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Some constructions of strictly ergodic non-regular Toeplitz flows

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Abstract. We give a necessary and sufficient condition for a Toeplitz flow to be strictly ergodic. Next we show that the regularity of a Toeplitz flow is not a topological invariant and define the "eventual regularity" as a sequence; its behavior at infinity is topologically invariant. A relation between regularity and topological entropy is given. Finally, we construct strictly ergodic Toeplitz flows with "good" cyclic approximation and non-discrete spectrum.

Introduction. Let A be a finite set of at least two elements. Denote by A^* the set of finite sequences, or words, over A. It has a semigroup structure for the concatenation of words; if $w \in A^*$, let $|w|$ denote its length.

Endow A with the discrete topology; then the set $\Omega = A^{\mathbb{Z}}$ is a compact metrizable space with the product topology. If $u \in \Omega$, $n \in \mathbb{Z}$ and $p \geq 1$, then let

$$u[n, n+p) = u(n)u(n+1)\dots u(n+p-1)$$

denote the word of length p appearing in u at position n.

According to [Ja-Ke], an element $\eta \in \Omega$ is called a *Toeplitz sequence* if it is not a periodic sequence and satisfies the following condition:

$$(\forall n \in \mathbb{Z})\ (\exists p \geq 2)\ (\forall k \in \mathbb{Z}) \quad \eta(n+kp) = \eta(n).$$

Define the shift transformation $S : \Omega \to \Omega$ by $Su(n) = u(n+1)$. With $u \in \Omega$ we associate its orbit $O(u) = \{S^n u : n \in \mathbb{Z}\}$ and the orbit closure $\bar{O}(u)$ for the product topology.

Now a *Toeplitz flow* is a pair $(\bar{O}(\eta), S)$ where $\eta \in \Omega$ is a Toeplitz sequence. Every Toeplitz flow is minimal; if it admits only one shift-invariant

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probability measure, we say it is *strictly ergodic*. We refer to [Wi] for further notation on Toeplitz flows.

The aim of the present paper is to continue the investigation of Toeplitz flows from three points of view: strict ergodicity, degree of non-regularity, and cyclic approximation with spectral implications. The paper is organized as follows.

In Section 1 we state in Theorem 1.1 a necessary and sufficient condition for a Toeplitz flow to be strictly ergodic.

In Section 2 we study the regularity $d(\eta)$ of a Toeplitz flow $(\bar{O}(\eta), S)$. We show in Example 2.2 that, perhaps surprisingly, $d(\eta)$ is not a topological invariant. A strong version of this is Example 2.3. We propose a notion of eventual regularity (Definition 2.1) which, using [Do-Kw-La], turns out to be "eventually invariant" for topological isomorphisms of Toeplitz flows (Proposition 2.1). It is shown that a Toeplitz flow with positive topological entropy has its eventual regularity $\delta(\eta)$ tending to zero (Corollary 2.1).

In Section 3 we construct Toeplitz flows as group extensions over their maximal equicontinuous factor, inspired by constructions from [Wi], and obtain such flows having good cyclic approximations. In our construction the only eigenvalues are those located on the maximal equicontinuous factor and the spectrum is partly continuous (Theorem 3.1).

1. Strict ergodicity. Let $\infty \notin A$ be an additional symbol, referred to as a *hole*. Let $\bar{A} = A \cup \{\infty\}$, and $\bar{\Omega} = \bar{A}^{\mathbb{Z}}$. Given $B \in \bar{A}^*$ define the periodic sequence B^∞ by letting $B^\infty(n) = B(m)$ whenever $n \equiv m \bmod |B|$.

Then a Toeplitz sequence $\eta \in \Omega$ can always be expressed as $\lim_n B_n^\infty$ in $\bar{\Omega}$ where $B_n \in \bar{A}^*$, $|B_n| = p_n$, the sequence p_n is increasing, and the following conditions are satisfied:

(1) $p_n \geq 2$, $p_n \mid p_{n+1}$,
(2) $B_{n+1}(j) = B_n(i)$ whenever $j \equiv i \bmod p_n$ and $B_n(i) \neq \infty$,
(3) p_n is the least period of B_n^∞.

The sequence $p_1, p_2, \ldots$ is a *period structure* of the Toeplitz sequence η (in the sense of [Wi]). As in [La1] and [Do-Kw-La], we define *t-symbols* of η. For any $t \geq 1$,

$$W_t(\eta) = \{\eta[kp_t, (k+1)p_t) : \ k \in \mathbb{Z}\}$$

will denote the set of t-symbols.

Now let $G_\eta = \varprojlim \mathbb{Z}_{p_n}$ be the compact monothetic group of (p_n)-adic integers (see [He-Ro]). The elements of G_η are represented as sequences $g = (g_n) \in \prod_{n=1}^\infty \{0, \ldots, p_n - 1\}$ such that $g_{n+1} \equiv g_n \bmod p_n$. The element $\hat{1} = (1, 1, \ldots, 1, \ldots)$ is a topological generator. We let $\tau_{\hat{1}}(g) = g + \hat{1}$, $g \in G_\eta$.

From [Wi], we know that for any $\omega \in \bar{O}(\eta)$, there exists a unique $g(\omega) = (g_n(\omega)) \in G_\eta$ such that for any $n \geq 1$, $k \in \mathbb{Z}$, we have $\omega[-g_n(\omega) + kp_n, -g_n(\omega) + (k+1)p_n) \in W_n(\eta)$. Thus ω is a bi-infinite concatenation of n-symbols at special places. We also know that the map $\pi : \bar{O}(\eta) \to G_\eta$ defined by $\pi(\omega) = g(\omega)$ is onto, continuous, and $\pi \circ S = \tau_{\hat{1}} \circ \pi$. This defines the maximal equicontinuous factor of the flow $(\bar{O}(\eta), S)$.

Now we take the set of t-symbols of η as an alphabet $A_t = W_t(\eta)$. Let $\eta^{(t)}$ be the bi-infinite sequence over A_t defined by

$$\eta^{(t)}(n) = \eta[np_t, (n+1)p_t).$$

Then since η is Toeplitz, so is $\eta^{(t)}$. In fact, if, given $\omega \in \bar{O}(\eta)$, we define $\omega^{(t)} \in \Omega_t = A_t^{\mathbb{Z}}$ by

$$\omega^{(t)}(n) = \omega[-g_t(\omega) + np_t, -g_t(\omega) + (n+1)p_t)$$

then the map $\Phi_t(\omega) = (g_t(\omega), \omega^{(t)})$ is a homeomorphism of $\bar{O}(\eta)$ onto $\mathbb{Z}_{p_t} \times \bar{O}(\eta^{(t)})$. Moreover, on Ω_t, we may once again define the shift, still denoted by S. We let $\varepsilon_t : \mathbb{Z}_{p_t} \to \{0, 1\}$, where $\varepsilon_t(g_t) = 1$ if $g_t = p_t - 1$ and $\varepsilon_t(g_t) = 0$ otherwise. The action $\widetilde{S}_t : \mathbb{Z}_{p_t} \times \bar{O}(\eta^{(t)}) \to \mathbb{Z}_{p_t} \times \bar{O}(\eta^{(t)})$ defined by $\widetilde{S}_t(g_t, u) = (g_t + 1, S^{\varepsilon_t(g_t)}(u))$ is a homeomorphism and

$$\Phi_t \circ S = \widetilde{S}_t \circ \Phi_t.$$

Let $B, C \in A^*$ be such that $|B| = \beta \leq \gamma = |C|$. Then we denote by $\mathrm{ap}(B, C)$ the frequency of appearances of B at positions $j\beta$ in C, in other words,

$$\mathrm{ap}(B, C) = \frac{1}{[\gamma/\beta]}(\#\{j : C[j\beta, (j+1)\beta) = B, \ 0 \leq j < [\gamma/\beta]\}).$$

Now we fix a period structure (p_t) for the Toeplitz sequence η and write W_t for $W_t(\eta)$.

THEOREM 1.1. *The Toeplitz flow $(\bar{O}(\eta), S)$ is strictly ergodic if and only if for any $s \geq 1$ and $B \in W_s$ there exists a number $\nu(B)$ such that*

$$\mathrm{ap}(B, C) \to \nu(B)$$

uniformly in $C \in W_t$ as $t \to \infty$.

Proof. First we prove the "if" part. For $w \in A^*$ define

$$[w] = \{u \in \Omega : u[0, |w|) = w\}.$$

It suffices to show that for every $w \in A^*$ the limit

$$\lim_k \frac{1}{k} \sum_{j=0}^{k-1} 1_{[w]}(S^{n+j}\eta)$$

exists uniformly in n (see [Ox]). Given w and $\varepsilon > 0$ we choose s such that $|w|/p_s < \varepsilon/3$. For any s-symbol B denote by $N(w, B)$ the number of occurrences of w in B.

Now we count the occurrences of w in $\eta[n, n+k)$. First observe that s-symbols B appear there at positions jp_s for at least $[k/p_s]-1$ values of j. Although they occur with frequencies depending on k, it follows from our assumption that the frequency of each B differs little from $\nu(B)$ if k is sufficiently large. We choose k_0 such that for $k \geq k_0$ the difference never exceeds δ, where

$$\delta < \frac{\varepsilon}{3}\left(\frac{1}{p_s}\sum_B N(w,B)\right)^{-1}.$$

Without loss of generality we may also assume $6p_s/k_0 < \varepsilon$. It is now clear that w appears in $\eta[n, n+k)$ at least

$$\sum_B ([k/p_s]-1)(\nu(B)-\delta)N(w,B)$$

times. The occurrences that are not taken into account are either those overlapping two consecutive s-symbols or those appearing at the extremities. Their total number does not exceed $[k/p_s]|w| + 2p_s$.

Consequently,

$$\left(\frac{1}{p_s}-\frac{2}{k}\right)\sum_B(\nu(B)-\delta)N(w,B)$$
$$\leq \frac{1}{k}\sum_{j=0}^{k-1} 1_{[w]}(S^{n+j}\eta) < \frac{1}{p_s}\sum_B(\nu(B)+\delta)N(w,B) + \frac{|w|}{p_s} + \frac{2p_s}{k},$$

which, by the choice of δ and k_0, implies that for $k \geq k_0$ the middle term differs from

$$\frac{1}{p_s}\sum_B \nu(B)N(w,B)$$

by less than ε. This is a uniform Cauchy condition which clearly ensures the required uniform convergence.

The "only if" part follows readily from the skew-product representation given by $\Phi_t : \bar{O}(\eta) \to \mathbb{Z}_{p_t} \times \bar{O}(\eta^{(t)})$. In fact, the strict ergodicity of the product implies that of $\bar{O}(\eta^{(t)})$ and B is a single letter in $\eta^{(t)}$.

As an immediate corollary we obtain the following criterion of strict ergodicity.

COROLLARY 1.1. *If* $\operatorname{ap}(B,C) = \operatorname{ap}(B,C')$ *for any* $B \in W_t$ *and* $C, C' \in W_{t+1}$ $(t = 1, 2, \ldots)$ *then the flow* $(\bar{O}(\eta), S)$ *is strictly ergodic.*

As an application of Corollary 1.1 we propose the following construction of strictly ergodic flows.

EXAMPLE 1.1. Let $A = \{0,1\}$. We will define three sequences of words, $B_n \in \bar{A}^*$ and $C_n, C_n' \in A^*$, with B_n satisfying (1)–(3) and such that $\eta = \lim_n B_n^\infty$ is a Toeplitz sequence with $W_t(\eta) = \{C_t, C_t'\}$ for any t.

We need the following notation. If $B \in \bar{A}^*$ and $w \in A^*$ is a word whose length coincides with the number of holes in B then we denote by $B(w)$ the word constructed from B by filling in its successive holes by the successive letters of w.

Now we fix a sequence r_n of positive integers. Let $B_1 = 0\infty\infty 1$, $C_1 = B_1(01)$, $C_1' = B_1(10)$. Assume that B_n, C_n and C_n' have been defined. Then define

$$B_{n+1} = C_n B_n^{r_n} B_n^{r_n} C_n',$$
$$C_{n+1} = C_n C_n^{r_n} C_n'^{r_n} C_n',$$
$$C_{n+1}' = C_n C_n'^{r_n} C_n^{r_n} C_n'.$$

It is easy to check that conditions (1)–(3) are satisfied and the sequence $\eta = \lim_n B_n^\infty$ is not periodic. It is easily seen that the assumptions of Corollary 1.1 are satisfied and therefore the flow $(\bar{O}(\eta), S)$ is strictly ergodic.

2. Regularity and topological entropy.

The regularity of the sequence η was introduced in [Ja-Ke] as follows.

Let B_n be a sequence of words in $\bar{A}^*$ associated with η, satisfying conditions (1)–(3) of the preceding section. Then, for any $n \geq 1$, let

$$d_n(\eta) = \frac{1}{p_n}(\#\{i : B_n(i) \neq \infty,\ 0 \leq i < p_n\}).$$

It is easy to deduce that the sequence $d_n(\eta)$ is strictly increasing. Thus it has a limit, denoted by $d(\eta)$, in $(0,1]$. This is the *regularity* of the sequence η:

$$d(\eta) = \lim_n d_n(\eta).$$

A Toeplitz sequence η is called *regular* if $d(\eta) = 1$.

EXAMPLE 2.1. Let us go back to the construction of Example 1.1. We can see that $p_1 = 4$, $p_{n+1} = |B_{n+1}| = (2r_n+2)|B_n|$, $d_1(\eta) = 1/2$, and

$$d_{n+1}(\eta) = \frac{2 + 2r_n d_n(\eta)}{2r_n+2} = d_n(\eta) + \frac{2(1-d_n(\eta))}{2r_n+2}$$

so

$$1 - d_{n+1}(\eta) = (1-d_n(\eta))\left(1 - \frac{1}{r_n+1}\right).$$

Thus

$$1 - d(\eta) = \frac{1}{2}\prod_{n=1}^{\infty}\left(1 - \frac{1}{r_n+1}\right).$$

Here we have $d(\eta) < 1$ if and only if $\sum 1/r_n < \infty$.

Remark 2.1. If η is Toeplitz, then $\bar{O}(\eta)$ contains many Toeplitz sequences different from η. However, it can be proved rather easily that for any $\widetilde{\eta} \in \bar{O}(\eta)$ which is Toeplitz, $d(\widetilde{\eta}) = d(\eta)$. Thus, in particular, it makes sense to speak of the regularity of a Toeplitz *flow*.

Recall that if η is regular, then it is strictly ergodic and measure-theoretically isomorphic to its maximal equicontinuous factor, which is the group of (p_t)-adic integers (see [Ja-Ke], [Wi]).

DEFINITION 2.1. The *eventual regularity* of the Toeplitz sequence η given a period structure p_t is the sequence $\delta(\eta) = (d(\eta^{(1)}), d(\eta^{(2)}), \ldots)$.

We say that two flows $(\bar{O}(\eta), S)$ and $(\bar{O}(\omega), S)$ are topologically isomorphic if there exists an invertible homeomorphism $f : \bar{O}(\eta) \to \bar{O}(\omega)$ such that $f \circ S = S \circ f$. In [Do-Kw-La] (see also [La1]) the following isomorphism criterion is proved for Toeplitz flows which are not necessarily over the same alphabet:

Two Toeplitz flows $(\bar{O}(\eta), S)$ and $(\bar{O}(\omega), S)$ are topologically isomorphic if and only if for some $t \geq 1$ there is a Toeplitz sequence $\widetilde{\omega} \in \bar{O}(\omega)$ (recall that $\bar{O}(\omega) = \bar{O}(\widetilde{\omega})$ from minimality) *such that*

$$\eta^{(t)} = \widetilde{\omega}^{(t)}$$

modulo a bijective map from $W_t(\eta)$ to $W_t(\widetilde{\omega})$.

From this and Remark 2.1 we obtain

PROPOSITION 2.1. *If $(\bar{O}(\eta), S)$ and $(\bar{O}(\omega), S)$ are topologically isomorphic, then one can find a choice of period structures such that the corresponding eventual regularities are eventually equal, i.e. there exists some t_0 such that $d(\eta^{(t)}) = d(\omega^{(t)})$ for any $t \geq t_0$.*

Here is a simple example which shows that the regularity $d(\eta)$ is not a topological invariant and illustrates the above proposition.

EXAMPLE 2.2. We take the same construction as in Example 2.1 except that we start with the following slight modification:

$$B_1 = 00\infty 1, \quad C_1 = B_1(0) = 0001, \quad C_1' = B_1(1) = 0011.$$

Then denote by ω the associated Toeplitz sequence. It is easy to compute, still calling η the sequence constructed in Example 1.1, that

$$1 - d(\omega) = \tfrac{1}{2}(1 - d(\eta)).$$

Then if $\sum 1/r_n < \infty$, both sequences are non-regular and moreover their regularities are different. But it is easy to see that the associated flows are topologically isomorphic by the above isomorphism criterion with $t = 1$ and $\widetilde{\omega} = \omega$. Thus, *the regularity is not a topological invariant.* On the other hand,

it easily follows from Proposition 2.1 that if $\bar{O}(\eta)$ and $\bar{O}(\omega)$ are topologically isomorphic and $d(\eta) = 1$, then $d(\omega) = 1$.

In view of this example, we can easily understand that it is possible, given finitely many rational numbers $0 < \varrho_1 < \ldots < \varrho_k < 1$, to construct Toeplitz flows $(\bar{O}(\eta^1), S), \ldots, (\bar{O}(\eta^k), S)$ with regularities $\varrho_1, \ldots, \varrho_k$ respectively and all being topologically isomorphic (we could for example construct Toeplitz flows for which the regularity appears as an infinite product of rational numbers; see [La2] for further information about such products).

In the following example we construct a countable family of Toeplitz flows that are topologically isomorphic and whose associated family of regularities is dense in the unit interval. Theorem 1.1 will be used to verify strict ergodicity.

EXAMPLE 2.3. This example is based on "Toeplitz sequences constructed from subshifts", due to S. Williams (the reader is referred to Section 4 in [Wi]). Our construction differs slightly from [Wi] in that we allow repetitions of words of the subshift Y used to fill up the holes in the p_i-skeleton of η. This does not influence the properties of the constructed flow so we use freely the results of Section 4 of [Wi].

Let $2 < p_1 < p_2 < \ldots$ be such that $p_n \mid p_{n+1}$. Put $p_0 = 1$, $\lambda_n = p_n/p_{n-1}$, and assume λ_n odd. For two sequences of positive integers $\kappa = (\kappa_n)$ and $\kappa' = (\kappa_n')$, we write $\kappa \sim \kappa'$ if the sequences coincide for some n onwards. Now for all κ such that

(a) $2 \leq \kappa_n < \lambda_n - 2$,
(b) κ_n is even,

we will construct 0-1 Toeplitz sequences $\eta = \eta^\kappa$ such that

(i) η is a Toeplitz sequence constructed from the subshift $Y = \{01^\infty, 10^\infty\}$,
(ii) η is strictly ergodic,
(iii) $d(\eta) = \lim_n d_n$, where $d_1 = \kappa_1/\lambda_1$, $d_{n+1} = d_n + (1 - d_n)\kappa_{n+1}/\lambda_{n+1}$,
(iv) $(\bar{O}(\eta^\kappa), S)$, $(\bar{O}(\eta^{\kappa'}), S)$ are topologically isomorphic if $(\kappa_n) \sim (\kappa_n')$.

We construct η by induction. First let $B_1 = b_0 b_1 \ldots b_{p_1 - 1}$ where

$$b_0 b_1 \ldots b_{\kappa_1 - 2} = 0101 \ldots 0, \quad b_{\kappa_1 - 1} = \ldots = b_{p_1 - 2} = \infty, \quad b_{p_1 - 1} = 1.$$

Then put $C_1 = B_1(0101 \ldots 010)$ and $C_1' = B_1(1010 \ldots 101)$ (this means that the holes of B_1 are filled in consecutively with the two words of length $\lambda_1 - \kappa_1$ in Y). For the inductive step assume B_n, C_n, and C_n' have been constructed. Then let $B_{n+1} = v_0 v_1 \ldots v_{\lambda_{n+1} - 1}$ where

$$v_0 \ldots v_{\kappa_{n+1} - 2} = C_n C_n' C_n C_n' \ldots,$$
$$v_{\kappa_{n+1} - 1} = \ldots = v_{\lambda_{n+1} - 2} = B_n, \quad v_{\lambda_{n+1} - 1} = C_n'.$$

Then define $C_{n+1} = B_{n+1}(0101 \ldots 010)$ and $C_{n+1}' = B_{n+1}(1010 \ldots 101)$.

We have $|C_n| = p_n$ and for every n the only n-symbols are C_n and C'_n. Since the κ_n are even and the $\lambda_n - \kappa_n$ odd, C_n appears at positions jp_n with frequency

$$\nu_n = \mathrm{ap}(C_n, C_{n+1}) = \frac{1}{2}\left(1 + \frac{1}{\lambda_{n+1}}\right)$$

in C_{n+1} and with frequency $\nu'_n = 1 - \nu_n$ in C'_{n+1}. By the same token C'_n appears at these positions with frequency ν'_n in C_{n+1} and ν_n in C'_{n+1}. Now let $\nu_{n,k}$ and $\nu'_{n,k}$ denote the frequencies of C_n (at positions jp_n) in C_{n+k} and C'_{n+k}, respectively. We have

$$\nu_{n,k+1} = \nu_{n,k}\nu_k + \nu'_{n,k}\nu'_k, \qquad \nu'_{n,k+1} = \nu_{n,k}\nu'_k + \nu'_{n,k}\nu_k$$

and check by induction that

$$\nu_{n,k} = \frac{1}{2}\left(1 + \frac{1}{\lambda_{n+1}\dots\lambda_{n+k}}\right), \qquad \nu'_{n,k} = \frac{1}{2}\left(1 - \frac{1}{\lambda_{n+1}\dots\lambda_{n+k}}\right).$$

Since $\lambda_n > 2$, we obtain

$$\lim_k \nu_{n,k} = \lim_k \nu'_{n,k} = 1/2.$$

Now the assumption of Theorem 1.1 is verified and (ii) follows.

Checking (i) and (iii) is easy and clearly (iv) follows from the criterion of [Do-Kw-La]. Finally, we have

$$1 - d(\eta^\kappa) = \prod_{n=1}^{\infty}\left(1 - \frac{\kappa_n}{\lambda_n}\right).$$

To get the denseness of the associated regularities $d(\eta^\kappa)$, it suffices to choose sequences (κ_n) and (p_n) such that

$$\lim_n \prod_{j=n}^{\infty}\left(1 - \frac{\kappa_j}{\lambda_j}\right) = 1,$$

which simply means $\sum \kappa_n/\lambda_n < \infty$, and next construct all possible sequences κ' that are eventually equal to κ and satisfy the conditions (a) and (b) for the chosen (p_n). The denseness is then obtained using classical arguments on infinite products (cf. [La2]).

We recall that given any $u \in \Omega$, the topological entropy $h(u)$ of the flow $(\bar{O}(u), S)$ is equal to $\lim_n n^{-1}\log N_n$, where N_n is the number of distinct words of length n in u. For the Toeplitz sequence η it is not hard to see that

$$\#W_t \le N_{p_t} \le p_{t-1}(\#W_{t-1})^{1+p_t/p_{t-1}},$$

which yields the following entropy formula (appearing in [La1] and [Do-Kw-La]):

$$h(\eta) = \lim_t \frac{1}{p_t}\log(\#W_t(\eta)).$$

PROPOSITION 2.2. *If η is a Toeplitz sequence, then*

$$h(\eta) \le (1 - d(\eta))\log(\#A).$$

Proof. Obviously

$$\#W_t(\eta) \le (\#A)^{p_t(1-d_t(\eta))},$$

so the assertion follows immediately from the entropy formula above.

The following is a corollary to Proposition 2.2.

COROLLARY 2.1. *If η is a Toeplitz sequence and $h(\eta) > 0$ then*

$$d(\eta^{(t)}) \to 0.$$

Proof. We have

$$h(\eta^{(t)}) \le (1 - d(\eta^{(t)}))\log(\#W_t(\eta)).$$

The entropy formula applied to the Toeplitz sequence $\eta^{(t)}$ over the alphabet A_t implies $h(\eta^{(t)}) = p_t h(\eta)$ (alternatively, the same is obtained from [Ne] with the help of the isomorphism Φ_t). This gives

$$1 - d(\eta^{(t)}) \ge h(\eta^{(t)})(\log(\#W_t(\eta)))^{-1} = h(\eta)\left(\frac{1}{p_t}\log(\#W_t(\eta))\right)^{-1} \to 1,$$

which means $d(\eta^{(t)}) \to 0$.

3. Cyclic approximation of Toeplitz flows. In this section we exploit William's "Toeplitz sequences constructed from subshifts" to obtain systems isomorphic to group extensions. This will enable us to construct strictly ergodic Toeplitz flows with partly continuous simple spectrum.

Let T be an invertible measure preserving transformation (automorphism) of a standard Lebesgue probability space (X, μ). We denote by ϵ the point partition of X.

According to Katok and Stepin (see e.g. [Co-Fo-Si], Chapter 15), an automorphism T admits *cyclic approximation* (*c.a.*) with speed $f(n)$ if there exist measurable partitions $\xi_n = \{C_0, \dots, C_{h_n-1}\} \to \epsilon$ and automorphisms T_n permuting cyclically the elements of ξ_n such that

$$\sum_{j=0}^{h_n-1} \mu(TC_j \,\triangle\, T_n C_j) < f(h_n).$$

If T admits c.a. with speed θ/n, $\theta < 1$, then it has simple spectrum ([Co-Fo-Si]). The c.a. with speed $o(1/n)$ implies rank one; moreover, if T admits

c.a. with speed $1/n^r$ then the spectral measure is concentrated on a set of Hausdorff dimension not exceeding $1/r$ (see [Iw]).

THEOREM 3.1. *Let $p_1 < p_2 < \dots$ be positive integers with $p_t \mid p_{t+1}$ and assume that there exists $s \geq 2$ such that, for every $k \geq 1$,*

$$s^k \mid p_t \quad \text{for all sufficiently large } t.$$

Let $f(n) > 0$ decrease to zero. Then there exists a strictly ergodic 0-1 Toeplitz flow which admits c.a. with speed $f(n)$ and has partly continuous spectrum with the only eigenvalues of the form $\exp(2\pi ik/p_t)$ (arising from the maximal equicontinuous factor).

Proof. Let Y be the shift orbit of a strictly s-periodic sequence, e.g. $10\dots0^\infty$. The number of words of a fixed length never exceeds s in Y. We construct a 0-1 Toeplitz sequence η by induction.

Step 1. Choose p_{n_1} with $s \mid p_{n_1}$ and find an integer $0 < p_1' < p_{n_1} - 1$ such that $(s, p_1') = 1$. Now fill up the positions $-1, 0, \dots, p_{n_1} - p_1' - 2$ with any sequence of $\beta_1 = p_{n_1} - p_1'$ symbols 0,1 using both 0 and 1. Then repeat the pattern with period p_{n_1} to obtain the p_{n_1}-skeleton of η. The density of the skeleton is equal to $d_1 = (p_{n_1} - p_1')/p_{n_1}$ so

$$p_1' = (1 - d_1)p_{n_1}$$

is the number of holes in each p_{n_1}-period. Finally, let

$$0 < \varepsilon_1 < f(sp_{n_1})/2.$$

Step 2. Choose an integer $\beta_2 > s$ with $(s, \beta_2) = 1$. Fill up the holes in the β_2 consecutive p_{n_1}-words

$$\eta[-p_{n_1}, 0),\ \eta[0, p_{n_1}),\ \dots,\ \eta[(\beta_2 - 2)p_{n_1}, (\beta_2 - 1)p_{n_1})$$

using all possible words of length p_1' in Y (words may be repeated). Now repeat the pattern with period p_{n_2}, where n_2 is chosen sufficiently large to ensure $s \mid \lambda_2$, where $\lambda_2 = p_{n_2}/p_{n_1}$, and $\beta_2 p_1'/p_{n_2} < \varepsilon_1$. We thus obtain the p_{n_2}-skeleton of η of density $d_2 = d_1 + (1 - d_1)\beta_2/\lambda_2$. Clearly $d_2 - d_1 < \varepsilon_1$ and the number of holes in each p_{n_2}-period equals

$$p_2' = (1 - d_2)p_{n_2} = p_1'(\lambda_2 - \beta_2)$$

so $(s, p_2') = 1$. Finally, we choose $\varepsilon_2 > 0$ such that

$$\varepsilon_1 + \varepsilon_2 < f(sp_{n_1})/2, \quad \varepsilon_2 < f(sp_{n_2})/2.$$

It is clear how to continue the process by induction. At the end of the ith step we choose $\varepsilon_i > 0$ to satisfy

$$\varepsilon_1 + \dots + \varepsilon_i < f(sp_{n_1})/2,$$
$$\varepsilon_2 + \dots + \varepsilon_i < f(sp_{n_2})/2,\ \dots,\ \varepsilon_i < f(sp_{n_i})/2.$$

We have $s \mid \lambda_i$ where $\lambda_i = p_{n_i}/p_{n_{i-1}}$. The density of the p_{n_i}-skeleton of η is $d_i = d_{i-1} + (1 - d_{i-1})\beta_i/\lambda_i$, where β_i is the number of words of length p_{i-1}' in Y used—with possible repetitions—to construct the p_{n_i}-skeleton. The condition $(s, \beta_i) = 1$ implies $(s, p_i') = 1$, where

$$p_i' = p_{i-1}'(\lambda_i - \beta_i)$$

is the number of holes in the p_{n_i}-skeleton. Moreover, $d_{i+1} - d_i < \varepsilon_i$ so $d - d_i \leq f(sp_i)/2$, where $d = \lim_i d_i$.

Denote by G the compact group of (p_{n_i})-adic integers. Clearly, G is isomorphic (as a topological group) to the group of (p_i)-adic integers so to simplify the notation we omit the double subscripts. G is the maximal equicontinuous factor of $\bar{O}(\eta)$ and according to [Wi], Section 4, there is a bi-measurable mapping ϕ of $G \times Y$ onto $\bar{O}(\eta)$ such that $S \circ \phi = \phi \circ T$, where T is the piecewise power skew product defined by $T(g, y) = (g + \hat{1}, S^{\theta(g)}y)$, with $\theta(g) = 1$ if $0 \in \mathrm{Aper}(g)$ and $\theta(g) = 0$ otherwise (where $\mathrm{Aper}(g)$ is the set of those $n \in \mathbb{Z}$ for which the condition defining a Toeplitz sequence is not satisfied for $\omega \in \pi^{-1}(g)$ at n).

In other words, T is the group extension of the rotation $(G, \tau_{\hat{1}})$ by means of the measurable cocycle $\psi : G \to \mathbb{Z}_s = Y$ defined by $\psi(g) = 0$ if $g \in \pi(C)$ and $\psi(g) = 1$ otherwise. Here, $C = \{\omega : 0 \notin \mathrm{Aper}\,(g(\omega))\}$. More precisely, let $G_i = \{g \in G :\ g_1 = \dots = g_i = 0\}$ and let $k_1, \dots, k_{p_i'}$ be the places where the holes appear in the p_i-period of η. Now

$$G \setminus \pi(C) = \bigcap_i \bigcup_{j=1}^{p_i'} (G_i + \hat{k}_j).$$

Moreover, ϕ establishes a one-to-one correspondence between the ergodic invariant measures on $(\bar{O}(\eta), S)$ and the ergodic T-invariant Borel measures on $G \times Y$; for any fixed invariant measure, ϕ becomes a measure-theoretic isomorphism of the two systems ([Wi], Theorem 4.5).

In particular, the strict ergodicity of the flow $(\bar{O}(\eta), S)$ will follow once it is shown that there is a unique T-invariant measure on $G \times Y$. Therefore, by a well known result of Furstenberg [Fu], to get the strict ergodicity, it suffices to prove that T is ergodic for the product measure μ on $G \times Y$ (with Haar measures on G and $\mathbb{Z}_s$). Consequently, as cyclic approximation with sufficiently good speed implies ergodicity ($f(n) \leq \theta/n$ with $\theta < 4$ will do, see [Co-Fo-Si]), the strict ergodicity will automatically follow from the rest of the theorem.

Now we prove that the skew-product group extension $(G \times Y, \mu, T)$ admits c.a. with speed $f(n)$. For $i = 1, 2, \dots$, we define the partition $\xi_i = \{C_0, \dots, C_{sp_i - 1}\}$ of $G \times Y$ by letting $C_0 = G_i \times \{0\}$ and $C_j = T_i^j C_0$ with

$T_i(g,y) = (g+\hat{1}, y+\psi_i(g))$ where

$$\psi_i(g) = \begin{cases} 1 & \text{if } g \in \bigcup_{j=1}^{p_i'}(G_i + \hat{k}_j), \\ 0 & \text{otherwise.} \end{cases}$$

Since the number of G_i-cosets on which $\psi_i(g) = 1$ is equal to p_i' and $(s, p_i') = 1$, we can see that T_i has period sp_i and permutes cyclically the elements of ξ_i. It is now clear that $\xi_i \to \epsilon$. To estimate the approximation error we observe that

$$\sum_{j=0}^{sp_i-1} \mu(TC_j \triangle T_iC_j) = 2\mu\{\psi \neq \psi_i\} = 2(d-d_i) \leq f(sp_i).$$

It remains to prove that ψ is a weakly mixing cocycle, i.e., the only eigenvalues of T are those occurring in the rotation $g \to g+\hat{1}$ of G. These are the numbers of the form $\exp(2i\pi k/p_j)$. Our argument will be similar to that of Keane (Lemma 7 in [Ke]). Suppose ζ is an eigenvalue. Then it corresponds to an eigenfunction of the form $f(g)\chi(y)$, where $f \in L^2(G)$ (in fact, $f \in L^\infty(G)$ by ergodicity) and χ is a character of the finite cyclic group $\mathbb{Z}_s = Y$. It is now clear that ζ^s corresponds to the eigenfunction $f^s(g)\chi^s(y) = f^s(g) \in L^2(G)$. Since T acts as the ergodic rotation by $\hat{1}$ on $L^2(G)$, we obtain $\zeta^s = \exp(2i\pi k/p_j)$ for some k and j. Therefore, $\zeta = \exp(2i\pi m/sp_j)$ for some $m \in \mathbb{Z}$. Now we write $\zeta = (\exp(2i\pi m/p_{j+1}))^{\lambda_{j+1}/s}$. By construction, $s \mid \lambda_{j+1}$ so ζ is an eigenvalue of the rotation $g \to g+\hat{1}$, which ends the proof of the theorem.

Remark 3.1. If $s = 2$ then the Toeplitz flow constructed above is (measure-theoretically) a $\mathbb{Z}_2$-extension of the rotation $(G, \tau_{\hat{1}})$. Therefore, by a result of M. Lemańczyk ([Le]), if $f(n) = o(1/n^2)$ then our Toeplitz flow is isomorphic to a (generalized) Morse flow. With some care Lemańczyk's proof can be improved to show that $o(1/n)$ implies Morse.

As shown in [Do-Kw-La], there exist strictly ergodic Toeplitz flows with positive entropy; this implies the presence of an infinite Lebesgue multiplicity in the spectrum. On the other hand, Toeplitz flows constructed in Theorem 3.1 are strictly ergodic and have simple spectrum with a highly singular, but non-trivial, continuous part. Moreover, there exist strictly ergodic non-regular Toeplitz flows with purely discrete spectrum ([Do-Iw], Remark 4). This suggests the following questions.

QUESTIONS. Can a strictly ergodic Toeplitz flow have an irrational eigenvalue? Do all eigenvalues arise from the maximal equicontinuous factor? Is it possible to construct a non-regular strictly ergodic Toeplitz flow measure-theoretically isomorphic to its maximal equicontinuous factor?

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