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Isomorphism of some anisotropic Besov and sequence spaces

by

A. KAMONT (Sopot)

Abstract. An isomorphism between some anisotropic Besov and sequence spaces is established, and the continuity of a Stieltjes-type integral operator, acting on some of these spaces, is proved.

1. Introduction. This paper gives a description of some anisotropic Besov spaces $B^{\underline{\alpha}}_{p,q}(I^d)$, $\underline{\alpha} = (\alpha_1, \ldots, \alpha_d)$. It is proved (Theorem A.1) that these spaces are isomorphic to some sequence spaces $b^{\underline{\alpha}}_{p,q}$, and the isomorphism is given by the coefficients of a function in the tensor product Franklin system of sufficiently high order. The one-dimensional version of Theorem A.2 was proved in [8]. In several dimensions the case of isotropic Besov spaces $B^s_{p,q}(I^d)$, $s \in \mathbb{R}$, was treated in [4] and [5]. It was proved in those papers that the isotropic Besov space $B^s_{p,q}(I^d)$ is isomorphic to some sequence space, and the isomorphism is given by the coefficients of a function in a specially constructed spline basis. The functions forming such a basis have the property that they are concentrated on small cubes, while the tensor products of Franklin functions, which are used in this paper, are concentrated on parallelepipeds.

The second part of Theorem A says that for some $\underline{\alpha}$ we can obtain another isomorphism of $B^{\underline{\alpha}}_{p,q}(I^d)$ and a sequence space by taking the coefficients of a function in the tensor product Schauder system (normalized in L^2). The one-dimensional version of Theorem A.2 was proved in [6].

Theorem B says that the Stieltjes-type integral operator $I(F,G)$,

$$I : B^{\underline{\beta}}_{p,1}(I^d) \times B^{\underline{\alpha}}_{p,\infty}(I^d) \to B^{\underline{\alpha}}_{p,\infty}(I^d),$$

with $1/p < \alpha_i < 1 - 1/p$, $\beta_i = 1 - \alpha_i$, is bounded as a bilinear operator. Its one-dimensional version was also proved in [6].

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Theory", Voneshta Voda (Bulgaria), June 18–24, 1993 ([7]).

The author thinks that the characterization of anisotropic Besov-type spaces of the type considered in this paper can be useful in investigation of fractional Wiener fields with multidimensional time parameter.

2. Preliminaries and notation. For $I=[0,1]$ we will denote by $W_p^0(I^d)$ the space $L^p(I^d)$ of all functions integrable with pth power for $1\le p<\infty$, and the space $C(I^d)$ of continuous functions on I^d for $p=\infty$.

Let $\underline{e}_i=(\delta_{1,i},\ldots,\delta_{d,i})\in\mathbb{R}^d$ for $i=1,\ldots,d$ be the unit vectors, and $\mathcal{D}=\{1,\ldots,d\}$. For $f:I^d\to\mathbb{R}$, $n\in\mathbb{N}$, $t\in\mathbb{R}$ and $i\in\mathcal{D}$ we define

$$I^d(n,t,i)=\{x\in I^d: x+nt\underline{e}_i\in I^d\},$$

$$\Delta_{t,i}^n f(x)=\begin{cases}\sum_{j=0}^n(-1)^{n+j}\binom{n}{j}f(x+jt\underline{e}_i) & \text{for } x\in I^d(n,t,i),\\ 0 & \text{for } x\in I^d\setminus I^d(n,t,i).\end{cases}$$

Let $A=\{i_1,\ldots,i_k\}\subset\mathcal{D}$, $\underline{t}=(t_1,\ldots,t_d)\in\mathbb{R}^d$ and $\underline{n}=(n_1,\ldots,n_d)\in\mathbb{N}^d$; then we set

$$\Delta_{\underline{t},A}^{\underline{n}}f=\Delta_{t_{i_1},i_1}^{n_{i_1}}\circ\ldots\circ\Delta_{t_{i_k},i_k}^{n_{i_k}}f.$$

The modulus of smoothness of order $\underline{n}$ in directions A in the L^p-norm is defined as follows:

$$\omega_{\underline{n},p,A}(f,\underline{t})=\sup_{|h_1|\le t_1}\ldots\sup_{|h_d|\le t_d}\|\Delta_{\underline{h},A}^{\underline{n}}f\|_p\quad\text{for } \underline{t}\in\mathbb{R}^d,\ 0<t_j\le 1/n_j.$$

For $A=\emptyset$ we put

$$\omega_{\underline{n},p,A}(f,\underline{t})=\|f\|_p.$$

For $\underline{h}=(h_1,\ldots,h_d)\in\mathbb{R}^d$ and $A=\{i_1,\ldots,i_k\}$ we will write $\underline{h}(A)=(\widetilde{h}_1,\ldots,\widetilde{h}_d)\in\mathbb{R}^d$, where $\widetilde{h}_i=h_i$ for $i\in A$ and $\widetilde{h}_i=0$ for $i\notin A$. For $\underline{n}=(n_1,\ldots,n_d)\in\mathbb{N}^d$ and $\underline{t}=(t_1,\ldots,t_d)\in\mathbb{R}^d$ the following abbreviations will be used:

$$\underline{t}^{\underline{n}}=\prod_{i=1}^d t_i^{n_i},\qquad D^{\underline{n}}f=\frac{\partial^{n_1}}{\partial x_{n_1}}\cdots\frac{\partial^{n_d}}{\partial x_{n_d}}f.$$

We will also need some spaces of spline functions with dyadic knots. Let us introduce the notation

$$s_0=0,\quad s_1=1,$$
$$s_n=\frac{2\nu-1}{2^{\mu+1}}\quad\text{for } n>1,\ n=2^\mu+\nu,\ \mu\ge0,\ 1\le\nu\le2^\mu. \tag{1}$$

Then for each $m\in\mathbb{N}\cup\{0\}$ and $n\ge -m$ we define

$$S_n^{(m)}(I)=\begin{cases}\text{for } -m\le n\le0: & \text{the space of polynomials of degree } n+m, \text{ restricted to } I,\\ \text{for } n>0: & \text{the space of spline functions on } I \text{ of degree } m+1 \text{ and maximal smoothness, with knots } s_0,\ldots,s_n.\end{cases}$$

Now the system of *Franklin functions* of order $m+2$ is defined as follows: $f_{-m}^{(m)}=1$, and for $n>-m$, $f_n^{(m)}\in S_n^{(m)}(I)$ and is orthogonal (in $L^2(I)$) to $S_{n-1}^{(m)}(I)$, $\|f_n^{(m)}\|_2=1$.

We will also need the Haar and Schauder systems on I:

- the *Haar system*: $h_1=1$, and for $n>1$, $n=2^\mu+\nu$, $\mu\ge0$, $1\le\nu\le2^\mu$,

$$h_n(t)=\begin{cases}2^{\mu/2} & \text{for } t\in\left[\frac{2\nu-2}{2^{\mu+1}},\frac{2\nu-1}{2^{\mu+1}}\right),\\ -2^{\mu/2} & \text{for } t\in\left[\frac{2\nu-1}{2^{\mu+1}},\frac{2\nu}{2^{\mu+1}}\right) \text{ if } \nu<2^\mu,\\ & \text{or } t\in\left[\frac{2\nu-1}{2^{\mu+1}},1\right] \text{ if } \nu=2^\mu,\\ 0 & \text{elsewhere in } I,\end{cases}$$

- the *Schauder system*:

$$\phi_0(t)=1,\quad \phi_n(t)=\int_0^t h_n(u)\,du,\quad n\ge1,$$

- the *normalized Schauder system*: $\phi_n^*=\phi_n/\|\phi_n\|_2$.

For $\underline{n}=(n_1,\ldots,n_d)$ we introduce the tensor product Franklin, Haar and Schauder systems on I^d:

$$\begin{aligned}f_{\underline{n}}^{(m)}&=f_{n_1}^{(m)}\otimes\ldots\otimes f_{n_d}^{(m)} &&\text{for } n_i\ge -m,\\ h_{\underline{n}}&=h_{n_1}\otimes\ldots\otimes h_{n_d} &&\text{for } n_i\ge1,\\ \phi_{\underline{n}}&=\phi_{n_1}\otimes\ldots\otimes\phi_{n_d} &&\text{for } n_i\ge0,\\ \phi_{\underline{n}}^*&=\phi_{n_1}^*\otimes\ldots\otimes\phi_{n_d}^* &&\text{for } n_i\ge0.\end{aligned}$$

It is well known that the tensor product Franklin system of order $m+2$, $\{f_{\underline{n}}^{(m)}: n_i\ge -m\}$, properly ordered (in the so-called rectangular order, described for instance in [3]) is a Schauder basis in $W_p^0(I^d)$ (cf. [3]). Similarly, $\{h_{\underline{n}}: n_i\ge1\}$ (in rectangular order) is a Schauder basis in $L^p(I^d)$ for $1\le p<\infty$, and $\{\phi_{\underline{n}}: n_i\ge0\}$ (also in rectangular order) is a Schauder basis in $C(I^d)$.

R e m a r k. Throughout this paper when we sum over a d-dimensional set of parameters ($\mathbb{N}^d$ or N_m^d), we always mean that this set is arranged in rectangular order.

3. Function and sequence spaces. Let $\underline{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$, $\underline{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$, $0 < \alpha_i < n_i$ for $i = 1, \ldots, d$, $1 \le p \le \infty$, $\underline{\gamma} = (1, \ldots, 1) \in \mathbb{N}^d$. For $f \in W_p^0(I^d)$ and $1 \le q < \infty$ we define

$$\|f\|_{p,q}^{(\underline{\alpha})} = \sum_{A \subset \mathcal{D}} \Bigl(\int_0^{1/n_1} \ldots \int_0^{1/n_d} \Bigl(\omega_{\underline{n},p,A}(f, \underline{t}) \underline{t}^{-\underline{\alpha}(A)} \Bigr)^q \underline{t}^{-\underline{\gamma}(A)} d\underline{t} \Bigr)^{1/q},$$

and for $q = \infty$ we put for $f \in W_p^0(I^d)$,

$$\|f\|_{p,q}^{(\underline{\alpha})} = \sum_{A \subset \mathcal{D}} \sup_{0 < t_1 \le 1/n_1} \ldots \sup_{0 < t_d \le 1/n_d} \underline{t}^{-\underline{\alpha}(A)} \omega_{\underline{n},p,A}(f, \underline{t}).$$

For $1 \le p, q \le \infty$ we consider the Besov-type function spaces

$$B_{p,q}^{\underline{\alpha}}(I^d) = \{ f \in W_p^0(I^d) : \|f\|_{p,q}^{(\underline{\alpha})} < \infty \}.$$

Remark. Let $\underline{n}, \underline{m} \in \mathbb{N}^d$ be such that $\alpha_i < n_i$ and $\alpha_i < m_i$ for $i = 1, \ldots, d$, and let $B_{p,q}^{\underline{\alpha}}(I^d)_{\underline{n}}$ and $B_{p,q}^{\underline{\alpha}}(I^d)_{\underline{m}}$ denote the spaces defined as above, corresponding to $\underline{n}$ and $\underline{m}$ respectively. Then it follows from the Marchaud-type inequalities for L^p-valued functions (cf. [4], Proposition 2.1) that $B_{p,q}^{\underline{\alpha}}(I^d)_{\underline{n}} = B_{p,q}^{\underline{\alpha}}(I^d)_{\underline{m}}$ (the sets are equal and the norms are equivalent).

Now we define the sequence spaces. For a given integer m define $N_m = \{-m-2, -m-1, -m, \ldots\}$, and for $j \in N_m$,

$$\widetilde{N}_j = \begin{cases} \{j+2\} & \text{for } j = -m-2, \ldots, -1, \\ \{2^j + k : k = 1, \ldots, 2^j\} & \text{for } j \ge 0. \end{cases}$$

For $\underline{j} = (j_1, \ldots, j_d) \in N_m^d$ define

$$\widetilde{N}_{\underline{j}} = \widetilde{N}_{j_1} \times \ldots \times \widetilde{N}_{j_d}.$$

Observe that $N_{m-2}^d = \bigcup_{\underline{j} \in N_m^d} \widetilde{N}_{\underline{j}}$.

For a real number $\alpha > 0$ and $1 \le p \le \infty$ set

$$c(j, \alpha, p) = \begin{cases} 1 & \text{for } j \le 0, \\ 2^{j(1/2 - 1/p + \alpha)} & \text{for } j > 0. \end{cases}$$

For $\underline{j} \in N_m^d$ and $\underline{\alpha} = (\alpha_1, \ldots, \alpha_d)$ put

$$c(\underline{j}, \underline{\alpha}, p) = c(j_1, \alpha_1, p) \ldots c(j_d, \alpha_d, p).$$

Now for a given sequence of real numbers $\underline{a} = (a_{\underline{k}})_{\underline{k} \in N_{m-2}^d}$ let

$$\|\underline{a}\|_{p,q}^{(\underline{\alpha})} = \Bigl(\sum_{\underline{j} \in N_m^d} \Bigl(c(\underline{j}, \underline{\alpha}, p) \Bigl(\sum_{\underline{k} \in \widetilde{N}_{\underline{j}}} |a_{\underline{k}}|^p \Bigr)^{1/p} \Bigr)^q \Bigr)^{1/q}$$

(with the sums of pth or qth powers replaced by suprema over the same set of indices if $p = \infty$ or $q = \infty$), and

$$b_{p,q}^{\underline{\alpha}} = \{ \underline{a} = (a_{\underline{k}})_{\underline{k} \in N_{m-2}^d} : \|\underline{a}\|_{p,q}^{(\underline{\alpha})} < \infty \}.$$

4. Results

THEOREM A. 1. *Let $m \in \mathbb{N} \cup \{0\}$, $\underline{\alpha} = (\alpha_1, \ldots, \alpha_d)$, $0 < \alpha_i < m+1$ for $i = 1, \ldots, d$, $1 \le p, q \le \infty$. Then the spaces $B_{p,q}^{\underline{\alpha}}(I^d)$ and $b_{p,q}^{\underline{\alpha}}$ are isomorphic, and the isomorphism is given by the coefficients of a function in the basis $\{f_{\underline{k}}^{(m)} : \underline{k} \in N_{m-2}^d\}$ of tensor products of Franklin functions of order $m+2$.*

2. *In case $1/p < \alpha_i < 1$ for $i = 1, \ldots, d$ another isomorphism of these spaces is given by the coefficients of a function in the basis $\{\phi_{\underline{k}}^* : \underline{k} \in N_{-2}^d\}$ of tensor products of Schauder functions, normalized in L^2.*

Remark. As $f_{\underline{k}}^{(m)} \in B_{p,q}^{\underline{\alpha}}(I^d)$ for $0 < \alpha_i < m+1$, $\underline{k} \in N_{m-2}^d$, it follows from Theorem A.1 that these functions form a Schauder basis in $B_{p,q}^{\underline{\alpha}}(I^d)$ for $1 \le q < \infty$, and in some separable subspace of $B_{p,\infty}^{\underline{\alpha}}(I^d)$. Analogously, it follows from Theorem A.2 that if $1/p < \alpha_i < 1$, then $\{\phi_{\underline{k}}^* : \underline{k} \in N_{-2}^d\}$ form a Schauder basis in $B_{p,q}^{\underline{\alpha}}(I^d)$ for $1 \le q < \infty$, and in their linear span in case $q = \infty$.

THEOREM B. *Let $1 \le p \le \infty$, $1/p + 1/p' = 1$, $\underline{\alpha} = (\alpha_1, \ldots, \alpha_d)$, $1/p < \alpha_i < 1/p'$, $\beta_i = 1 - \alpha_i$, $\underline{\beta} = (\beta_1, \ldots, \beta_d)$. For $F \in B_{p,1}^{\underline{\beta}}(I^d)$ and $G \in B_{p,\infty}^{\underline{\alpha}}(I^d)$,*

$$F = \sum_{\underline{j} \in N_1^d} \sum_{\underline{k} \in \widetilde{N}_{\underline{j}}} F_{\underline{k}} h_{\underline{k}}, \qquad G = \sum_{\underline{\xi} \in N_1^d} \sum_{\underline{\eta} \in \widetilde{N}_{\underline{\xi}}} G_{\underline{\eta}} \phi_{\underline{\eta}},$$

and $\underline{s} = (s_1, \ldots, s_d) \in I^d$ define

$$I(F, G)(\underline{s}) = \sum_{\underline{j} \in N_1^d} \sum_{\underline{\xi} \in N_1^d} \sum_{\underline{k} \in \widetilde{N}_{\underline{j}}} \sum_{\underline{\eta} \in \widetilde{N}_{\underline{\xi}}} F_{\underline{k}} G_{\underline{\eta}} \prod_{i=1}^d \int_0^{s_i} h_{k_i}(u_i) h_{\eta_i}(u_i) \, du_i.$$

There exists a constant $C = C(\underline{\alpha}, p)$ such that for all $F \in B_{p,1}^{\underline{\beta}}(I^d)$ and $G \in B_{p,\infty}^{\underline{\alpha}}(I^d)$,

$$\|I(F, G)\|_{p,\infty}^{(\underline{\alpha})} \le C \|F\|_{p,1}^{(\underline{\beta})} \|G\|_{p,\infty}^{(\underline{\alpha})}.$$

Remark. As $D^{\underline{\gamma}} \phi_{\underline{n}} = h_{\underline{n}}$ for $\underline{n} \in \mathbb{N}^d$, we can write

$$I(F, G)(\underline{s}) = \int_0^{s_1} \ldots \int_0^{s_d} F \, dG.$$

This operator can be useful in investigation of multidimensional Stratonovitch integrals.

5. Properties of the moduli of smoothness. The following properties of $\omega_{\underline{n},p,A}(f,\underline{t})$ will be needed:

5.1. *For each* $B \subset A$, $f \in W_p^{\underline{n}(B)}(I^d)$ *and* $\underline{t} = (t_1,\ldots,t_d) \in \mathbb{R}^d$, $0 < t_i < 1/n_i$,

$$\omega_{\underline{n},p,A}(f,\underline{t}) \le \underline{t}^{\underline{n}(B)} \omega_{\underline{n},p,A\setminus B}(D^{\underline{n}(B)} f, \underline{t}).$$

5.2. *For each* $f \in W_p^0(I^d)$, $\underline{l} = (l_1,\ldots,l_d) \in \mathbb{N}^d$, $\underline{t} = (t_1,\ldots,t_d)$, $0 < l_i t_i \le 1/n_i$, $\underline{t}^{(l)} = (l_1 t_1,\ldots,l_d t_d)$ *and* $A = \{i_1,\ldots,i_k\} \subset \mathcal{D}$,

$$\omega_{\underline{n},p,A}(f,\underline{t}^{(\underline{l})}) \le \underline{l}^{\underline{n}(A)} \omega_{\underline{n},p,A}(f,\underline{t}).$$

The following extension lemma will be useful.

LEMMA 5.3. *Let* $Q = [a_1, a_1+l_1] \times \ldots \times [a_d, a_d + l_d]$ *and* S *be two compact parallelepipeds in* $\mathbb{R}^d$ *with* $Q \subset S$. *Then there exists an extension operator* $T : W_p^0(Q) \to W_p^0(S)$ *such that*

$$\|Tf\|_p(S) \le C\|f\|_p(Q), \quad \omega_{\underline{n},p,A}(Tf,\underline{t})(S) \le C\omega_{\underline{n},p,A}(f,\underline{t})(Q)$$

for all $f \in W_p^0(Q)$, $A = \{i_1,\ldots,i_k\} \subset \mathcal{D}$ *and* $\underline{t} \in \mathbb{R}^d$ *with* $0 < t_i \le l_i/n_i$.

Proof. It is enough to prove the lemma for Q, S of the form

$$Q = [-a, 0] \times Q_0, \quad S = [-a, a] \times Q_0$$

(where Q_0 is a compact parallelepiped in $\mathbb{R}^{d-1}$). Then for Whitney's extension

$$Tf(x_1,x') = \begin{cases} f(x_1,x') & \text{for } -a \le x_1 \le 0, \\ \sum_{j=0}^{n_1} a_j f(-2^{-j}x_1, x') & \text{for } 0 < x_1 \le a, \end{cases}$$

where $\sum_{j=0}^{n_1} a_j(-1/2)^{jk} = 1$ for $k = 0,\ldots,n_1$, there is a constant C such that for all $f \in W_p^0(Q)$, $\underline{t}$ as above and $A_1 = \{1\}$,

$$\|Tf\|_p(S) \le C\|f\|_p(Q), \tag{2}$$

$$\omega_{\underline{n},p,A_1}(Tf,\underline{t})(S) \le C\omega_{\underline{n},p,A_1}(f,\underline{t})(Q) \tag{3}$$

(cf. Proposition 2.9 of [4]). Observe that for $A' \subset \mathcal{D}$ with $1 \notin A'$ we have

$$\Delta^{\underline{n}}_{\underline{t},A'} \circ T(f) = T \circ \Delta^{\underline{n}}_{\underline{t},A'}(f),$$

so from (2) we get

$$\omega_{\underline{n},p,A'}(Tf,\underline{t})(S) \le C\omega_{\underline{n},p,A'}(f,\underline{t})(Q).$$

If $A = \{1\} \cup A'$ then

$$\Delta^{\underline{n}}_{\underline{t},A} \circ T(f) = \Delta^{n_1}_{t_1,1} \circ T \circ \Delta^{\underline{n}}_{\underline{t},A'}(f),$$

and it follows from (3) that

$$\sup_{|h_1|\le t_1} \|\Delta^{n_1}_{h_1,1} \circ T \circ \Delta^{\underline{n}}_{\underline{t},A'}(f)\|_p(S) \le C \sup_{|h_1|\le t_1} \|\Delta^{n_1}_{h_1,1} \circ \Delta^{\underline{n}}_{\underline{t},A'}(f)\|_p(Q),$$

so the lemma follows from the definition of $\omega_{\underline{n},p,A}(f,\underline{t})$. ■

In the sequel we will need the equivalence between the modulus of smoothness $\omega_{\underline{n},p,A}(f,\underline{t})$ and the K-functional defined by the formula

$$K_{\underline{n},A,p}(f,\underline{t}) = \inf\Big\{\Big\|f - \sum_{\emptyset\ne B\subset A} g_B\Big\|_p + \sum_{\emptyset\ne B\subset A} \underline{t}^{\underline{n}(B)}\omega_{\underline{n},p,A\setminus B}(D^{\underline{n}(B)} g_B, \underline{t}) : g_B \in W_p^{\underline{n}(B)}(I^d),\ \emptyset \ne B \subset A\Big\}$$

for $f \in W_p^0(I^d)$ and $\underline{t} = (t_1,\ldots,t_d) \in \mathbb{R}^d$ with $t_i > 0$.

LEMMA 5.4. *Let* $1 \le p \le \infty$, $\underline{n} = (n_1,\ldots,n_d) \in \mathbb{N}^d$ *and* $A = \{i_1,\ldots,i_k\} \subset \mathcal{D}$ *be given. There exist constants* $C_k = C_k(\underline{n},p,A)$, $k = 1,2$, *such that*

$$C_1\omega_{\underline{n},p,A}(f,\underline{t}) \le K_{\underline{n},A,p}(f,\underline{t}) \le C_2\omega_{\underline{n},p,A}(f,\underline{t})$$

for every $f \in W_p^0(I^d)$ *and* $\underline{t} = (t_1,\ldots,t_d) \in \mathbb{R}^d$ *with* $0 < t_i \le 1/n_i$.

Proof. The left inequality is a consequence of 5.1.

Now let $\widetilde{f}$ be the extension of f to the parallelepiped $S = [0, n_1^2+1] \times \ldots \times [0, n_d^2 + 1]$, given by Lemma 5.3; then for $\underline{t} = (t_1,\ldots,t_d)$ with $0 < t_i \le 1/n_i$,

$$\omega_{\underline{n},p,A}(\widetilde{f},\underline{t})(S) \le C\omega_{\underline{n},p,A}(f,\underline{t}).$$

For $B \subset A$, $B = \{i_{j_1},\ldots,i_{j_m}\}$, define a function $g_B \in W_p^{\underline{n}(B)}(I^d)$ by the Steklov means

$$g_B(x) = -\sum_{k_{i_{j_1}}=1}^{n_{i_{j_1}}} \ldots \sum_{k_{i_{j_m}}=1}^{n_{i_{j_m}}} (-1)^{k_{i_{j_1}}+\ldots+k_{i_{j_m}}} \binom{n_{i_{j_1}}}{k_{i_{j_1}}} \ldots \binom{n_{i_{j_m}}}{k_{i_{j_m}}}$$
$$\times \int_0^1 \ldots \int_0^1 \widetilde{f}\Big(x + \sum_{l=1}^m k_{i_{j_l}} t_{i_{j_l}} (s_1^{(i_{j_l})} + \ldots + s_{n_{i_{j_l}}}^{(i_{j_l})}) \underline{e}_{i_{j_l}}\Big)\, ds_1^{(i_{j_1})} \ldots ds_{n_{i_{j_m}}}^{(i_{j_m})}.$$

Then

$$\Big\|f - \sum_{\emptyset\ne B\subset A} g_B\Big\|_p \le C_1\omega_{\underline{n},p,A}(\widetilde{f},\underline{t})(S)$$

and it follows from 5.2 that for some constant $C_2 > 0$,

$$\underline{t}^{\underline{n}(B)}\omega_{\underline{n},p,A\setminus B}(D^{\underline{n}(B)} g_B, \underline{t}) \le C_2\omega_{\underline{n},p,A}(\widetilde{f},\underline{t})(S),$$

which together with Lemma 5.3 completes the proof. ■

6. Proof of Theorem A. Let us start with the following definitions. For $g \in W_p^0(I)$, $m \in \mathbb{N} \cup \{0\}$ and $\mu \in N_m$ let

$$P_\mu^{(m)} g = \sum_{j=-m}^{\mu} \sum_{k \in \widetilde{N}_j} (g, f_k^{(m)}) f_k^{(m)}.$$

The following one-dimensional results will be needed (cf. [2], [3]).

LEMMA 6.1. *Let $m \in \mathbb{N} \cup \{0\}$. There exists a constant $C = C(m)$ such that for all $1 \le p \le \infty$ and $\mu \in N_m$,*

$$\|P_\mu^{(m)}\|_p \le C.$$

LEMMA 6.2. *Let $m \in \mathbb{N} \cup \{0\}$ and $1 \le p \le \infty$. Then there exists a constant $C > 0$ such that for all $\mu \ge 0$ and $f \in W_p^{m+1}(I)$,*

$$\|f - P_\mu^{(m)} f\|_p \le C \frac{1}{2^{\mu(m+1)}} \|D^{m+1} f\|_p.$$

For $f \in W_p^0(I^d)$ and $i \in \mathcal{D}$ let

$$P_{\mu,i}^{(m)} f(x_1, \ldots, x_d) = \sum_{j=-m}^{\mu} \sum_{k \in \widetilde{N}_j} \int_I f(x_1, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_d) f_k^{(m)}(u)\, du\, f_k^{(m)}(x_i)$$

and for $A = \{i_1, \ldots, i_k\}$ and $\underline{\mu} = (\mu_1, \ldots, \mu_d)$,

$$P_{\underline{\mu},A}^{(m)} = \mathrm{Id} - (\mathrm{Id} - P_{\mu_{i_1},i_1}^{(m)}) \circ \ldots \circ (\mathrm{Id} - P_{\mu_{i_k},i_k}^{(m)}).$$

Define

$$N_m^*(\underline{\mu}, A) = \{\underline{k} = (k_1, \ldots, k_d) \in N_{m-2}^d : \exists_{i_s \in A}\, \exists_{\xi \le \mu_{i_s}}\ k_{i_s} \in \widetilde{N}_\xi\},$$
$$S_{\underline{\mu},A,p}^{(m)} = \mathrm{span}_{W_p^0(I^d)}\{f_{\underline{k}}^{(m)} : \underline{k} \in N_m^*(\underline{\mu}, A)\}.$$

For $f \in W_p^0(I^d)$ let us introduce

$$E_{\underline{\mu},A,p}^{(m)}(f) = \inf_{g \in S_{\underline{\mu},A,p}^{(m)}} \|f - g\|_p.$$

Observe that $P_{\underline{\mu},A}^{(m)}$ is a projection of $W_p^0(I^d)$ onto $S_{\underline{\mu},A,p}^{(m)}$. As a consequence of Lemma 6.1 we obtain

LEMMA 6.3. *Let $m \in \mathbb{N} \cup \{0\}$. There exists a constant $C = C(m, d)$ such that for all $1 \le p \le \infty$, $\underline{\mu} = (\mu_1, \ldots, \mu_d) \in N_m^d$ and $\emptyset \ne A \subset \mathcal{D}$,*

$$\|P_{\underline{\mu},A}^{(m)}\|_p \le C.$$

For $\underline{\mu} \in (\mathbb{N} \cup \{0\})^d$ let $\underline{t_\mu} = (1/2^{\mu_1}, \ldots, 1/2^{\mu_d})$.

LEMMA 6.4. *Let $m \in \mathbb{N} \cup \{0\}$, $1 \le p \le \infty$, $\underline{n} = (m+1, \ldots, m+1) \in \mathbb{N}^d$, $\lambda \in \mathbb{N}$, $2^{\lambda-1} \le m+1 < 2^\lambda$. There exists a constant $C = C(m, d, p) > 0$ such that for each $\underline{\mu} = (\mu_1, \ldots, \mu_d)$ with $\mu_i \ge \lambda$, $A \subset \mathcal{D}$ and $f \in W_p^0(I^d)$,*

$$\|f - P_{\underline{\mu},A}^{(m)} f\|_p \le C \omega_{\underline{n},p,A}(f, \underline{t_\mu}).$$

Proof. The proof is by induction on $\#A$ (i.e. the cardinality of A).

For $\#A = 1$ this lemma is a consequence of Proposition 7.15 of [5] and Lemma 6.3.

Now let $\#A > 1$, $A = \{i_1, \ldots, i_k\}$, $g \in C^\infty(I^d)$ and $\emptyset \ne B \subset A$. As $D^{(m+1)\underline{e}_i} P_{\nu,j}^{(m)} g = P_{\nu,j}^{(m)} D^{(m+1)\underline{e}_i} g$ for $i \ne j$, from Lemma 6.2 we obtain

$$\|g - P_{\underline{\mu},A}^{(m)} g\|_p = \|(\mathrm{Id} - P_{\mu_1,i_1}^{(m)}) \circ \ldots \circ (\mathrm{Id} - P_{\mu_k,i_k}^{(m)}) g\|_p$$
$$\le C^{\#B} \underline{t_\mu}^{\underline{n}(B)} \|D^{\underline{n}(B)} g - P_{\underline{\mu},A\setminus B}^{(m)} D^{\underline{n}(B)} g\|_p.$$

As $\#(A \setminus B) < \#A$, it follows from the induction hypothesis that for any $g \in C^\infty(I^d)$,

$$\|g - P_{\underline{\mu},A}^{(m)} g\|_p \le C_1 \underline{t_\mu}^{\underline{n}(B)} \omega_{\underline{n},p,A\setminus B}(D^{\underline{n}(B)} g, \underline{t_\mu}).$$

Now let $f \in W_p^0(I^d)$ and $g_B \in C^\infty(I^d)$ for each $\emptyset \ne B \subset A$. Using the last inequality and Lemma 6.3 we obtain

$$\|f - P_{\underline{\mu},A}^{(m)} f\|_p \le \Big\|f - \sum_{\emptyset \ne B \subset A} g_B\Big\|_p + \sum_{\emptyset \ne B \subset A} \|g_B - P_{\underline{\mu},A}^{(m)} g_B\|_p + \Big\|P_{\underline{\mu},A}^{(m)}\Big(\sum_{\emptyset \ne B \subset A} g_B - f\Big)\Big\|_p$$
$$\le C_2\Big(\Big\|f - \sum_{\emptyset \ne B \subset A} g_B\Big\|_p + \sum_{\emptyset \ne B \subset A} \underline{t_\mu}^{\underline{n}(B)} \omega_{\underline{n},p,A\setminus B}(D^{\underline{n}(B)} g, \underline{t_\mu})\Big).$$

As this inequality holds for every choice of $g_B \in C^\infty(I^d)$, we obtain

$$\|f - P_{\underline{\mu},A}^{(m)} f\|_p \le C_2 K_{\underline{n},A,p}(f, \underline{t_\mu}).$$

This, together with Lemma 5.4, completes the proof. ∎

Now we are ready to prove one of the inequalities needed in Theorem A.1. For $f \in W_p^0(I^d)$ and $\underline{j} \in N_m^d$ we introduce the notation

$$F_{\underline{j},p}^{(m)}(f) = \Big\|\sum_{\underline{k} \in \widetilde{N}_{\underline{j}}} (f, f_{\underline{k}}^{(m)}) f_{\underline{k}}^{(m)}\Big\|_p, \quad \tau_{\underline{j},p}^{(m)}(f) = \Big(\sum_{\underline{k} \in \widetilde{N}_{\underline{j}}} |(f, f_{\underline{k}}^{(m)})|^p\Big)^{1/p}.$$

It follows from the properties of one-dimensional Franklin functions (cf. [2]) that

$$F_{\underline{j},p}^{(m)}(f) \sim \prod_{i=1}^{d} 2^{j_i(1/2-1/p)} \tau_{\underline{j},p}^{(m)}(f). \tag{4}$$

For $\underline{\mu} = (\mu_1, \ldots, \mu_d)$ define $\mathcal{D}_{\underline{\mu}} = \{i \in \mathcal{D} : \mu_i \geq -m-2\}$. Observe that

$$\Big|\sum_{\underline{k}\in\widetilde{N}_{\underline{j}}} (f, f_{\underline{k}}^{(m)}) f_{\underline{k}}^{(m)}\Big| = \Big|\sum_{\mu_1=j_1-1}^{j_1} \cdots \sum_{\mu_d=j_d-1}^{j_d} (-1)^{\Sigma_{i=1}^d (j_i-\mu_i)} P_{\underline{\mu},\mathcal{D}_{\underline{\mu}}}^{(m)}(f)\Big|.$$

Let $\lambda \in \mathbb{N}$ be chosen as in Lemma 6.4 and $A_{\underline{j}} = \{i : j_i \geq 2^\lambda\} = \{i_1, \ldots, i_k\}$, $\# A_{\underline{j}} = k$. Then from the definition of $P_{\underline{\mu},\mathcal{D}_{\underline{\mu}}}^{(m)}$ Lemmas 6.3, 6.4 and (5.2) we obtain, for $\underline{n} = (m+1, \ldots, m+1) \in \mathbb{N}^d$,

$$\begin{aligned} F_{\underline{j},p}^{(m)}(f) &= \Big\|\sum_{\mu_1=j_1-1}^{j_1} \cdots \sum_{\mu_d=j_d-1}^{j_d} (-1)^{\Sigma_{i=1}^d (j_i-\mu_i)} P_{\underline{\mu},\mathcal{D}_{\underline{\mu}}}^{(m)}(f)\Big\|_p \\ &\leq \sum_{\mu_1=j_1-1}^{j_1} \cdots \sum_{\mu_d=j_d-1}^{j_d} \|f - P_{\underline{\mu},\mathcal{D}_{\underline{\mu}}}^{(m)}(f)\|_p \\ &\leq C_1 \sum_{\mu_{i_1}=j_{i_1}-1}^{j_{i_1}} \cdots \sum_{\mu_{i_k}=j_{i_k}-1}^{j_{i_k}} \|f - P_{\underline{\mu},A_{\underline{j}}}^{(m)}(f)\|_p \leq C_2 \omega_{\underline{n},p,A_{\underline{j}}}(f, \underline{t}_{\underline{j}}). \end{aligned}$$

If $A_{\underline{j}} = \emptyset$ then $F_{\underline{j},p}^{(m)}(f) \leq C_3\|f\|_p$. Note that for a given $A \subset \mathcal{D}$ and $\underline{t} = (1/2^{a_1}, \ldots, 1/2^{a_d})$ there is only a finite (independent of $\underline{t}$) number of $\underline{j}$'s such that $A_{\underline{j}} = A$ and $\underline{t}_{\underline{j}}(A_{\underline{j}}) = \underline{t}(A)$, so it follows that for $\underline{\alpha} = (\alpha_1, \ldots, \alpha_d)$ with $0 < \alpha_i < m+1$ there exists a constant $C > 0$ such that for all $f \in B_{p,q}^{\underline{\alpha}}(I^d)$,

$$\Big(\sum_{\underline{j}\in N_m^d} \Big(c(\underline{j}, \underline{\alpha}, p)\Big(\sum_{\underline{k}\in\widetilde{N}_{\underline{j}}} |(f, f_{\underline{k}}^{(m)})|^p\Big)^{1/p}\Big)^q\Big)^{1/q} \leq C\|f\|_{p,q}^{(\underline{\alpha})}.$$

Now the reverse inequality will be proved. Let $\underline{\mu} = (\mu_1, \ldots, \mu_d)$, $\mu_i \geq \lambda$, $A \subset \mathcal{D}$ and $\underline{h} = (h_1, \ldots, h_d) \in \mathbb{R}^d$ with $|h_i| \leq 1/2^{\mu_i}$. It follows from the properties of spline functions of one variable (cf. [2], Lemma 9.2) and Fubini's theorem that there exists $C > 0$ such that for all $\underline{j} \in N_m^d$ and $(a_{\underline{k}})_{\underline{k}\in\widetilde{N}_{\underline{j}}}$,

$$\Big\|\Delta_{\underline{h},A}^{\underline{n}}\Big(\sum_{\underline{k}\in\widetilde{N}_{\underline{j}}} a_{\underline{k}} f_{\underline{k}}^{(m)}\Big)\Big\|_p \leq C|\underline{h}^{\underline{n}(A)}|\underline{t}_{\underline{j}}^{-\underline{n}(A)}\Big\|\sum_{\underline{k}\in\widetilde{N}_{\underline{j}}} a_{\underline{k}} f_{\underline{k}}^{(m)}\Big\|_p.$$

Defining $W(\underline{\mu}, B) = \{\underline{j} \in N_m^d : j_i \leq \mu_i \text{ for } i \in B,\ j_i > \mu_i \text{ for } i \notin B\}$ for $B \subset A$, we obtain, for $f \in W_p^0(I^d)$,

$$\begin{aligned} \|\Delta_{\underline{h},A}^{\underline{n}} f\|_p &\leq \sum_{\underline{j}\in N_m^d} \Big\|\Delta_{\underline{h},A}^{\underline{n}}\Big(\sum_{\underline{k}\in\widetilde{N}_{\underline{j}}} (f, f_{\underline{k}}^{(m)}) f_{\underline{k}}^{(m)}\Big)\Big\|_p \\ &\leq C \sum_{B\subset A} \sum_{\underline{j}\in W(\underline{\mu},B)} |\underline{h}^{\underline{n}(B)}|\underline{t}_{\underline{j}}^{-\underline{n}(B)} \Big\|\Delta_{\underline{h},A\setminus B}^{\underline{n}}\Big(\sum_{\underline{k}\in\widetilde{N}_{\underline{j}}} (f, f_{\underline{k}}^{(m)}) f_{\underline{k}}^{(m)}\Big)\Big\|_p \\ &\leq C_1 \underline{t}_{\underline{\mu}}^{\underline{n}(A)} \sum_{\underline{j}\in N_m^d} \prod_{i\in A} 2^{(m+1)\min(\mu_i,j_i)} F_{\underline{j},p}^{(m)}(f), \end{aligned}$$

which gives

$$\omega_{\underline{n},p,A}(f, \underline{t}_{\underline{\mu}}) \leq C_1 \underline{t}_{\underline{\mu}}^{\underline{n}(A)} \sum_{\underline{j}\in N_m^d} \prod_{i\in A} 2^{(m+1)\min(\mu_i,j_i)} F_{\underline{j},p}^{(m)}(f).$$

Using this inequality and (4) we obtain

$$\omega_{\underline{n},p,A}(f, \underline{t}_{\underline{\mu}}) \leq C_2 \underline{t}_{\underline{\mu}}^{\underline{n}(A)} \sum_{\underline{j}\in N_m^d} c(\underline{j}, \underline{\alpha}, p)\tau_{\underline{j},p}^{(m)}(f) \prod_{i=1}^{d} 2^{-j_i\alpha_i} \prod_{i\in A} 2^{(m+1)\min(\mu_i,j_i)}.$$

Observe that

$$r_{\underline{\mu},A} = \sum_{\underline{j}\in N_m^d} \prod_{i=1}^{d} 2^{-\alpha_i j_i} \prod_{i\in A} 2^{(m+1)\min(\mu_i,j_i)} \sim \prod_{i\in A} 2^{\mu_i(m+1-\alpha_i)}.$$

Using Jensen's inequality we get

$$\begin{aligned} &\Big(\prod_{i\in A} 2^{\alpha_i\mu_i}\omega_{\underline{n},p,A}(f, \underline{t}_{\underline{\mu}})\Big)^q \\ &\quad \leq C_3 \prod_{i\in A} 2^{q\mu_i(\alpha_i-m-1)} r_{\underline{\mu},A}^q \\ &\qquad \times \Big(\sum_{\underline{j}\in N_m^d} c(\underline{j}, \underline{\alpha}, p)\tau_{\underline{j},p}^{(m)}(f) \frac{\prod_{i=1}^d 2^{-\alpha_i j_i} \prod_{i\in A} 2^{(m+1)\min(\mu_i,j_i)}}{r_{\underline{j}}}\Big)^q \\ &\quad \leq C_4 \prod_{i\in A} 2^{\mu_i(\alpha_i-m-1)} \\ &\qquad \times \sum_{\underline{j}\in N_m^d} (c(\underline{j}, \underline{\alpha}, p)\tau_{\underline{j},p}^{(m)}(f))^q \prod_{i\in A} 2^{(m+1)\min(\mu_i,j_i)-\alpha_i j_i} \prod_{i\notin A} 2^{-\alpha_i j_i}. \end{aligned}$$

Then

$$\sum_{\mu_{i_1}=\lambda}^{\infty}\dots\sum_{\mu_{i_k}=\lambda}^{\infty}\Big(\prod_{i\in A}2^{\alpha_i\mu_i}\omega_{\underline{n},p,A}(f,\underline{t}_{\underline{\mu}})\Big)^q$$
$$\le C_4\sum_{\mu_{i_1}=\lambda}^{\infty}\dots\sum_{\mu_{i_k}=\lambda}^{\infty}\sum_{\underline{j}\in N_m^d}(c(\underline{j},\underline{\alpha},p)\tau_{\underline{j},p}^{(m)}(f))^q$$
$$\times\prod_{i\in A}2^{(m+1)\min(\mu_i,j_i)-\alpha_i j_i-\mu_i(m+1-\alpha_i)}\prod_{i\notin A}2^{-\alpha_i j_i}$$
$$\le C_5\sum_{\underline{j}\in N_m^d}(c(\underline{j},\underline{\alpha},p)\tau_{\underline{j},p}^{(m)}(f))^q$$
$$\times\sum_{\mu_{i_1}=\lambda}^{\infty}\dots\sum_{\mu_{i_k}=\lambda}^{\infty}\prod_{i\in A}2^{(m+1)\min(\mu_i,j_i)-\alpha_i j_i-\mu_i(m+1-\alpha_i)}.$$

There exists a constant $C_6>0$ such that for all $\underline{j}\in N_m^d$,

$$\sum_{\mu_{i_1}=\lambda}^{\infty}\dots\sum_{\mu_{i_k}=\lambda}^{\infty}\prod_{i\in A}2^{(m+1)\min(\mu_i,j_i)-\alpha_i j_i-\mu_i(m+1-\alpha_i)}\le C_6,$$

so

$$\sum_{\mu_{i_1}=\lambda}^{\infty}\dots\sum_{\mu_{i_k}=\lambda}^{\infty}\Big(\prod_{i\in A}2^{\alpha_i\mu_i}\omega_{\underline{n},p,A}(f,\underline{t}_{\underline{\mu}})\Big)^q\le C_7\sum_{\underline{j}\in N_m^d}(c(\underline{j},\underline{\alpha},p)\tau_{\underline{j},p}^{(m)}(f))^q,$$

which (together with 5.2) implies

$$\Big(\int_0^{1/(m+1)}\dots\int_0^{1/(m+1)}\Big(\frac{\omega_{\underline{n},p,A}(f,\underline{t})}{\underline{t}^{\underline{\alpha}(A)}}\Big)^q\underline{t}^{-\underline{\gamma}(A)}d\underline{t}\Big)^{1/q}$$
$$\le C_8\Big(\sum_{\underline{j}\in N_m^d}(c(\underline{j},\underline{\alpha},p)\tau_{\underline{j},p}^{(m)}(f))^q\Big)^{1/q},$$

and the proof of Theorem A.1 is complete.

Now Theorem A.2 will be proved. Its proof is based on the main idea of the proof of Theorem III.6 of [6]. For convenience we set $f_{\underline{k}}=f_{\underline{k}}^{(0)}$. Observe first that if $1/p<\alpha_i<1$ for all $i=1,\dots,d$, $f\in B_{p,q}^{\underline{\alpha}}(I^d)$ and $f=\sum_{\underline{j}\in N_0^d}\sum_{\underline{k}\in\widetilde{N}_{\underline{j}}}a_{\underline{k}}f_{\underline{k}}$, then

$$\Big\|\sum_{\underline{k}\in\widetilde{N}_{\underline{j}}}a_{\underline{k}}f_{\underline{k}}\Big\|_\infty\le C\prod_{i=1}^d 2^{j_i/2}\sup_{\underline{k}\in\widetilde{N}_{\underline{j}}}|a_{\underline{k}}|$$

$$\le C\prod_{i=1}^d 2^{j_i(1/p-\alpha_i)}c(\underline{j},\underline{\alpha},p)\Big(\sum_{\underline{k}\in\widetilde{N}_{\underline{j}}}|a_{\underline{k}}|^p\Big)^{1/p}$$
$$\le C\prod_{i=1}^d 2^{j_i(1/p-\alpha_i)}\sup_{\underline{j}\in N_0^d}c(\underline{j},\underline{\alpha},p)\Big(\sum_{\underline{k}\in\widetilde{N}_{\underline{j}}}|a_{\underline{k}}|^p\Big)^{1/p},$$

and this implies $f\in C(I^d)$.

Now let $f\in B_{p,q}^{\underline{\alpha}}(I^d)$,

$$f=\sum_{\underline{j}\in N_0^d}\sum_{\underline{k}\in\widetilde{N}_{\underline{j}}}a_{\underline{k}}f_{\underline{k}}=\sum_{\underline{j}\in N_0^d}\sum_{\underline{k}\in\widetilde{N}_{\underline{j}}}b_{\underline{k}}\phi_{\underline{k}}^*$$

(recall that $\phi_{\underline{k}}^*$ denotes a tensor product Schauder function, normalized in $L^2(I^d)$). To prove the second part of Theorem A it is enough to show the existence of constants $M_1,M_2>0$ such that for each $f\in B_{p,q}^{\underline{\alpha}}(I^d)$,

$$M_1\|\underline{a}\|_{p,q}^{(\underline{\alpha})}\le\|\underline{b}\|_{p,q}^{(\underline{\alpha})}\le M_2\|\underline{a}\|_{p,q}^{(\underline{\alpha})},\tag{5}$$

where $\underline{a}=(a_{\underline{k}})_{\underline{k}\in N_{-2}^d}$, $\underline{b}=(b_{\underline{k}})_{\underline{k}\in N_{-2}^d}$. Set

$$\tau_{\underline{j},p}^{(m)}(\underline{a})=\Big(\sum_{\underline{k}\in\widetilde{N}_{\underline{j}}}|a_{\underline{k}}|^p\Big)^{1/p},\quad \tau_{\underline{j},p}^{(m)}(\underline{b})=\Big(\sum_{\underline{k}\in\widetilde{N}_{\underline{j}}}|b_{\underline{k}}|^p\Big)^{1/p}.$$

First we will show the existence of M_1.

Observe that (cf. (4))

$$\tau_{\underline{j},p}^{(m)}(\underline{a})\sim\prod_{i=1}^d 2^{-j_i(1/2-1/p)}\Big\|\sum_{\underline{k}\in\widetilde{N}_{\underline{j}}}a_{\underline{k}}f_{\underline{k}}\Big\|_p.$$

It follows from Lemma 6.3 that $\|f-P_{\underline{\mu},D}^{(0)}f\|_p\sim E_{\underline{\mu},D,p}^{(0)}(f)$, so we have

$$\Big\|\sum_{\underline{k}\in\widetilde{N}_{\underline{j}}}a_{\underline{k}}f_{\underline{k}}\Big\|_p=\Big\|\sum_{\mu_1=j_1}^{j_1+1}\dots\sum_{\mu_d=j_d}^{j_d+1}(-1)^{\Sigma_{i=1}^d(j_i+1-\mu_i)}P_{\underline{\mu},D}^{(0)}(f)\Big\|_p$$
$$\le C_1\sum_{\mu_1=j_1}^{j_1+1}\dots\sum_{\mu_d=j_d}^{j_d+1}E_{\underline{\mu},D,p}^{(0)}(f)\le C_2E_{\underline{j},D,p}^{(0)}(f)$$
$$\le C_2\Big\|f-\sum_{\underline{k}\in N_0^*(\underline{j},A)}b_{\underline{k}}\phi_{\underline{k}}^*\Big\|_p$$

$$\le C_2 \sum_{\xi_1=j_1}^{\infty} \dots \sum_{\xi_d=j_d}^{\infty} \Big\| \sum_{\underline{\eta}\in \widetilde{N}_{\underline{\xi}}} b_{\underline{\eta}} \phi^*_{\underline{\eta}} \Big\|_p$$

$$\le C_3 \sum_{\xi_1=j_1}^{\infty} \dots \sum_{\xi_d=j_d}^{\infty} \prod_{i=1}^{d} 2^{\xi_i(1/2-1/p)} \tau^{(m)}_{\underline{\xi},p}(\underline{b})$$

(because for each pair $\underline{\eta}_1 \ne \underline{\eta}_2$, $\underline{\eta}_1, \underline{\eta}_2 \in \widetilde{N}_{\xi}$ the supports of the functions $\phi^*_{\underline{\eta}_1}$ and $\phi^*_{\underline{\eta}_2}$ are disjoint), so

$$\prod_{i=1}^{d} 2^{j_i(1/2-1/p)} \tau^{(m)}_{\underline{j},p}(\underline{a}) \le C_4 \sum_{\xi_1=j_1}^{\infty} \dots \sum_{\xi_d=j_d}^{\infty} \prod_{i=1}^{d} 2^{\xi_i(1/2-1/p)} \tau^{(m)}_{\underline{\xi},p}(\underline{b}).$$

As $\sum_{\xi_1=j_1}^{\infty} \dots \sum_{\xi_d=j_d}^{\infty} \prod_{i=1}^{d} 2^{-\alpha_i \xi_i} \sim \prod_{i=1}^{d} 2^{-\alpha_i j_i}$, from Jensen's inequality we get

$$\begin{aligned}(c(\underline{j},\underline{\alpha},p)&\tau^{(m)}_{\underline{j},p}(\underline{a}))^q \\ &\le C_5 \Big(\prod_{i=1}^{d} 2^{\alpha_i j_i}\Big)^q \Big(\sum_{\xi_1=j_1}^{\infty} \dots \sum_{\xi_d=j_d}^{\infty} c(\underline{\xi},\underline{\alpha},p)\tau^{(m)}_{\underline{\xi},p}(\underline{b}) \prod_{i=1}^{d} 2^{-\alpha_i \xi_i}\Big)^q \\ &\le C_6 \prod_{i=1}^{d} 2^{\alpha_i j_i} \sum_{\xi_1=j_1}^{\infty} \dots \sum_{\xi_d=j_d}^{\infty} \prod_{i=1}^{d} 2^{-\alpha_i \xi_i} (c(\underline{\xi},\underline{\alpha},p)\tau^{(m)}_{\underline{\xi},p}(\underline{b}))^q.\end{aligned}$$

This implies

$$\begin{aligned}\sum_{\underline{j}\in N_0^d} &(c(\underline{j},\underline{\alpha},p)\tau^{(m)}_{\underline{j},p}(\underline{a}))^q \\ &\le C_6 \sum_{\underline{\xi}\in N_0^d} (c(\underline{\xi},\underline{\alpha},p)\tau^{(m)}_{\underline{\xi},p}(\underline{b}))^q \prod_{i=1}^{d} 2^{-\alpha_i \xi_i} \sum_{j_1\le \xi_1} \dots \sum_{j_d \le \xi_d} \prod_{i=1}^{d} 2^{\alpha_i j_i} \\ &\le C_7 \sum_{\underline{\xi}\in N_0^d} (c(\underline{\xi},\underline{\alpha},p)\tau^{(m)}_{\underline{\xi},p}(\underline{b}))^q,\end{aligned}$$

which gives the left inequality in (5).

To prove the right inequality in (5) we will need the formula for the coefficients of a function $f \in C(I^d)$ in the basis $\{\phi^*_{\underline{k}}\}$:

$$b_{\underline{k}}(f) = \widetilde{\Delta}_{k_1,1} \circ \dots \circ \widetilde{\Delta}_{k_d,d} f, \tag{6}$$

where

$$\widetilde{\Delta}_{k,i} f(x_1,\dots,x_d) = \begin{cases} f(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_d) & \text{for } k=0, \\ \frac{1}{\sqrt{3}}(f(x_1,\dots,1,\dots,x_d) - f(x_1,\dots,0,\dots,x_d)) & \text{for } k=1, \\ \frac{1}{\sqrt{3\cdot 2^j}}\big(f\big(x_1,\dots,\frac{2\nu-1}{2^{j+1}},\dots,x_d\big) - \frac{1}{2}\big(f\big(x_1,\dots,\frac{\nu}{2^j},\dots,x_d\big) & \\ \quad + f\big(x_1,\dots,\frac{\nu-1}{2^j},\dots,x_d\big)\big)\big) & \text{for } 2^j < k \le 2^{j+1},\ k = 2^j+\nu. \end{cases}$$

(This formula follows from the formulae for functions of one variable and the fact that linear combinations of tensor products of continuous functions of one variable are dense in $C(I^d)$.)

The exponential estimates for $f_{\underline{k}}$ will also be needed: there exist $C>0$ and $0<\vartheta<1$ such that for all $\underline{j}\in N_0^d$, $\underline{k}\in\widetilde{N}_{\underline{j}}$ and $\underline{t}\in I^d$,

$$|f_{\underline{k}}(\underline{t})| \le C \prod_{i=1}^{d} 2^{j_i/2} \vartheta^{|2^{j_i} t_i - k_i + 2^{j_i}|}. \tag{7}$$

(This is a straightforward consequence of exponential estimates for Franklin functions of one variable, cf. [2], [1].)

It follows from (6), (7) and the definition of $f_{\underline{k}}$'s that for $\underline{k}\in\widetilde{N}_{\underline{j}}$, $\underline{\eta}\in\widetilde{N}_{\underline{\xi}}$,

$$b_{\underline{k}}(f_{\underline{\eta}}) = 0 \quad \text{if } j_i > \xi_i \text{ for some } 1\le i\le d,$$

$$b_{\underline{k}}(f_{\underline{\eta}}) \le C \prod_{i=1}^{d} 2^{(\xi_i-j_i)/2} \vartheta^{|2^{\xi_i-j_i}(k_i-2^{j_i})-\eta_i+2^{\xi_i}|} \quad \text{if } j_i\le \xi_i \text{ for all } 1\le i\le d,$$

so for $f = \sum_{\underline{\xi}\in N_0^d}\sum_{\underline{\eta}\in\widetilde{N}_{\underline{\xi}}} a_{\underline{\eta}} f_{\underline{\eta}}$ and $\underline{k}\in\widetilde{N}_{\underline{j}}$ we get

$$|b_{\underline{k}}(f)| \le C \sum_{\xi_1=j_1}^{\infty} \dots \sum_{\xi_d=j_d}^{\infty} \prod_{i=1}^{d} 2^{(\xi_i-j_i)/2} \sum_{\underline{\eta}\in\widetilde{N}_{\underline{\xi}}} |a_{\underline{\eta}}| \vartheta^{|2^{\xi_i-j_i}(k_i-2^{j_i})-\eta_i+2^{\xi_i}|}.$$

Defining

$$z(\underline{\xi},\underline{k}) = \sum_{\underline{\eta}\in\widetilde{N}_{\underline{\xi}}} |a_{\underline{\eta}}| \vartheta^{|2^{\xi_i-j_i}(k_i-2^{j_i})-\eta_i+2^{\xi_i}|}$$

we have

$$\tau^{(m)}_{\underline{j},p}(\underline{b}) \le C \sum_{\xi_1=j_1}^{\infty} \dots \sum_{\xi_d=j_d}^{\infty} \prod_{i=1}^{d} 2^{(\xi_i-j_i)/2} \Big(\sum_{\underline{k}\in\widetilde{N}_{\underline{j}}} |z(\underline{\xi},\underline{k})|^p\Big)^{1/p}.$$

But for $\Theta(\underline{j}) = \prod_{i=1}^{d} \vartheta^{|j_i|}$ for $\underline{j}\in\mathbb{Z}^d$,

$$A(\underline{\xi},\underline{n}) = \begin{cases} |a_{\underline{n}}| & \text{if } \underline{n}\in\widetilde{N}_{\underline{\xi}}, \\ 0 & \text{if } \underline{n}\notin\widetilde{N}_{\underline{\xi}}, \end{cases}$$

and $\underline{n}(\underline{\xi},\underline{k}) = (2^{\xi_1-j_1}(k_1-2^{j_1})+2^{\xi_1},\ldots,2^{\xi_d-j_d}(k_d-2^{j_d})+2^{\xi_d})$ we have

$$z(\underline{\xi},\underline{k}) = (A(\underline{\xi},\cdot)*\Theta)(\underline{n}(\underline{\xi},\underline{k})),$$

so

$$\Big(\sum_{\underline{k}\in\widetilde{N}_{\underline{j}}} |z(\underline{\xi},\underline{k})|^p\Big)^{1/p} \le \Big(\sum_{\underline{n}\in\mathbb{Z}^d} |(A(\underline{\xi},\cdot)*\Theta)(\underline{n})|^p\Big)^{1/p} \le \|\Theta\|_1\|A(\underline{\xi},\cdot)\|_p = C_1\tau^{(m)}_{\underline{\xi},p}(\underline{a}).$$

This implies

$$\prod_{i=1}^d 2^{j_i/2}\tau^{(m)}_{\underline{j},p}(\underline{b}) \le C_2\sum_{\xi_1=j_1}^\infty\cdots\sum_{\xi_d=j_d}^\infty \prod_{i=1}^d 2^{\xi_i/2}\tau^{(m)}_{\underline{\xi},p}(\underline{a})$$

and

$$c(\underline{j},\underline{\alpha},p)\tau^{(m)}_{\underline{j},p}(\underline{b}) \le C_2\prod_{i=1}^d 2^{j_i(\alpha_i-1/p)}\sum_{\xi_1=j_1}^\infty\cdots\sum_{\xi_d=j_d}^\infty\prod_{i=1}^d 2^{\xi_i(1/p-\alpha_i)}c(\underline{\xi},\underline{\alpha},p)\tau^{(m)}_{\underline{\xi},p}(\underline{a}).$$

As $1/p<\alpha_i$ for all $1\le i\le d$, it follows that

$$\sum_{\xi_1=j_1}^\infty\cdots\sum_{\xi_d=j_d}^\infty\prod_{i=1}^d 2^{\xi_i(1/p-\alpha_i)} \sim \prod_{i=1}^d 2^{j_i(1/p-\alpha_i)},$$

and from Jensen's inequality we obtain

$$(c(\underline{j},\underline{\alpha},p)\tau^{(m)}_{\underline{j},p}(\underline{b}))^q \le C_3\prod_{i=1}^d 2^{j_i(\alpha_i-1/p)}\sum_{\xi_1=j_1}^\infty\cdots\sum_{\xi_d=j_d}^\infty\prod_{i=1}^d 2^{\xi_i(1/p-\alpha_i)}(c(\underline{\xi},\underline{\alpha},p)\tau^{(m)}_{\underline{\xi},p}(\underline{a}))^q$$

and

$$\sum_{\underline{j}\in N_0^d}(c(\underline{j},\underline{\alpha},p)\tau^{(m)}_{\underline{j},p}(\underline{b}))^q \le C_4\sum_{\underline{\xi}\in N_0^d}(c(\underline{\xi},\underline{\alpha},p)\tau^{(m)}_{\underline{\xi},p}(\underline{a}))^q\prod_{i=1}^d 2^{\xi_i(1/p-\alpha_i)}\sum_{j_1\le\xi_1}\cdots\sum_{j_d\le\xi_d}\prod_{i=1}^d 2^{j_i(\alpha_i-1/p)} \le C_5\sum_{\underline{\xi}\in N_0^d}(c(\underline{\xi},\underline{\alpha},p)\tau^{(m)}_{\underline{\xi},p}(\underline{a}))^q,$$

which gives the right inequality in (5) and completes the proof of Theorem A.2. ■

7. Proof of Theorem B. Recall that for $\underline{\alpha}=(\alpha_1,\ldots,\alpha_d)$ with $1/p<\alpha_i<1/p'$, $\beta_i=1-\alpha_i$, $\underline{\beta}=(\beta_1,\ldots,\beta_d)$, $F\in B^{\underline{\beta}}_{p,1}(I^d)$, $G\in B^{\underline{\alpha}}_{\overline{p},\infty}(I^d)$,

$$F=\sum_{\underline{j}\in N_1^d}\sum_{\underline{k}\in\widetilde{N}_{\underline{j}}}F_{\underline{k}}h_{\underline{k}},\qquad G=\sum_{\underline{\xi}\in N_1^d}\sum_{\underline{\eta}\in\widetilde{N}_{\underline{\xi}}}G_{\underline{\eta}}\phi_{\underline{\eta}},$$

and $\underline{s}=(s_1,\ldots,s_d)\in I^d$ we have defined

$$I(F,G)(\underline{s})=\sum_{\underline{j}\in N_1^d}\sum_{\underline{\xi}\in N_1^d}\sum_{\underline{k}\in\widetilde{N}_{\underline{j}}}\sum_{\underline{\eta}\in\widetilde{N}_{\underline{\xi}}}F_{\underline{k}}G_{\underline{\eta}}\prod_{i=1}^d\int_0^{s_i}h_{k_i}(u_i)h_{\eta_i}(u_i)\,du_i.$$

We are to prove that there exists a constant $C=C(\underline{\alpha},p)$ such that for all $F\in B^{\underline{\beta}}_{p,1}(I^d)$ and $G\in B^{\underline{\alpha}}_{\overline{p},\infty}(I^d)$,

$$\|I(F,G)\|^{(\underline{\alpha})}_{\overline{p},\infty}\le C\|F\|^{(\underline{\beta})}_{p,1}\|G\|^{(\underline{\alpha})}_{\overline{p},\infty}.$$

Recall that by s_k, $k\in\mathbb{N}\cup\{0\}$, we denote the dyadic points in I, described by (1).

There are the following formulae for inner products of one-dimensional Haar functions:

- for $k\in\widetilde{N}_j$ and $\eta\in\widetilde{N}_\xi$ with $\xi<j$:

$$\int_0^s h_k(u)h_\eta(u)\,du=h_\eta(s_k)\phi_k(s),$$

- for $k,\eta\in\widetilde{N}_j$, $j>-1$:

$$\int_0^s h_k(u)h_\eta(u)\,du=\delta_{k,\eta}\Big(\sum_{\zeta=-1}^{j-1}\sum_{\varrho\in\widetilde{N}_\zeta}h_\varrho(s_k)\phi_\varrho(s)\Big),$$

- for $k=\eta=1$: $\int_0^s h_k(u)h_\eta(u)\,du=\phi_1(s)$.

Define

$$J(F,G)(\underline{s})=\sum_{\underline{j}\in N_1^d}\sum_{\underline{\xi}\in N_1^d}\sum_{\underline{k}\in\widetilde{N}_{\underline{j}}}\sum_{\underline{\eta}\in\widetilde{N}_{\underline{\xi}}}|F_{\underline{k}}||G_{\underline{\eta}}|\prod_{i:j_i<\xi_i}|h_{k_i}(s_{\eta_i})|\phi_{\eta_i}(s_i)\times\prod_{i:j_i=\xi_i}\delta_{k_i,\eta_i}\Big(\sum_{\zeta_i=-1}^{j_i}\sum_{\varrho_i\in\widetilde{N}_{\zeta_i}}|h_{\varrho_i}(s_{k_i})|\phi_{\varrho_i}(s_i)\Big)\times\prod_{i:j_i>\xi_i}|h_{\eta_i}(s_{k_i})|\phi_{k_i}(s_i).$$

As for each $\underline{s}\in I^d$ this is a series with non-negative components, it can be rearranged as

$$J(F,G)(\underline{s}) = \sum_{\underline{j}\in N_1^d} \sum_{\underline{k}\in \widetilde{N}_{\underline{j}}} J^*_{\underline{k}} \phi^*_{\underline{k}}(\underline{s}).$$

Observe that for $\underline{F} = (F_{\underline{k}})$ we have

$$\|\underline{F}\|^{(\beta)}_{p,1} \le C_1 \|F\|^{(\beta)}_{p,1}, \tag{8}$$

where $C_1 > 0$ does not depend on $F \in B^{\beta}_{p,1}(I^d)$ (the proof of this statement is a simple calculation and is omitted here). Also, set $\underline{G}^* = (G^*_{\underline{k}})$, $G^*_{\underline{k}} = \|\phi_{\underline{k}}\|_2 G_{\underline{k}}$ and $\underline{J}^* = (J^*_{\underline{k}})$. It will be shown that

$$\|\underline{J}^*\|^{(\alpha)}_{p,\infty} \le C_2 \|\underline{F}\|^{(\beta)}_{p,1} \|\underline{G}^*\|^{(\alpha)}_{p,\infty}, \tag{9}$$

where $C_2 > 0$ does not depend on $F \in B^{\beta}_{p,1}(I^d)$ and $G \in B^{\alpha}_{p,\infty}(I^d)$. Then it will follow from Theorem A.2 that $J(F,G) \in B^{\alpha}_{p,\infty}(I^d) \subset C(I^d)$. Moreover, we will show that the series defining $I(F,G)(\underline{s})$ is absolutely convergent for each $\underline{s} \in I^d$, and therefore it can be rearranged as

$$I(F,G)(\underline{s}) = \sum_{\underline{j}\in N_1^d} \sum_{\underline{k}\in \widetilde{N}_{\underline{j}}} I^*_{\underline{k}} \phi^*_{\underline{k}}(\underline{s}),$$

with $|I^*_{\underline{k}}| \le J^*_{\underline{k}}$, so for $\underline{I}^* = (I^*_{\underline{k}})$ we will have $\underline{I}^* \in b^{\alpha}_{p,\infty}$, and

$$\|\underline{I}^*\|^{(\alpha)}_{p,\infty} \le \|\underline{J}^*\|^{(\alpha)}_{p,\infty},$$

and Theorem B will follow from (8) and Theorem A.2.

It remains to prove inequality (9).

For $A, B, C \subset \mathcal{D}$ with $A\cap B = A\cap C = B \cap C = \emptyset$ and $A \cup B \cup C = \mathcal{D}$, let

$$N(A,B,C) = \{(\underline{j},\underline{\xi}) \in N_1^d \times N_1^d : \ j_i < \xi_i \text{ for } i \in A, \ j_i = \xi_i \text{ for } i \in B \text{ and } j_i > \xi_i \text{ for } i \in C\},$$

and

$$J_{A,B,C}(\underline{s}) = \sum_{(\underline{j},\underline{\xi})\in N(A,B,C)} \sum_{\underline{k}\in\widetilde{N}_{\underline{j}}} \sum_{\underline{\eta}\in\widetilde{N}_{\underline{\xi}}} |F_{\underline{k}}||G_{\underline{\eta}}| \prod_{i\in A} |h_{k_i}(s_{\eta_i})||\phi_{\eta_i}(s_i)|$$
$$\times \prod_{i\in B} \delta_{k_i,\eta_i} \Big(\sum_{\zeta_i=-1}^{j_i} \sum_{\varrho_i\in\widetilde{N}_{\zeta_i}} |h_{\varrho_i}(s_{k_i})||\phi_{\varrho_i}(s_i)|\Big) \prod_{i\in C} |h_{\eta_i}(s_{k_i})||\phi_{k_i}(s_i)|.$$

Setting $S(\underline{j}) = \{\underline{\xi} = (\xi_1,\ldots,\xi_d) : \ \xi_i < j_i \text{ for } i \in A\cup C, \ \xi_i \ge j_i \text{ for } i \in B\}$,

and for $\underline{k} \in \widetilde{N}_{\underline{j}}$ and $\underline{\eta} \in \widetilde{N}_{\underline{\xi}}$,

$$\underline{a}(\underline{k},\underline{\eta}) = \begin{cases} \eta_i & \text{for } i \in A\cup B, \\ k_i & \text{for } i \in C, \end{cases} \qquad \underline{b}(\underline{k},\underline{\eta}) = \begin{cases} k_i & \text{for } i \in A, \\ \eta_i & \text{for } i \in B \cup C, \end{cases}$$

and denoting by χ_n the characteristic function of $\{t \in I : h_n(t) \ne 0\}$ we get

$$J_{A,B,C}(\underline{s}) = \sum_{\underline{j}} \sum_{\underline{k}\in\widetilde{N}_{\underline{j}}} W_{\underline{k}} \phi_{\underline{k}}(\underline{s}),$$

where for $\underline{k} \in \widetilde{N}_{\underline{j}}$,

$$W_{\underline{k}} = \sum_{\underline{\xi}\in S(\underline{j})} \prod_{i\in A\cup C} 2^{\xi_i/2} \prod_{i\in B} 2^{j_i/2}$$
$$\times \sum_{\underline{\eta}\in\widetilde{N}_{\underline{\xi}}} |F_{\underline{a}(\underline{k},\underline{\eta})}||G_{\underline{b}(\underline{k},\underline{\eta})}| \prod_{i\in A\cup C} \chi_{\eta_i}(s_{k_i}) \prod_{i\in B} \chi_{k_i}(s_{\eta_i}).$$

As for given $\underline{k} \in \widetilde{N}_{\underline{j}}$ and $\underline{\xi} \in S(\underline{j})$ we have

$$\#\Big\{\underline{\eta} \in \widetilde{N}_{\underline{\xi}} : \ \prod_{i\in A\cup C} \chi_{\eta_i}(s_{k_i}) \prod_{i\in B} \chi_{k_i}(s_{\eta_i}) \ne 0\Big\} = \prod_{i\in B} 2^{\xi_i - j_i},$$

we obtain

$$\sum_{\underline{\eta}\in\widetilde{N}_{\underline{\xi}}} |F_{\underline{a}(\underline{k},\underline{\eta})}||G_{\underline{b}(\underline{k},\underline{\eta})}| \prod_{i\in A\cup C} \chi_{\eta_i}(s_{k_i}) \prod_{i\in B} \chi_{k_i}(s_{\eta_i})$$
$$\le \prod_{i\in B} 2^{(\xi_i-j_i)(1/p'-1/p)} \Big(\sum_{\underline{\eta}\in\widetilde{N}_{\underline{\xi}}} |F_{\underline{a}(\underline{k},\underline{\eta})}|^p \prod_{i\in A\cup C} \chi_{\eta_i}(s_{k_i}) \prod_{i\in B} \chi_{k_i}(s_{\eta_i})\Big)^{1/p}$$
$$\times \Big(\sum_{\underline{\eta}\in\widetilde{N}_{\underline{\xi}}} |G_{\underline{b}(\underline{k},\underline{\eta})}|^p \prod_{i\in A\cup C} \chi_{\eta_i}(s_{k_i}) \prod_{i\in B} \chi_{k_i}(s_{\eta_i})\Big)^{1/p}$$
$$= \prod_{i\in B} 2^{(\xi_i-j_i)(1/p'-1/p)} \Big(\sum_{i\in A\cup B} \sum_{\eta_i\in\widetilde{N}_{\xi_i}} |F_{\underline{a}(\underline{k},\underline{\eta})}|^p \prod_{i\in A} \chi_{\eta_i}(s_{k_i}) \prod_{i\in B} \chi_{k_i}(s_{\eta_i})\Big)^{1/p}$$
$$\times \Big(\sum_{i\in B\cup C} \sum_{\eta_i\in\widetilde{N}_{\xi_i}} |G_{\underline{b}(\underline{k},\underline{\eta})}|^p \prod_{i\in B} \chi_{k_i}(s_{\eta_i}) \prod_{i\in C} \chi_{\eta_i}(s_{k_i})\Big)^{1/p}$$
$$\le \prod_{i\in B} 2^{(\xi_i-j_i)(1/p'-1/p)} \Big(\sum_{i\in A\cup B} \sum_{\eta_i\in\widetilde{N}_{\xi_i}} |F_{\underline{a}(\underline{k},\underline{\eta})}|^p \prod_{i\in B} \chi_{k_i}(s_{\eta_i})\Big)^{1/p}$$
$$\times \Big(\sum_{i\in B\cup C} \sum_{\eta_i\in\widetilde{N}_{\xi_i}} |G_{\underline{b}(\underline{k},\underline{\eta})}|^p \prod_{i\in B} \chi_{k_i}(s_{\eta_i})\Big)^{1/p}.$$

Using this inequality we obtain

$$\Big(\sum_{\underline{k}\in\widetilde{N}_{\underline{j}}}|W_{\underline{k}}|^p\Big)^{1/p}\le\sum_{\underline{\xi}\in S(\underline{j})}\prod_{i\in A\cup C}2^{\xi_i/2}\prod_{i\in B}2^{j_i(1/2+1/p-1/p')+\xi_i(1/p'-1/p)}$$
$$\times\Big(\sum_{\underline{k}\in\widetilde{N}_{\underline{a}(\underline{j},\underline{\xi})}}|F_{\underline{k}}|^p\Big)^{1/p}\Big(\sum_{\underline{\eta}\in\widetilde{N}_{\underline{b}(\underline{j},\underline{\xi})}}|G_{\underline{\eta}}|^p\Big)^{1/p}.$$

Writing

$$\mathcal{F}_{\underline{j}}=c(\underline{j},\underline{\beta},p)\Big(\sum_{\underline{k}\in\widetilde{N}_{\underline{j}}}|F_{\underline{k}}|^p\Big)^{1/p},\quad \mathcal{G}^*_{\underline{j}}=c(\underline{j},\underline{\alpha},p)\Big(\sum_{\underline{k}\in\widetilde{N}_{\underline{j}}}|G^*_{\underline{\eta}}|^p\Big)^{1/p},$$
$$\mathcal{W}^*_{\underline{j}}=c(\underline{j},\underline{\alpha},p)\Big(\sum_{\underline{k}\in\widetilde{N}_{\underline{j}}}(W^*_{\underline{k}})^p\Big)^{1/p},$$

where $W^*_{\underline{k}}=\|\phi_{\underline{k}}\|_2W_{\underline{k}}$ and $\underline{W}^*(A,B,C)=(W^*_{\underline{k}})$, we obtain from the last inequality

$$\mathcal{W}^*_{\underline{j}}\le M\prod_{i\in B}2^{j_i(\alpha_i-1/p')}\prod_{i\in C}2^{2j_i(\alpha_i-1)}$$
$$\times\sum_{\underline{\xi}\in S(\underline{j})}\mathcal{F}_{\underline{a}(\underline{j},\underline{\xi})}\mathcal{G}^*_{\underline{b}(\underline{j},\underline{\xi})}\prod_{i\in A}2^{\xi_i(\alpha_i-1/p')}\prod_{i\in C}2^{\xi_i(1+1/p-\alpha_i)}.$$

As $\alpha_i<1/p'$ for all $i=1,\ldots,d$, this implies

$$\mathcal{W}^*_{\underline{j}}\le M\|\underline{G}^*\|^{(\underline{\alpha})}_{p,\infty}\prod_{i\in C}2^{2j_i(\alpha_i-1)}\sum_{\underline{\xi}\in S(\underline{j})}\mathcal{F}_{\underline{a}(\underline{j},\underline{\xi})}\prod_{i\in C}2^{\xi_i(1+1/p-\alpha_i)}$$

and

$$\|\underline{W}^*(A,B,C)\|^{(\underline{\alpha})}_{p,\infty}\le M_1\|\underline{F}\|^{(\underline{\beta})}_{p,1}\|\underline{G}^*\|^{(\underline{\alpha})}_{p,\infty}.$$

As $J(F,G)=\sum_{A,B,C}J_{A,B,C}$, we get

$$\|\underline{J}^*\|^{(\underline{\alpha})}_{p,\infty}\le M_2\|\underline{F}\|^{(\underline{\beta})}_{p,1}\cdot\|\underline{G}^*\|^{(\underline{\alpha})}_{p,\infty}.$$

This completes the proof of Theorem B. ∎

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MATHEMATICAL INSTITUTE
POLISH ACADEMY OF SCIENCES
GDAŃSK BRANCH
ABRAHAMA 18
81-825 SOPOT, POLAND

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