Note on semigroups generated by positive Rockland operators on graded homogeneous groups

by

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Abstract. Let $L$ be a positive Rockland operator of homogeneous degree $d$ on a graded homogeneous group $G$ and let $p_t$ be the convolution kernels of the semigroup generated by $L$. We prove that if $\tau(x)$ is a Riemannian distance of $x$ from the unit element, then there are constants $c > 0$ and $C$ such that $|p_t(x)| \leq C \exp(-c\tau(x)^{d/(d-1)})$. Moreover, if $G$ is not stratified, more precise estimates of $p_t$ at infinity are given.

1. Introduction. Let $L$ be a positive Rockland operator on a homogeneous group $G$ (cf. [FS]) and let $d$ be the homogeneous degree of $L$ (cf. Section 2).

The operator $L$ satisfies the following subelliptic estimates proved by B. Helffer and J. Nourrigat [HN]: for every multi-index $I$ there are constants $C$ and $k$ such that

$$
\|X^I f\|_{L^2(G)} \leq C(\|L^k f\|_{L^2(G)} + \|f\|_{L^2(G)}), \quad f \in C_c^\infty(G).
$$

Theorem (4.25) of [FS] asserts that the closure $-\bar{L}$ of the essentially selfadjoint operator $-L$ is the infinitesimal generator of a semigroup of linear operators on $L^2(G)$ which has the form

$$
T_t f = f \ast p_t, \quad t > 0,
$$

where the $p_t$ belong to the Schwartz space $S(G)$.

The homogeneity of $L$ implies

$$
p_t(x) = t^{-Q/d}p_t(\delta_{t^{-1/d}}x),
$$

where $Q$ is the homogeneous dimension of $G$ and $\delta_t$ is the family of dilations associated with $G$ (cf. Section 2).

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It has been proved in [D] that the kernels \( p_t \) (and their derivatives) decay exponentially at infinity, that is, for every submultiplicative function \( \omega \) and every left-invariant differential operator \( \partial \) on \( G \),

\[
\int_G |\partial p_t(x)|^2 \omega(x) \, dx \leq C_{t,\omega,\partial} < \infty,
\]

and, consequently,

\[
\sup_{x \in G} |\partial p_t(x)\omega(x)| \leq C'_{t,\omega,\partial} < \infty.
\]

On the other hand, the result in [He] shows that if a positive Rockland operator \( L \) is a sum of even powers of left-invariant vector fields, that is,

\[
L = \sum (-1)^{m_j} X_{2m_j}^j,
\]

then there are constants \( C_0 \) and \( c > 0 \) such that

\[
|p_t(x)| \leq C_0 t^{-d/4} \exp(-c |x|^{d/(d-1)} t^{1/(d-1)}),
\]

where \( |\cdot| \) is a homogeneous norm on \( G \).

Semigroups of linear operators generated by differential operators of the form (1.6) in the setting of arbitrary Lie groups have been investigated in [Hei].

The purpose of the present paper is to study the behaviour at infinity of the kernels \( p_t \) of the semigroup generated by an arbitrary positive Rockland operator. We prove that the estimate (1.7) also holds in the general case. Moreover, we obtain estimates of \( p_t \) in various directions as \( x \to \infty \). These seem to be optimal.

2. Preliminaries. Let \( G \) be a graded nilpotent Lie algebra of step \( q \), that is,

\[
G = \bigoplus_{j=1}^s V_j,
\]

and \( [V_j, V_i] \subset V_{i+j} \) for every \( 1 \leq j, i \leq s \). We assume that \( \dim V_i \geq 1 \).

A dilation structure on a graded Lie algebra \( G \) is a one-parameter group \( \{ \delta_t \}_{t>0} \) of automorphisms of \( G \) determined by

\[
\delta_t X = t^{a_i} X \quad \text{for} \quad X \in V_j,
\]

where \( \alpha_1 < \alpha_2 < \cdots < \alpha_s \) are rational numbers called the exponents of homogeneity.

If we consider \( G \) as a nilpotent Lie group with multiplication given by the Campbell–Hausdorff formula

\[
xy = x + y + \frac{1}{2}[x, y] + \ldots,
\]

then \( \{ \delta_t \} \) forms a group of automorphisms on the group \( G \), and the nilpotent Lie group \( G \) equipped with the dilations \( \{ \delta_t \} \) is said to be a graded homogeneous group.

A Lie algebra \( G \) is called stratified if \( G \) is graded and \( V_1 \) generates \( G \) as an algebra.

The homogeneous dimension of \( G \) is the number \( Q \) determined by

\[
\int_G f(\delta_t x) \, dx = t^{-Q} \int_G f(x) \, dx,
\]

where \( dx \) is a right-invariant Haar measure on \( G \). It is evident that

\[
Q = \sum_{j=1}^s \alpha_j \cdot \dim V_j.
\]

A left-invariant differential operator \( L \) on \( G \) is called a Rockland operator if \( L \) is homogeneous of some degree \( d > 0 \), that is,

\[
L(f \circ \delta_t) = t^d L(f) \circ \delta_t \quad \text{for} \quad f \in C_0^\infty(G),
\]

and for every non-trivial irreducible unitary representation \( \pi \) of \( G \) the operator \( \pi(L) \) is injective on \( C_0^\infty \) vectors.

We choose and fix a homogeneous norm on \( G \), that is, a continuous, positive, symmetric function \( x \to |x| \) which is smooth away from 0, vanishes only for \( x = 0 \), and satisfies \( |x| = |y| \). Henceforth we will assume that our homogeneous norm is subadditive, that is, \( |x| + |y| \leq |x + y| \) (cf. e.g. [HS]).

Note that if \( \{ e_{k,(j)} \}_{j=1, \ldots, s; k=1, \ldots, \dim V_j} \) is a homogeneous basis of \( G \) (i.e. \( e_{k,(j)} \in V_j \)), then there is a constant \( C > 0 \) such that

\[
C^{-1} |x| \leq \sum_{j=1}^s \sum_{k=1}^{\dim V_j} |a_{k,(j)}|^{1/a_j} \leq C |x|, \quad \text{where} \quad x = \sum_{j=1}^s \sum_{k=1}^{\dim V_j} a_{k,(j)} e_{k,(j)}.
\]

If \( e_{k,(j)} \) is a fixed homogeneous basis of \( G \), then define left-invariant vector fields \( X_{k,(j)} \) by

\[
X_{k,(j)} f(x) = \frac{d}{dt} |_{t=0} f(x e_{k,(j)}) \cdot e_{k,(j)}.
\]

If \( I = (i_1, i_2, \ldots, i_s) \) is a multi-index, then put

\[
X^I = X_{1,(i_1)} X_{2,(i_2)} \cdots X_{s,(i_s)}.
\]

The number \( |I| = \sum_{j=1}^s \sum_{k=1}^{\dim V_j} i_j \cdot a_j \cdot \dim V_j \) is called the homogeneous length of \( I \) and determines the homogeneous degree of the operator \( X^I \).

3. Riemannian distance on graded homogeneous groups. This section is devoted to describing the behaviour at infinity of a right-invariant Riemannian distance on a graded homogeneous group \( G \) in some special
coordinates. The facts presented here are known. We shall use the approach of A. Hulanicki and J. W. Jenkins [HJ] which was developed by many authors afterwards (cf. [NRS]).

Let \( Y_1, \ldots, Y_n \) be a homogeneous basis of \( G \) and \( d_1, \ldots, d_n \) be their exponents of homogeneity. We assume that \( 1 = d_1 \leq \ldots \leq d_n \). Let \( \tilde{G} \) be the free nilpotent Lie algebra of step \( q \) with free generators \( \tilde{Y}_1, \ldots, \tilde{Y}_n \). We can consider \( \tilde{G} \) as a graded Lie algebra:

\[
\tilde{G} = \bigoplus_{j=1}^t \tilde{V}_j,
\]

where \( \tilde{V}_1 = \text{lin}\{\tilde{Y}_1, \ldots, \tilde{Y}_n\} \), \( \tilde{V}_2 = \text{lin}\{\tilde{Y}_i, \tilde{Y}_j\} : 1 \leq j \leq n, 1 \leq i \leq n, \ldots \).

Obviously, \( \tilde{G} \) is stratified.

Let \( \delta_k \) denote the natural dilations on \( \tilde{G} \) defined by

\[
\delta_k \tilde{x} = t^k \tilde{x} \quad \text{for} \quad \tilde{x} \in \tilde{V}_j.
\]

There is the canonical homomorphism \( \kappa : \tilde{G} \to G \) such that \( \kappa(\tilde{Y}_j) = Y_j \).

Let \( \tilde{H} \) be the kernel of \( \kappa \). Following Hulanicki and Jenkins [HJ] define

\[
\tilde{V}_j^0 = \left\{ \tilde{x} \in \tilde{V}_j : \tilde{x} + \tilde{y} \in \tilde{H} \text{ for some } \tilde{y} \in \bigoplus_{i \neq j} \tilde{V}_i \right\}, \quad j = 1, \ldots, t.
\]

Let \( \tilde{W}_j^0 \) be linear complements of \( \tilde{V}_j^0 \) in \( \tilde{V}_j \) such that each \( \tilde{W}_j^0 \) has a basis consisting of vectors \( \tilde{e}_k^{(j)} \) of the form

\[
\tilde{e}_k^{(j)} = [\bar{Y}_{i_1}, [\bar{Y}_{i_2}, \ldots, [\bar{Y}_{i_{j-1}}, \bar{Y}_{i_j}] \ldots]].
\]

Let \( \tilde{k} = \bigoplus_{j=1}^t \tilde{W}_j^0 \). Then

\[
\tilde{G} = \tilde{k} \oplus \tilde{H},
\]

and the algebra \( G \) can be identified with \( \tilde{H} \cong \tilde{k} \).

In the remaining part of the paper we fix the basis \( \{\tilde{e}_k^{(j)}\} \) of \( \tilde{k} \) of the form (3.3). This basis will be treated as a basis of \( G \). Notice that the homogeneity of the vector \( \tilde{e}_k^{(j)} \) as an element of \( G \) is \( d_k^{(j)} = d_1 + d_2 + \ldots + d_j \) while its homogeneity as a vector of \( \tilde{G} \) is \( j \). Clearly, \( d_k^{(j)} \geq j \). One should not confuse the basis \( \tilde{e}_k^{(j)} \) from Section 2 with the basis \( e_k^{(j)} \).

Note that there exists a constant \( C \) such that

\[
C^{-1} |x| \leq \sum_{j=1}^t \sum_{k=1}^{\dim \tilde{W}_j^0} |a_k^{(j)}|^{1/\alpha} \leq C |x|
\]

and

\[
C^{-1} |x| \leq \sum_{j=1}^t \sum_{k=1}^{\dim \tilde{W}_j^0} |a_k^{(j)}|^{1/\alpha} \leq C |x|.
\]

where \( | \cdot |^{-\alpha} \) is a fixed homogeneous norm on \( \tilde{G} \).

Consider \( \tilde{G} \) as a graded homogeneous group and \( \tilde{H} \) as its normal subgroup. Note that \( \tilde{H} \) need not be preserved under the dilations \( \delta_k \) of \( \tilde{G} \). Obviously \( G \) is isomorphic to \( \tilde{H} \backslash \tilde{G} \).

Let \( r \) be a right-invariant Riemannian distance on \( \tilde{G} \). Since \( \tilde{G} \) is stratified, it follows e.g. from [J, Theorem 4] and [H, Section 1] that for every compact neighbourhood \( \tilde{K} \) of 0 there is a constant \( C \) such that

\[
C^{-1} \tilde{r}(x, y) \leq |xy|^{-\alpha} \leq C \tilde{r}(x, y) \quad \text{for } xy \notin \tilde{K}.
\]

Now, define a right-invariant distance function \( \rho \) on \( G = \tilde{H} \backslash \tilde{G} \) by

\[
\rho(\tilde{h}, \tilde{h}') = \inf \{ \tilde{r}(\tilde{h}z, \tilde{h}'z) : \tilde{h}, \tilde{h}' \in \tilde{H} \}.
\]

Let \( r \) denote a fixed right-invariant Riemannian distance on \( G \). Obviously, for every compact neighbourhood \( K \) of 0 in \( G \) there is a constant \( C_0 > 0 \) such that

\[
C^{-1} r(x, y) \leq \rho(x, y) \leq C r(x, y) \quad \text{for } xy \notin K.
\]

For \( r > 0 \) let \( \tilde{B}(r) = \{ x \in \tilde{G} : |x|^{-\alpha} < r \} \) be the homogeneous ball of radius \( r \). Proposition 2.1 of [HJ] asserts that for every \( \alpha > 0 \) there are constants \( a_1 \) and \( b_2 \) such that

\[
\tilde{B}(r) \subset \tilde{B}(ar) \cap \tilde{k} \oplus (\tilde{B}(ar) \cap \tilde{H}) \subset \tilde{B}(br) \quad \text{for } r \geq \alpha.
\]

This combined with (3.7), (3.6), and (3.8) implies that for any compact neighbourhood \( K \) of 0 in \( G \) there is a constant \( C_0 > 0 \) such that if \( G = \tilde{k} \cup \bigcup_{j=1}^t \sum_{k=1}^{\dim \tilde{W}_j^0} e_k^{(j)} \), then

\[
C^{-1} \rho(x, 0) \leq \sum_{j=1}^t \sum_{k=1}^{\dim \tilde{W}_j^0} |a_k^{(j)}|^{1/\alpha} \leq C \rho(x, 0) \quad \text{whenever } x \notin K.
\]

Finally, from (3.11) and (3.9), we see that for any compact neighbourhood \( K \).
of 0 in \(G\) there is a constant \(C_1\) such that for \(x = \sum_{j=1}^{l} \sum_{k=1}^{\dim \tilde{W}_j^0} a_k^{(j)} e_k^{(j)} \notin K\),

\[
\tag{3.12} G_1^{-1} \tau(x, 0) \leq \sum_{j=1}^{l} \sum_{k=1}^{\dim \tilde{W}_j^0} |a_k^{(j)}|^{1/j} \leq C \tau(x, 0).
\]

We say that a Borel symmetric function \(\gamma\) on \(G\) is subadditive if

(a) \(\gamma \geq 0\) and \(\gamma\) is bounded on compact subsets of \(G\),

(b) \(\gamma(xy) \leq \gamma(x) + \gamma(y)\).

A Borel symmetric function \(w\) on \(G\) is said to be submultiplicative if

(a') \(w \geq 1\) and \(w\) is bounded on compact subsets of \(G\),

(b') \(w(xy) \leq w(x)w(y)\) for \(x, y \in G\).

Clearly, if \(w'\) satisfies (a') and (b') with a constant \(C \geq 1\), that is, \(w'(xy) \leq Cw'(x)w'(y)\), then the function \(w = Cw'\) is submultiplicative.

Moreover, if \(w\) is submultiplicative then so is \(w(t)(x) = w(\delta(x) t)\) for \(t > 0\).

If \(|\cdot|\) is a subadditive homogeneous norm on \(G\), then the function \(x \mapsto \exp(C|\tau(x)|)\) is submultiplicative for any positive constant \(C\).

The second example of a submultiplicative function is \(x \mapsto \exp(C\tau(x))\), where \(\tau(x) = \tau(x, 0), C > 0\).

The following proposition has been proved in [H].

PROPOSITION 1. For any subadditive function \(\gamma\) on \(G\) there is a constant \(C > 0\) such that \(\gamma(x) \leq C \tau(x) + C\).

4. Semigroups on weighted spaces. For real \(m > 0\) and a function \(f\) on \(G\) define

\[
(D_m f)(x) = m^{-Q} f(\delta_{m^{-1}} x).
\]

Obviously, \(D_m(f \ast g) = (D_m f) \ast (D_m g)\).

For a submultiplicative function \(w\) we shall denote by \(L^2(w)\) the Hilbert space of functions on \(G\) with the norm

\[
\|f\|^2_w = \int_G |f(x)|^2 w(x) \, dx.
\]

Clearly,

\[
D_m : L^2(w)[m] \rightarrow L^2(w) \quad \text{and} \quad \|D_m f\|_{L^2(w)} = m^{-Q/2} \|f\|_{L^2(w)[m]}.
\]

The following theorem has been actually proved in [D] (Proposition (5.11), Theorem (1.3)):

THEOREM 2. For every submultiplicative \(w\) the closure of \(-L\) considered on \(C_c^\infty(G)\) in the norm \(\|\cdot\|_w\) is the infinitesimal generator of a holomorphic semigroup \(\{T_z\}_{\Re z > 0}\) of operators on \(L^2(w)\).

As a consequence of Theorem 2, we get

LEMMA 3. For every submultiplicative \(w\) there are \(\lambda > 0, \theta > 0\) and \(C_2 > 0\) such that the closure of \(-L + \lambda I\) considered on \(C_c^\infty(G)\) in the norm \(\|\cdot\|_w\) is the infinitesimal generator of a uniformly bounded holomorphic semigroup \(\{S_z\}\) of operators on \(L^2(w)\) in the sector \(\Delta_{\theta} = \{z : \Arg z < \theta\}\) and

\[
\|S_z f\|_w \leq C_2 \|f\|_w \quad \text{for} \quad z \in \Delta_{\theta}.
\]

From Lemma 3 one can deduce

LEMMA 4. For every \(m > 0\) the operator \(-L + \lambda m^d I\) is the infinitesimal generator of a uniformly bounded holomorphic semigroup \(\{S_z[m]\}\) of operators on \(L^2(w)[m]\) in the sector \(\Delta_{\theta}\) and

\[
\|S_z[m] f\|_{w[m]} \leq C_2 \|f\|_{w[m]} \quad \text{for} \quad z \in \Delta_{\theta},
\]

with the same \(C_2\) and \(\theta\) as in Lemma 3.

Proof. Set \(S_z[m] = D_m^{-1} S_{m^{-1}} D_m\) and note that the \(S_z[m]\) form a uniformly bounded holomorphic semigroup whose infinitesimal generator is \(-L + \lambda m^d I\).

Obviously,

\[
(4.1) \quad S_z[m] = e^{-\lambda m^d T_z}.
\]

The Cauchy integral formula, Lemma 4, and (4.1) imply that there is a constant \(C_3\) such that for every natural number \(k \geq 0\) and every real \(m > 0\)

\[
\|L^k T_{\frac{1}{m}}\|_{L^2(w[m]) \rightarrow L^2(w[m])} \leq C_3 k! e^{C_3 m^d} \quad \text{for} \quad 1/8 < t < 2.
\]

The next lemma is a weighted version of subelliptic estimates for \(L\) (cf. [D, Section 4]).

LEMMA 5. For every multi-index \(I\) and every submultiplicative \(w\) there exist a constant \(C_4\) and a natural number \(k\) such that

\[
\|X^I f\|^2_{L^2(w)} \leq C_4 (\|L^k f\|^2_{L^2(w)} + \|f\|^2_{L^2(w)}) \quad \text{for} \quad f \in C_c^\infty(G).
\]

Since \(L^k\) is also a Rockland operator, Theorem 2 implies

COROLLARY 6. The estimates (4.3) hold for \(f\) in the domain \(D(L^k)\) of \(L^k\), where \(L^k\) is the infinitesimal generator of the semigroup (1.2) considered on \(L^2(w)\).

Using the operators \(D_m\) and Corollary 6, we obtain
Lemma 7. For every multi-index $I$ and for every submultiplicative $w$ there exist a positive integer $k$ and a constant $C_5$ such that for every $m > 0$,
\[
\| X^I f \|_{L^2_w(w^{-m})}^2 \\
\leq C_5 m^{2|I|} (m^{-2d} \| L^k f \|_{L^2_w(w^{-m})}^2 + \| f \|_{L^2_w(w^{-m})}^2) \quad \text{for } f \in D(L^k_{w^{-m}}). \]
We are now in a position to prove the following

Proposition 8. For every submultiplicative function $\eta$ there exists a constant $C_6$ such that for every multi-index $I$ there is a constant $C_7$ such that
\[
\sup_{x \in G} \{ |X^I p_1(x) | \eta^{|m|}(x) \} \leq C_7 \exp \{ C_6 m^d \}.
\]

Proof. Let $w$ be the submultiplicative function defined by $w(x) = |\eta(x) \big|^2$. Since $p_1 \in \bigcap_{k=1}^{\infty} D(L^k_{w^{-m}})$ (cf. (1.4) and Theorem 2), by Lemma 7 and (4.2), we get
\[
\| X^I p_1/2 \|_{L^2_w(w^{-m})}^2 = \| X^I T_{I/2 - \varepsilon} p_{1/2} \|_{L^2_w(w^{-m})}^2 \leq C_6 m^{2|I|} (m^{-2d} \| L^k T_{I/2 - \varepsilon} p_{1/2} \|_{L^2_w(w^{-m})}^2 + \| T_{I/2 - \varepsilon} p_{1/2} \|_{L^2_w(w^{-m})}^2) \leq C \exp \{ 3C_6 m^d \} \| p_1 \|_{L^2_w(w^{-m})}^2 \quad \text{for } \varepsilon < 1/4, \ m \geq 1.
\]

In virtue of (1.3), we obtain
\[
\| p_1 \|_{L^2_w(w^{-m})}^2 = \varepsilon^{-Q/d} \int_G | p_1(x) |^2 w(\delta_{mc^1/d} x) \ dx.
\]

Putting $\varepsilon = m^{-d}$ ($m > 2$), we have
\[
\| X^I p_1/2 \|_{L^2_w(w^{-m})}^2 \leq C m^{Q/d} \exp \{ 2C_6 m^d \} \| p_1 \|_{L^2_w(w^{-m})} \leq C' \exp \{ 3C_6 m^d \}.
\]

Since
\[
| X^I p_1(x) | \eta^{|m|}(x) \leq \int_G | p_1(xw^{-1}) X^I p_1/2(y) | \eta^{|m|}(xw^{-1}) \eta^{|m|}(y) \ dy,
\]

the Schwarz inequality and (4.5) imply (4.4) with $m > 2$.

Since the function $z \mapsto \sup_{m \in G} \{ \eta^{|m|}(z) \}$ is submultiplicative, the estimate (1.5) ends our proof.

Corollary 9. For any subadditive $\gamma$ there is a constant $C_6$ such that for every multi-index $J$ there is a constant $C_7$ such that
\[
| X^J p_1(z) | \leq C_7 e^{-\phi(z)} \quad \text{where } \phi(z) = \sup_{m \geq 1} \{ \gamma(\delta_{m} z) - C_6 m^d \}.
\]

Proof. Setting $\eta(x) = e^{\phi(x)}$ and applying Proposition 8, we get (4.6).

5. Pointwise estimates. Now we can make several choices of $\gamma$ in Corollary 9 and obtain estimates on the corresponding functions $\psi$, and, consequently, on the kernels $X^J p_{1}$. However, because $\gamma(x) \leq C(\tau(x) + 1)$ (cf. Proposition 1), the function
\[
\psi_p(x) = \sup_{m \geq 0} \{ r(\delta_{m} x) - C_6 m^d \}
\]
is essentially the largest (that is, $\psi_p(x) \leq C \psi_{\psi_p} - C_6 m^d$ for some constant $C > 0$).

Obtaining precise estimates on $\psi_p$ requires some effort, so we begin with simpler estimates which involve the homogeneous norm.

Let $\tau_{\psi_p}(x) = |x|$. Then
\[
\psi_{\psi_p}(x) = \sup_{m \geq 0} \{ m |x| - C_6 m^d \} = c |x|^{d/(d-1)},
\]
with $c = \frac{d-1}{d} (C_6 d)^{1/(1-d)} > 0$.

Corollary 9 combined with (5.2) leads to

Theorem 10. Let $p_{1}$ be the kernels associated with the semigroup generated by a positive Rockland operator $L$. Let $d$ be the homogeneous degree of $L$. Then there is a constant $c > 0$ such that for every multi-index $J$ there exists a constant $C_J$ such that
\[
| X^J p_{1}(x) | \leq C_J \exp \{ -c |x|^{d/(d-1)} \}.
\]

Remark 11. Using dilations (cf. (1.3)) and the fact that $\partial p_{1} = -L p_{1}$, we get the following estimates which generalize (1.7) to semigroups associated with arbitrary Rockland operators:
\[
| \partial_{x} X^J p_{1} | \leq C_{J} e^{-Q/d |x|^{d/(d-1)}} \exp \{ -c |x|^{d/(d-1)} \}.
\]

The same argument can be used to extend the estimates for $X_J p_{1}$ in Theorems 10 and 12 below to $\partial_{x} X_J p_{1}$.

Now we turn to studying the function $\psi_p$. First we give an easy estimate from below, then a more complex one which is essentially optimal.

Since the $\delta_{m}$ are linear automorphisms of $G$ (cf. (2.2)), there exists a constant $c > 0$ such that $\tau(\delta_{m} z) \geq cm^d(z)$ for $m \geq 1$, which implies
\[
\psi_p(x) \geq \sup_{m \geq 2} \{ cm^d(x) - C_6 m^d \}.
\]

By (5.4), there is a constant $c_1 > 0$ such that for $\tau(x)$ sufficiently large
\[
\psi_p(x) \geq c_1 \tau(x)^{d/(d-1)}.
\]

The estimate (5.5) and Corollary 9 give

Theorem 12. There exists a constant $c_1 > 0$ such that for every multi-index $J$ there exists a constant $C_J$ such that
\[
| X^J p_{1}(x) | \leq C_J e^{-c_1 \tau(x)^{d/(d-1)}}.
\]
In order to show precise estimates on $\psi_+$ we will proceed more carefully. Fix for a moment a coordinate $e_{k}^{(i)}$. By the definition of $\psi_+$ and (3.12), for a compact neighbourhood $K$ of $0$ there is a constant $c > 0$ such that

$$\psi_+(x) \geq \sup_{m \geq 1} \left\{ \text{cm} \frac{d^{(i)}}{m \gamma} \left| a_{k}^{(i)} \right|^{1/4} - C_0 m^d \right\} \text{ for } x = \sum_{j=1}^{l} \sum_{k=1}^{m} a_{k}^{(j)} e_{k}^{(j)} \notin K.$$ 

A simple calculation shows that there exists a constant $c > 0$ such that

$$1 + \psi_+(x) \geq c \sum_{j=1}^{l} \sum_{k=1}^{m} |a_{k}^{(j)}|^{d/(d-j)} e_{k}^{(j)}, \quad x = \sum_{j=1}^{l} \sum_{k=1}^{m} a_{k}^{(j)} e_{k}^{(j)} \in G.$$ 

Taking the arithmetical mean we can assert that

$$1 + \psi_+(x) \geq c \sum_{j=1}^{l} \sum_{k=1}^{m} |a_{k}^{(j)}|^{d/(d-j)} e_{k}^{(j)},$$

with a constant $c > 0$.

Now we show estimates from above for $\psi_+$ which are of the same type as (5.8). By (3.12), we have

$$\psi_+(x) = \sup_{m > 0} \left\{ \tau(\delta_{m} x) - C_0 m^d \right\}$$

$$\leq C_8 + \sup_{m > 0} \left\{ \sum_{j=1}^{l} \sum_{k=1}^{m} C_9 m^{d^{(j)}} |a_{k}^{(j)}|^{1/4} - C_0 m^d \right\}$$

$$\leq C_8 + \sum_{j=1}^{l} \sum_{k=1}^{m} \sup_{m > 0} \left\{ C_9 m^{d^{(j)}} |a_{k}^{(j)}|^{1/4} - C_0 m^d \right\}$$

$$\leq C_8 + C_{10} \sum_{j=1}^{l} \sum_{k=1}^{m} |a_{k}^{(j)}|^{d/(d-j)}.$$ 

As an immediate consequence of (5.8) and Corollary 9 we get

**Theorem 13.** There is a constant $c > 0$ such that for every multi-index $J$ there exists a constant $C_J$ such that

$$|X^{j} p_1(x)| \leq C_J \exp \left( - c \sum_{j=1}^{l} \sum_{k=1}^{m} |a_{k}^{(j)}|^{d/(d-j)} \right),$$

where $x = \sum_{j=1}^{l} \sum_{k=1}^{m} a_{k}^{(j)} e_{k}^{(j)} \in G.$

We already know that $\psi_+(x) \leq C \psi_+(x) + C$ (cf. (5.1)). Hence Theorem 13 is at least as strong as Theorem 10. Moreover, by Proposition 1, the estimates in Theorem 12 are at least as strong as the estimates in Theorem 10. The question is: When do we get an improvement? To this end we write the estimates (5.3) and (5.6) in the coordinates $e_{k}^{(j)}$. Let us state them in the following two remarks.

**Remark 14.** Applying (3.5) we can rewrite the estimate (5.3) as

$$|X^{j} p_1(x)| \leq C_J \exp \left( - c \sum_{j=1}^{l} \sum_{k=1}^{m} |a_{k}^{(j)}|^{d/(d-j)} \right),$$

for $x$ as before, with a constant $c > 0$ independent of $J$.

**Remark 15.** By (3.12), in our coordinates the estimate (5.6) can be expressed as

$$|X^{j} p_1(x)| \leq C_J \exp \left( - c \sum_{j=1}^{l} \sum_{k=1}^{m} |a_{k}^{(j)}|^{d/(d-j)} \right),$$

where $c$ is a strictly positive constant. 

Note that

$$d/(d^{(j)} - d_{k}^{(j)}) \leq d/(j d - j),$$

and

$$d/(j d - j) \leq d/(d^{(j)} - d_{k}^{(j)}).$$

One can prove that $G$ is not stratified if and only if there exists a coordinate $e_{k}^{(j)}$ such that $d_{k}^{(j)} > j$, and, consequently, the inequalities (5.12) and (5.13) are strict. So, we conclude that if $G$ is not stratified, then the estimate (5.11) strictly improves (5.10). Analogously, in this case, the estimate (5.9) strictly improves (5.10) and (5.11).

We expect that the estimates (5.9) are in some sense optimal. Our expectations are based on the fact that if $e_{k}^{(j)} \notin [G, G]$, then for $M$ large enough

$$\int_{G} |p_1(x)| \exp(M |a_{k}^{(j)}|^{d/(d-j)}) \, dx = \infty,$$

where $x = \sum_{j=1}^{l} \sum_{k=1}^{m} a_{k}^{(j)} e_{k}^{(j)}$, whereas for small $M > 0$ the integral (5.14) is obviously convergent (cf. (5.9)). This follows by a Fourier transform computation.

It seems likely that (5.14) holds for each direction $e_{k}^{(j)}$. 


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References


On the characterization of Hardy–Besov spaces on the dyadic group and its applications

by

JUN TATEOKA (Akita)

Dedicated to Professor C. Watari on the occasion of his sixtieth birthday

Abstract. C. Watari [12] obtained a simple characterization of Lipschitz classes $\text{Lip}^{\alpha}(W)$ ($1\leq p\leq \infty$, $\alpha > 0$) on the dyadic group using the $L^p$-modulus of continuity and the best approximation by Walsh polynomials. One sees and Weyl [4] characterized homogeneous Besov spaces $B^\alpha_{p,q}$ on locally compact Vilenkin groups, but there are still some gaps to be filled up. Our purpose is to give the characterization of Besov spaces $B^\alpha_{p,q}$ by oscillations, atoms and others on the dyadic groups. As applications, we show a strong capacity inequality of the type of the Hausdorff inequality, a weak type estimate for maximal Cesàro means and a sufficient condition of absolute convergence of Walsh–Fourier series.

0. Introduction and notation. The dyadic group, $2^\mathbb{N}$, is viewed classically as the set of all sequences of 0's and 1's with addition (mod 2) defined pointwise, and is supplied with the usual product topology. Our results are stated in the situation that $2^\mathbb{N}$ is the additive subgroup of the ring of integers in the 2-series field $K$ of formal laurent series in one variable over $GF(2)$ (see [9]). Such a field $K$ is a particular instance of a local field; that is, a locally compact, totally disconnected, non-discrete, complete field. The results of this paper have extensions to any local field.

We need to set some basic notation. It is taken from [9] where the fundamentals are detailed. For the additive subgroup $K^+$ of the 2-series field $K$, we may choose a Haar measure $dx$. Let $d(\alpha x) = |\alpha| dx$ and call $|\alpha|$ the valuation of $\alpha$. Let $|0| = 0$. The mapping $x \mapsto |x|$ has the following properties: $|x| = 0 \Leftrightarrow x = 0$, $|xy| = |x| \cdot |y|$, $|x + y| \leq \max(|x|, |y|)$.

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