsilon}{2}$$

and therefore

$$\Big\|\sum_{i=1}^{m} y(\eta_i)\Delta\tau_i\Big\| < \frac{\varepsilon}{2} + \|(t - t_1)y(t_1)\|.$$

Choose $\delta_1 > 0$ so that if $|\alpha| < \delta_1$ then $\|\alpha y(t_1)\| < \varepsilon/2$. Putting $\delta_2 = \min\{\delta, \delta_1\}$, we find that if $|t - t_1| < \delta_2$ then for each partition T_1 of $[t_1, t]$ (or $[t, t_1]$) and each $\eta_i \in [\tau_{i-1}, \tau_i]$, where τ_i are the points of T_1, we have

$$\Big\|\sum_{i=1}^{m} y(\eta_i)\Delta\tau_i\Big\| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\sum_{i=1}^{m} y(\eta_i)\Delta\tau_i$ is an arbitrary Riemann sum for y on $[t_1, t]$ (or $[t, t_1]$), we conclude that

$$\|x(t) - x(t_1)\| - \Big\|\int_{t_1}^{t} y(s)\,ds\Big\| \le \varepsilon. \ \blacksquare$$

COROLLARY 1.3. *Let $y : [a,b] \to X$ be integrable on $[a,b]$ and $\{a_n\}_{n=0}^{\infty}$ be a numerical sequence satisfying $a < a_{n+1} < a_n < a_0 = b$ for $n \ge 1$ and $\lim_n a_n = a$. Then the series*

$$\sum_{n=1}^{\infty} \int_{a_n}^{a_{n-1}} y(t)\,dt = \int_a^b y(t)\,dt$$

converges in X.

The following two propositions investigate the connections between convergence of improper integrals and integrability, for the needs of Section 2. We omit their proofs which are natural and straightforward.

PROPOSITION 1.4. *Let X be a non-locally convex F-space. There exists a continuous function $x : [0,1] \to X$ such that*

(a) $x(0) = 0$;

(b) *x is integrable on $[\varepsilon, 1]$ for every $\varepsilon \in (0,1)$ and the limit*

$$\lim_{\varepsilon \to 0} \int_{\varepsilon}^{1} x(t)\,dt$$

exists;

(c) *x is not integrable on $[0,1]$.*

PROPOSITION 1.5. *Let X be an F-space and $x : [a,b] \to X$ be a bounded function. Suppose that for some sequence $T_n \searrow a$, $T_n \in [a,b]$, the following hold:*

(i) *x is integrable on $[T_n, b]$ for each n and the limit*

$$I = \lim_n \int\limits_{T_n}^{b} x(t)\,dt$$

exists;

(ii) *for every $\varepsilon > 0$ there is a $\delta > 0$ such that for each partition $T_n = \tau_0 < \tau_1 < \ldots < \tau_m = b$ with $\max_k \Delta\tau_k = \max_k(\tau_k - \tau_{k-1}) < \delta$ and each $\xi_k \in [\tau_{k-1}, \tau_k]$ the Riemann sum*

$$\mathfrak{S}_0 = \sum_{k=1}^{m} x(\xi_k)\Delta\tau_k$$

satisfies the condition

$$\Big\|\mathfrak{S}_0 - \int\limits_{T_n}^{1} x(t)\,dt\Big\| < \varepsilon.$$

Then x is integrable on $[a, b]$ and

$$\int\limits_{a}^{b} x(t)\,dt = I.$$

2. An integrable continuous function having non-differentiable integral function. We show that if the space l_p with $0 < p < 1$ embeds isomorphically in an F-space X then there exists a function $y : [0,1] \to X$ as in the title of this section. Clearly, we may simply assume that $X = l_p$. However, we do not know whether there exists a continuous integrable function for which the integral function is non-differentiable at each point or even almost everywhere.

THEOREM 2.1. *There exists a continuous Riemann integrable function $y : [0,1] \to l_p$ $(0 < p < 1)$ such that the function*

$$x(t) = \int\limits_{0}^{t} y(s)\,ds$$

does not have a right derivative at $t = 0$.

For the proof of Theorem 2.1 we need some facts.

Denote by $\{e_n\}_{n=1}^{\infty}$ the standard basis in l_p. Put

$$c_n = 2^{-n},\ a_1 = 1,\ b_1 = 1/2,\ a_2 = a_3 = 2^{-(1-p)/2},\ b_2 = b_3 = c_2/2,$$
$$a_{(n-1)n/2+1} = a_{(n-1)n/2+2} = \ldots = a_{n(n+1)/2} = n^{-(1-p)/2},$$
$$b_{(n-1)n/2+1} = b_{(n-1)n/2+2} = \ldots = b_{n(n+1)/2} = c_n/n.$$

Obviously,

$$\sum_{k=1}^{\infty} b_k = \sum_{n=1}^{\infty} c_n = 1.$$

Now put

$$t_1 = 1, \quad t_n = 1 - \sum_{k=1}^{n-1} b_k = \sum_{k=n}^{\infty} b_k \ (n \ge 2), \quad d_k = b_k/2 \ (k \ge 1).$$

LEMMA 2.2. (a) *$t_{k+1} + d_{k+1} \le t_k - d_k$ for all $k \ge 1$;*

(b) *for the function $y(t)$ defined on $[t_k - d_k, t_k + d_k]$ by*

$$y(t) = y_k(1 - |t - t_k|/d_k)$$

where $y_k = a_k^{1/p} e_k$, every Riemann sum $\mathfrak{S}_k$ on this interval is estimated as

$$\|\mathfrak{S}_k\| \le 2^p d_k^p;$$

(c) *$\sum_{k=1}^{\infty} d_k^p < \infty$.*

Proof. (a) Since $t_k - t_{k+1} = b_k$, we have

$$t_{k+1} + d_{k+1} = t_{k+1} + \frac{b_{k+1}}{2} \le t_{k+1} + \frac{b_k}{2} = t_{k+1} + \frac{t_k - t_{k+1}}{2}$$
$$= t_k - \frac{t_k - t_{k+1}}{2} = t_k - \frac{b_k}{2} = t_k - d_k.$$

(b) Let $t_k - d_k = s_0 < s_1 < \ldots < s_n = t_k + d_k$ be any partition, and let $\xi_i \in [s_{i-1}, s_i]$, $\Delta s_i = s_i - s_{i-1}$. Then

$$\mathfrak{S}_k = \sum_{i=1}^{n} y(\xi_i)\Delta s_i = \sum_{i=1}^{n} y_k(1 - |\xi_i - t_k|/d_k)\Delta s_i$$
$$= y_k \sum_{i=1}^{n}(1 - |\xi_i - t_k|/d_k)\Delta s_i.$$

Then by the definition of y_k,

$$\|\mathfrak{S}_k\| = \Big|a_k^{1/p} \sum_{i=1}^{n}(1 - |\xi_i - t_k|/d_k)\Delta s_i\Big|^p$$
$$\le \Big|\sum_{i=1}^{n} \Delta s_i\Big|^p a_k,$$

since $|1 - |\xi_i - t_k|/d_k| \le 1$. Thus, $\|\mathfrak{S}_k\| \le a_k(2d_k)^p \le 2^p d_k^p$.

(c) Note that

$$\begin{aligned}\sum_{k=1}^{n(n+1)/2} b_k^p &= \sum_{i=1}^{n} \sum_{j=(i-1)i/2+1}^{i(i+1)/2} b_j^p \\ &= \sum_{i=1}^{n} \sum_{j=(i-1)i/2+1}^{i(i+1)/2} \left(\frac{c_j}{i}\right)^p = \sum_{i=1}^{n} i^{1-p} c_i^p \\ &= \sum_{i=1}^{n} \frac{i^{1-p}}{2^{ip}} \le \sum_{i=1}^{\infty} \frac{i^{1-p}}{2^{ip}}.\end{aligned}$$

This implies the convergence of $\sum_{k=1}^{\infty} b_k^p$ and hence that of $\sum_{k=1}^{\infty} d_k^p$. The lemma is proved.

Define $y(t)$ on $[0,1]$ by putting

$$y(t) = y_k(1 - |t - t_k|/d_k)$$

for $t \in [t_k - d_k, t_k + d_k] \cap [0,1]$, $k = 1, 2, \ldots$, and $y(t) = 0$ for $t \in [0,1] \setminus \bigcup_{k=1}^{\infty} [t_k - d_k, t_k + d_k]$. Clearly, y is continuous on $[0,1]$ and $\|y(t)\| \le 1$ for each $t \in [0,1]$.

LEMMA 2.3. *Let* $\lambda \in (0,1)$ *and* $0 = \tau_0 < \tau_1 < \ldots < \tau_s = \lambda$ *be an arbitrary partition of* $[0,\lambda]$, $\xi_i \in [\tau_{i-1}, \tau_i]$, $i = 1, \ldots, s$, *and* $\Delta\tau_i = \tau_i - \tau_{i-1}$. *Let* n_0 *be an integer such that* $\lambda \le t_{n_0}$. *Then the Riemann sum* $\mathfrak{S} = \sum_{i=1}^{s} y(\xi_i)\Delta\tau_i$ *of the function* y *defined above has the following estimate*:

$$\|\mathfrak{S}\| \le (2^{p+1} + 4 + 2^p) \sum_{k=n_0}^{\infty} d_k^p.$$

Proof. Decompose $\mathfrak{S}$ as

$$\mathfrak{S} = \mathfrak{S}' + \mathfrak{S}'' + \sum_{k=n_0}^{\infty} \mathfrak{S}_k$$

where $\mathfrak{S}_k$ is the sum of $y(\xi_i)\Delta\tau_i$ over all i for which $[\tau_{i-1}, \tau_i] \subset [t_k - d_k, t_k + d_k]$, $\mathfrak{S}'$ is the sum over i for which $y(\xi_i) = 0$, and $\mathfrak{S}''$ over the remaining i (i.e. over those i for which $[\tau_{i-1}, \tau_i]$ lies partly in $\widehat{I} = \bigcup_{k=n_0}^{\infty} [t_k - d_k, t_k + d_k]$, and partly in $[0,\lambda] \setminus \widehat{I}$).

We now give an upper estimate of the sums $\mathfrak{S}'$, $\mathfrak{S}''$ and $\mathfrak{S}_k$ (of course, there are only finitely many non-zero elements among $\mathfrak{S}_k$). Clearly, $\mathfrak{S}' = 0$. To estimate $\|\mathfrak{S}''\|$, denote by I the set of corresponding indices i. Let $i \in I$. Since $y(\xi_i) \ne 0$, there is a (unique) $k = k(i) \ge n_0$ such that $\xi_i \in [t_k - d_k, t_k + d_k]$. Now put

$$K = \{k \ge n_0 : (\exists i \in I)\ k = k(i)\}.$$

Suppose that $K = \{k_1, \ldots, k_r\}$ where $n_0 \le k_1 < \ldots < k_r$. Then

$$\begin{aligned}(2)\qquad \|\mathfrak{S}''\| &= \Big\|\sum_{i\in I} y(\xi_i)\Delta\tau_i\Big\| \le \sum_{i\in I} \|y(\xi_i)\|(\Delta\tau_i)^p \\ &\le \sum_{j=1}^{r} \sum_{\substack{i\in I\\ k(i)=k_j}} \|y(\xi_i)\|(\Delta\tau_i)^p \le \sum_{j=1}^{r} \sum_{\substack{i\in I\\ k(i)=k_j}} \|y_{k_j}\|(\Delta\tau_i)^p.\end{aligned}$$

Note that if $k(i) = k_j$ then

$$\Delta\tau_i \le t_{k_{j-1}} + d_{k_{j-1}} - t_{k_{j+1}} + d_{k_{j+1}}$$

since the definitions of $k_1, \ldots, k_r$ and of I and K imply

$$t_{k_{j+1}} - d_{k_{j+1}} < \tau_{i-1} < \tau_i < \tau_{k_{j-1}} + d_{k_{j-1}}$$

($k_0 = n_0$ is understood). Hence the right hand side of (2) is bounded by

$$(3)\qquad \sum_{j=1}^{r} \sum_{\substack{i\in I\\ k(i)=k_j}} \|y_{k_j}\|(t_{k_{j-1}} - t_{k_{j+1}} + d_{k_{j-1}} + d_{k_{j+1}})^p.$$

Note that for a given $k \in K$ there are at most two indices i', i'' in I such that $k = k(i') = k(i'')$ since $[\tau_{i'-1}, \tau_{i'}]$ and $[\tau_{i''-1}, \tau_{i''}]$ should intersect $[t_k - d_k, t_k + d_k]$ without being contained in it. Thus, (3) is estimated by

$$\begin{aligned}&\le 2\sum_{j=1}^{r} \|y_{k_j}\|(t_{k_{j-1}} - t_{k_{j+1}} + d_{k_{j-1}} + d_{k_{j+1}})^p \\ &\le 2\sum_{j=1}^{r} \|y_{k_j}\|(t_{k_{j-1}} - t_{k_{j+1}})^p + 2\sum_{j=1}^{r} \|y_{k_j}\| d_{k_{j-1}}^p + 2\sum_{j=1}^{r} \|y_{k_j}\| d_{k_{j+1}}^p \\ &\le 2\sum_{j=1}^{r} \|y_{k_j}\|(t_{k_{j-1}} - t_{k_{j+1}})^p + 4\sum_{k=n_0}^{\infty} d_k^p \\ &= 2\sum_{j=1}^{r} \|y_{k_j}\| \Big(\sum_{m=k_{j-1}}^{k_{j+1}-1} b_m\Big)^p + 4\sum_{k=n_0}^{\infty} d_k^p \\ &\le 2\sum_{j=1}^{r} \|y_{k_j}\| \sum_{m=k_{j-1}}^{k_{j+1}-1} b_m^p + 4\sum_{k=n_0}^{\infty} d_k^p \\ &\le 2\sum_{k=n_0}^{\infty} b_k^p + 4\sum_{k=n_0}^{\infty} d_k^p = (2^{p+1} + 4)\sum_{k=n_0}^{\infty} d_k^p.\end{aligned}$$

To estimate $\mathfrak{S}_k$, note that it is a Riemann sum for y on $[t_k - d_k, t_k + d_k]$ (the missing terms of the form $y(\xi)(\tau_{i-1} - (t_k - d_k))$ and $y(\eta)((t_k + d_k) - \tau_i)$ are considered to be zero since $y(\xi) = y(\eta) = 0$ for $\xi = t_k - d_k$ and $\eta = t_k + d_k$). Thus, by Lemma 2.2(b),

$$\|\mathfrak{S}_k\| \le 2^p d_k^p,$$

and so

$$\Big\|\sum_{k=n_0}^{\infty} \mathfrak{S}_k\Big\| \le \sum_{k=n_0}^{\infty} \|\mathfrak{S}_k\| \le 2^p \sum_{k=n_0}^{\infty} d_k^p.$$

Combining all the above estimates we obtain the assertion of the lemma.

Proof of Theorem 2.1. Now we prove that y satisfies the assumptions of Proposition 1.5 with $T_n = t_n - d_n$ and $[a,b] = [0,1]$ (the definition of y was given before Lemma 2.3).

Since y is piecewise linear on $[T_n, 1]$, it is integrable. In order to prove the existence of the limit in (i), we calculate the integral

$$\begin{aligned}\int_{T_k}^{T_{k-1}} y(t)\,dt &= \int_{t_k-d_k}^{t_k+d_k} y_k(1-|t-t_k|/d_k)\,dt\\ &= y_k \int_{t_k-d_k}^{t_k+d_k} (1-|t-t_k|/d_k)\,dt = d_k y_k.\end{aligned}$$

Hence if $n > n_0$ then

$$\int_{T_n}^{T_{n_0}} y(t)\,dt = \sum_{k=n_0+1}^{n} \int_{T_k}^{T_{k-1}} y(t)\,dt = \sum_{k=n_0+1}^{n} d_k y_k \tag{4}$$

and also

$$\Big\|\int_{T_n}^{T_{n_0}} y(t)\,dt\Big\| \le \sum_{k=n_0+1}^{n} d_k^p \|y_k\| < \sum_{k=n_0+1}^{\infty} d_k^p.$$

This means that $\int_{T_n}^{1} y(t)\,dt$ is a Cauchy sequence. Define

$$I = \lim_n \int_{T_n}^{1} y(t)\,dt.$$

To prove assumption (ii) of Proposition 1.5, fix $\varepsilon > 0$ and choose n_0 so that every Riemann sum for y on $[0, T_{n_0}]$ is less than $\varepsilon/4$ (this is possible by Lemma 2.3) and so that for each $n \ge n_0$,

$$\Big\|\int_{T_n}^{T_{n_0}} y(t)\,dt\Big\| < \frac{\varepsilon}{4}.$$

Now pick $\delta > 0$ so that every Riemann sum $\mathfrak{S}_1$ for y on $[T_{n_0}, 1]$, corresponding to any partition U of $[T_{n_0}, 1]$ with $\operatorname{diam} U < \delta$, satisfies

$$\Big\|\mathfrak{S}_1 - \int_{T_{n_0}}^{1} y(t)\,dt\Big\| < \frac{\varepsilon}{4},$$

and also that $\delta^p < \varepsilon/4$. Let $n > n_0$ and let $S = \{T_n = \tau_0 < \tau_1 < \ldots < \tau_m = 1\}$ be a partition of $[T_n, 1]$ with $\operatorname{diam} S < \delta$. Put

$$\mathfrak{S}_0 = \sum_{k=1}^{m} y(\xi_k)\Delta\tau_k$$

where $\Delta\tau_k = \tau_k - \tau_{k-1}$ and $\xi_k \in [\tau_{k-1}, \tau_k]$. Choose $j \in \{1, \ldots, m\}$ so that $T_{n_0} \in [\tau_{j-1}, \tau_j]$. Note that $\mathfrak{S}_1 = \sum_{k=j+1}^{m} y(\xi_k)\Delta\tau_k$ is a Riemann sum for y on $[T_{n_0}, 1]$ since we can pick $\xi = T_{n_0} \in [T_{n_0}, \tau_j]$ with $y(\xi) = 0$. Finally, $\mathfrak{S}' = \sum_{k=1}^{j-1} y(\xi_k)\Delta\tau_k$ is a Riemann sum for y on $[0, T_{n_0}]$ since $y(0) = y(T_{n_0}) = 0$. Thus,

$$\begin{aligned}\Big\|\mathfrak{S}_0 - \int_{T_n}^{1} y(t)\,dt\Big\| &\le \|\mathfrak{S}'\| + \Big\|\int_{T_n}^{T_{n_0}} y(t)\,dt\Big\|\\ &\quad + \|y(\xi_j)\Delta\tau_j\| + \Big\|\mathfrak{S}_1 - \int_{T_{n_0}}^{1} y(t)\,dt\Big\|\\ &< 4\frac{\varepsilon}{4} = \varepsilon.\end{aligned}$$

Suppose that $n \le n_0$, i.e. $T_{n_0} \le T_n$. It is easily seen that (since y is piecewise linear on $[T_{n_0}, T_n]$) there is a Riemann sum $\mathfrak{S}''$ for y on $[T_{n_0}, T_n]$ such that

$$\mathfrak{S}'' = \int_{T_{n_0}}^{T_n} y(t)\,dt.$$

Hence also for $n \le n_0$ we have

$$\Big\|\mathfrak{S}_0 - \int_{T_n}^{1} y(t)\,dt\Big\| = \Big\|\mathfrak{S}_0 + \mathfrak{S}'' - \int_{T_{n_0}}^{1} y(t)\,dt\Big\| < \frac{\varepsilon}{4} < \varepsilon,$$

since $\mathfrak{S}_0 + \mathfrak{S}''$ is a Riemann sum for y on $[T_{n_0}, 1]$.

Thus, we have proved that y is integrable on $[0,1]$. Now we prove that the function

$$x(t) = \int_{0}^{t} y(s)\,ds$$

does not have a right derivative at 0. By (4),

$$x(T_{n_0}) = \lim_n (x(T_{n_0}) - x(T_n)) = \sum_{k=n_0+1}^{\infty} d_k y_k.$$

Hence

$$\|T_{n_0}^{-1}(x(T_{n_0}) - x(0))\| = T_{n_0}^{-p}\|x(T_{n_0})\| = T_{n_0}^{-p}\Big\|\sum_{k=n_0+1}^{\infty} d_k y_k\Big\|.$$

By the definition of y_k,

$$\begin{aligned}\Big\|\sum_{k=(n-1)n/2+1}^{\infty} d_k y_k\Big\| &= \sum_{k=(n-1)n/2+1}^{\infty} d_k^p \|y_k\| \\ &= 2^{-p} \sum_{k=(n-1)n/2+1}^{\infty} b_k^p a_k = 2^{-p} \sum_{i=1}^{\infty} \sum_{k=(i-1)i/2+1}^{i(i+1)/2} b_k^p a_k \\ &= 2^{-p} \sum_{i=1}^{\infty} \sum_{k=(i-1)i/2+1}^{i(i+1)/2} \frac{c_i^p}{i^p}\left(\frac{1}{i}\right)^{(1-p)/2} = 2^{-p} \sum_{i=1}^{\infty} c_i^p i^{1-p-(1-p)/2} \\ &= 2^{-p} \sum_{i=n}^{\infty} c_i^p i^{(1-p)/2} \geq 2^{-p} n^{(1-p)/2} \sum_{i=n}^{\infty} c_i^p \\ &\geq 2^{-p} n^{(1-p)/2} \Big(\sum_{i=n}^{\infty} c_i\Big)^p = \frac{n^{(1-p)/2}}{2^p}\Big(\sum_{k=(n-1)n/2+1}^{\infty} b_k\Big)^p \\ &= 2^{-p} n^{(1-p)/2} t_{(n-1)n/2+1}^p = 2^{-p} n^{(1-p)/2} (T_{(n-1)n/2} - d_{(n-1)n/2})^p.\end{aligned}$$

Thus,

$$\begin{aligned}&\|T_{(n-1)n/2}^{-1}(x(T_{(n-1)n/2}) - x(0))\| \\ &\quad \geq T_{(n-1)n/2}^{-p} 2^{-p} n^{(1-p)/2} (T_{(n-1)n/2} - d_{(n-1)n/2})^p \\ &\quad = 2^{-p} n^{(1-p)/2} \left(1 - \frac{\frac{1}{2} b_{(n-1)n/2}}{\sum_{k=(n-1)n/2}^{\infty} b_k - \frac{1}{2} b_{(n-1)n/2}}\right)^p \\ &\quad \geq 2^{-p} n^{(1-p)/2} \left(1 - \frac{\frac{1}{2} b_{(n-1)n/2}}{b_{(n-1)n/2+1} + b_{(n-1)n/2+2}}\right)^p \\ &\quad = 2^{-p} n^{(1-p)/2} \left(1 - \frac{\frac{1}{2} c_{n-1}/(n-1)}{2c_n/n}\right)^p = 2^{-p} n^{(1-p)/2} \left(1 - \frac{n}{2(n-1)}\right)^p.\end{aligned}$$

Thus, the absence of the right derivative at 0 for x is proved.

3. On the impossibility of passage to a limit under the derivation

THEOREM 3.1. *Let $x, y : [a,b] \to L_p$ $(0 < p < 1)$ be continuous. There exists a sequence $x_n : [a,b] \to L_p$, $n \geq 1$, of functions differentiable on $[a,b]$ such that x_n tends to x uniformly on $[a,b]$ and x_n' tends to y uniformly on $[a,b]$.*

For the proof we need a few lemmas.

LEMMA 3.2. *For each $x_0 \in L_p \setminus \{0\}$ $(0 < p < 1)$ there exists a constant $M < \infty$ such that for each $y_0 \in L_p$ there is a $T \in \mathcal{L}(L_p)$ with $Tx_0 = y_0$ and $\|T\| \leq M\|y_0\|$.*

Lemma 3.2 can be obtained as a consequence of the results of [1] or [3, p. 151] (notion of bounded transitivity).

LEMMA 3.3. *The set of all functions $x : [a,b] \to L_p$ $(0 < p < 1)$ with zero derivative is dense in the space $C([a,b], L_p)$ of all continuous functions from $[a,b]$ into L_p.*

Proof. Fix $y \in C([a,b], L_p)$, $\varepsilon > 0$ and $\alpha < \beta$. First we show that there is a constant M such that for each $x, y \in L_p$ there exists a differentiable function $z : [\alpha, \beta] \to L_p$ with the properties

(i) $z(\alpha) = x$, $z(\beta) = y$,
(ii) the oscillation of z on $[\alpha, \beta]$ satisfies

$$\omega(z, [\alpha, \beta]) := \sup_{t,s \in [\alpha,\beta]} \|z(t) - z(s)\| \leq M\|x - y\|,$$

(iii) $z'(t) = 0$ for each $t \in [\alpha, \beta]$.

Let $u : [a,b] \to L_p$ be some non-constant differentiable function with zero derivative (such a function exists in each F-space with trivial dual [2]). By continuity of u, there are numbers α_1, β_1 $(a \leq \alpha_1 < \beta_1 \leq b)$ such that

$$0 < \omega(u, [\alpha_1, \beta_1]) \leq 1.$$

Again by continuity of u, there are α_2, β_2 $(\alpha_1 \leq \alpha_2 < \beta_2 \leq \beta_1)$ with

$$\omega(u, [\alpha_1, \beta_1]) = \|u(\alpha_2) - u(\beta_2)\|$$

and hence

$$0 < \omega(u, [\alpha_2, \beta_2]) = \|u(\alpha_2) - u(\beta_2)\| \leq 1. \tag{5}$$

By Lemma 3.2, for $x_0 = u(\beta_2) - u(\alpha_2)$, choose $M < \infty$ so that for each $y_0 \in L_p$ there is a $T \in \mathcal{L}(L_p)$ with $T(u(\beta_2) - u(\alpha_2)) = y_0$ and $\|T\| \leq M\|y_0\|$. Putting $y_0 = y - x$, choose $T \in \mathcal{L}(L_p)$ with the above properties. For each $t \in [\alpha_2, \beta_2]$ put

$$v(t) = T(u(t) - u(\alpha_2)) + x.$$

It is not hard to see that v satisfies the following conditions:

(i′) $v(\alpha_2) = x$, $v(\beta_2) = y$,
(iii′) $v'(t) = 0$ for each $t \in [\alpha_2, \beta_2]$.

We show

(ii′) $\omega(v, [\alpha_2, \beta_2]) \le M\|x - y\|$.

Fix any $t, s \in [\alpha_2, \beta_2]$. Then

$$\|v(t) - v(s)\| = \|T(u(t) - u(s))\|.$$

By (5) we obtain $\|u(t) - u(s)\| \le 1$, hence

$$\|v(t) - v(s)\| \le \|T\| \le M\|y - x\|.$$

Finally, define z as the composition of the linear bijection of $[\alpha, \beta]$ onto $[\alpha_2, \beta_2]$ and the function v. Thus, (i′), (ii′), (iii′) imply (i), (ii), (iii) for z.

Since y is uniformly continuous on $[a, b]$, we can decompose $[a, b]$ into small intervals $a = t_0 < \ldots < t_n = b$ so that for each $k = 1, \ldots, n$,

$$\omega(y, [t_{k-1}, t_k]) \le \frac{\varepsilon}{M+1}.$$

For each $k = 1, \ldots, n$, choose $z_k : [t_{k-1}, t_k] \to L_p$ so that

$(\mathrm{i})_k$ $z_k(t_{k-1}) = y(t_{k-1})$, $z_k(t_k) = y(t_k)$,
$(\mathrm{ii})_k$ $\omega(z_k, [t_{k-1}, t_k]) \le M\|y(t_k) - y(t_{k-1})\|$,
$(\mathrm{iii})_k$ $z_k'(t) = 0$ for each $t \in [t_{k-1}, t_k]$.

Then we piece together the functions z_k:

$$x(t) = z_k(t) \quad \text{if } t \in [t_{k-1}, t_k],\ k = 1, \ldots, n.$$

Thus, x is defined on $[a, b]$ and has zero derivative. To estimate $\|x - y\|$, fix any $t \in [a, b]$; say, $t \in [t_{k-1}, t_k]$. Then

$$\begin{aligned}
\|x(t) - y(t)\| &= \|z_k(t) - y(t)\| \\
&\le \|z_k(t) - z_k(t_k)\| + \|y(t_k) - y(t)\| \\
&\le \omega(z_k, [t_{k-1}, t_k]) + \omega(y, [t_{k-1}, t_k]) \\
&\le M\|y(t_k) - y(t_{k-1})\| + \varepsilon/(M+1) \\
&\le M\omega(y, [t_{k-1}, t_k]) + \frac{\varepsilon}{M+1} \le \frac{\varepsilon}{M+1}(M+1) = \varepsilon. \ ■
\end{aligned}$$

LEMMA 3.4. *Let y_1 be a continuous piecewise linear function from $[a, b]$ into an F-space X. Then y_1 is integrable on $[a, b]$ and has a primitive of the form*

$$z(t) = \int\limits_a^t y_1(s)\, ds.$$

The proof is straightforward.

Proof of Theorem 3.1. Fix $\varepsilon > 0$. Since y is uniformly continuous on $[a, b]$, we can construct a continuous piecewise linear function $y_1 : [a, b] \to L_p$ such that $\|y_1(t) - y(t)\| < \varepsilon$ for each $t \in [a, b]$. Let z be a primitive of y_1 on $[a, b]$. By Lemma 3.3, choose a differentiable function z_ε with zero derivative such that $\|z_\varepsilon(t) - x(t) + z(t)\| < \varepsilon$ for each $t \in [a, b]$. Finally, put $x_\varepsilon(t) = z(t) + z_\varepsilon(t)$ for each $t \in [a, b]$. Thus, we obtain $\|x_\varepsilon(t) - x(t)\| < \varepsilon$ and $\|x_\varepsilon'(t) - y(t)\| = \|y_1(t) - y(t)\| < \varepsilon$ for each $t \in [a, b]$. ■

Remark. We can prove Theorem 3.1 in a more general case. Recall that an F-space X is called a *quasi-Banach space* if there exists an F-norm on X equivalent to the original one which is p-homogeneous for some $p \in (0, 1]$ (i.e. $\|\lambda x\| = |\lambda|^p\|x\|$). In this case X is also called a *p-Banach space*. If X is a p-Banach space then the space $\mathcal{L}(X)$ of all continuous linear operators $T : X \to X$ is also a p-Banach space with respect to the p-norm $\|T\| = \sup\{\|Tx\| : \|x\| \le 1\}$. A quasi-Banach space X is called *boundedly transitive* [3, p. 151] if there is a constant $M < \infty$ such that if $x, y \in X$ with $\|x\| = \|y\| = 1$ then there exists a $T \in \mathcal{L}(X)$ with $Tx = y$ and $\|T\| \le M$. But we need a weaker property of X. We say that a quasi-Banach space X is *pointwise-boundedly transitive* if for each $x_0 \in X \setminus \{0\}$ there exists a constant $M < \infty$ such that for each $y_0 \in X$ there is a $T \in \mathcal{L}(X)$ with $Tx_0 = y_0$ and $\|T\| \le M\|y_0\|$. Now we are ready to formulate an exact result.

THEOREM 3.1′. *Let X be a pointwise-boundedly transitive quasi-Banach space for which there exists a non-constant function $u : [a, b] \to X$ with zero derivative on $[a, b]$. Let $x, y : [a, b] \to X$ be continuous. Then there exists a sequence $x_n : [a, b] \to X$, $n \ge 1$, of functions differentiable on $[a, b]$ such that x_n tends to x uniformly on $[a, b]$ and x_n' tends to y uniformly on $[a, b]$.*

The proof is just the same.

THEOREM 3.5. *Let X be a quasi-Banach space satisfying the conditions of Theorem 3.1′. Then there exists a differentiable function $x : [a, b] \to X$ with derivative having a point of discontinuity of the first kind.*

Proof. Fix $t_0 \in (a, b)$, $x_0 \in X$ and construct a differentiable function $x : [a, b] \to X$ with $x'(t_0) = x_0$ and $x'(t) = 0$ for $t \in [a, b] \setminus \{t_0\}$. For this purpose, choose any sequence $\delta_n \searrow 0$ with $a < t_0 - \delta_1$ and $t_0 + \delta_1 < b$. Using Lemma 3.3 for the space X instead of L_p, for $n = 1, 2, \ldots$ construct a function

$$x : [t_0 - \delta_n, t_0 - \delta_{n+1}] \cup [t_0 + \delta_{n+1}, t_0 + \delta_n] \to X$$

having zero derivative such that $x(s) = sx_0$ for $s = t_0 \pm \delta_n$ and $s = t_0 \pm \delta_{n+1}$, and

$$\left\| \frac{x(t) - tx_0}{\delta_{n+1}} \right\| < \frac{1}{n}$$

for each t with $\delta_{n+1} \le |t-t_0| \le \delta_n$. Finally, define $x(t_0)=t_0x_0$. We show that $x'(t_0)=x_0$. If $\delta_{n+1} \le |\Delta t| \le \delta_n$, then

$$\left\|\frac{x(t_0+\Delta t)-x(t_0)}{\Delta t}-x_0\right\| = \left\|\frac{x(t_0+\Delta t)-(t_0+\Delta t)x_0}{\Delta t}\right\| < \frac{1}{n}. \blacksquare$$

PROBLEM. Let X be an F-space with trivial dual. Does every continuous function from $[a,b]$ into X have a primitive? What happens for $X=L_p$ with $0 \le p < 1$?

Addendum (January 1994). Recently Professor N. J. Kalton sent me his short preprint "The existence of primitives for continuous functions in quasi-Banach space" which contains an affirmative answer to the Problem in the setting of quasi-Banach spaces.

References

[1] N. J. Kalton, *The compact endomorphisms of* L_p $(0 \le p \le 1)$, Indiana Univ. Math. J. 27 (1978), 353–381.
[2] —, *Curves with zero derivatives in F-spaces*, Glasgow Math. J. 22 (1981), 19–29.
[3] N. J. Kalton, N. T. Peck and J. W. Roberts, *An F-space Sampler*, London Math. Soc. Lecture Note Ser. 89, Cambridge Univ. Press, Cambridge, 1984.
[4] S. Mazur and W. Orlicz, *Sur les espaces métriques linéaires I*, Studia Math. 10 (1948), 184–208.
[5] S. Rolewicz, *Metric Linear Spaces*, PWN, Warszawa, 1985.

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A recurrence theorem for square-integrable martingales

by

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Abstract. Let $(M_n)_{n\ge0}$ be a zero-mean martingale with canonical filtration $(\mathcal{F}_n)_{n\ge0}$ and stochastically L_2-bounded increments $Y_1, Y_2, \ldots$, which means that

$$P(|Y_n|>t \mid \mathcal{F}_{n-1}) \le 1-H(t) \quad \text{a.s. for all } n \ge 1,\ t>0$$

and some square-integrable distribution H on $[0,\infty)$. Let $V^2=\sum_{n\ge1}E(Y_n^2 \mid \mathcal{F}_{n-1})$. It is the main result of this paper that each such martingale is a.s. convergent on $\{V<\infty\}$ and recurrent on $\{V=\infty\}$, i.e. $P(M_n \in [-c,c] \text{ i.o.} \mid V=\infty)=1$ for some $c>0$. This generalizes a recent result by Durrett, Kesten and Lawler [4] who consider the case of only finitely many square-integrable increment distributions. As an application of our recurrence theorem, we obtain an extension of Blackwell's renewal theorem to a fairly general class of processes with independent increments and linear positive drift function.

1. Introduction. Let $(S_n)_{n\ge0}$ be a random walk with i.i.d. zero-mean, non-vanishing increments $X_1, X_2, \ldots$ Then $(S_n)_{n\ge0}$ is recurrent with recurrence set $\Re=\mathbb{R}$ in case of non-arithmetic increments, and $\Re=d\mathbb{Z}$ if $X_1, X_2, \ldots$ are d-arithmetic for some $d>0$. In any case

$$P(|S_n| \le c \text{ i.o.})=1 \tag{1.1}$$

for all $c>0$. Dispensing with the stationarity assumption on $X_1, X_2, \ldots$, (1.1) need no longer be true. Durrett, Kesten and Lawler [4] give an example of a random walk $(S_n)_{n\ge0}$ which converges a.s. to ∞, even though its increments are independent and drawn from a set of merely two zero-mean distributions. On the other hand, they also show that (1.1) holds true for sufficiently large c provided that $X_1, X_2, \ldots$ are independent and drawn from a finite set of distributions with mean 0 and finite, positive variances. In fact, their result is even stated for so-called controlled random walks, that is, general martingales with square-integrable conditional increment distributions drawn from a finite set. Although their proof uses the finiteness of the latter

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