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Note on semigroups generated by positive Rockland operators on graded homogeneous groups

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Abstract. Let L be a positive Rockland operator of homogeneous degree d on a graded homogeneous group G and let p_t be the convolution kernels of the semigroup generated by L. We prove that if $\tau(x)$ is a Riemannian distance of x from the unit element, then there are constants $c>0$ and C such that $|p_1(x)|\le C\exp(-c\tau(x)^{d/(d-1)})$. Moreover, if G is not stratified, more precise estimates of p_1 at infinity are given.

1. Introduction. Let L be a positive Rockland operator on a homogeneous group G (cf. [FS]) and let d be the homogeneous degree of L (cf. Section 2).

The operator L satisfies the following subelliptic estimates proved by B. Helffer and J. Nourrigat [HN]: for every multi-index I there are constants C and k such that

$$\|X^I f\|_{L^2(G)}\le C(\|L^k f\|_{L^2(G)}+\|f\|_{L^2(G)}),\quad f\in C_c^\infty(G). \tag{1.1}$$

Theorem (4.25) of [FS] asserts that the closure $-\bar{L}$ of the essentially selfadjoint operator $-L$ is the infinitesimal generator of a semigroup of linear operators on $L^2(G)$ which has the form

$$T_t f = f * p_t,\quad t>0, \tag{1.2}$$

where the p_t belong to the Schwartz space $\mathcal{S}(G)$.

The homogeneity of L implies

$$p_t(x)=t^{-Q/d}p_1(\delta_{t^{-1/d}}x), \tag{1.3}$$

where Q is the homogeneous dimension of G and δ_t is the family of dilations associated with G (cf. Section 2).

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It has been proved in [D] that the kernels p_t (and their derivatives) decay exponentially at infinity, that is, for every submultiplicative function w and every left-invariant differential operator ∂ on G,

$$\int_G |\partial p_t(x)|^2 w(x)\,dx \le C_{t,w,\partial} < \infty, \tag{1.4}$$

and, consequently,

$$\sup_{x\in G} |\partial p_t(x) w(x)| \le C'_{t,w,\partial} < \infty. \tag{1.5}$$

On the other hand, the result in [He] shows that if a positive Rockland operator L is a sum of even powers of left-invariant vector fields, that is,

$$L = \sum (-1)^{n_j} X_j^{2n_j}, \tag{1.6}$$

then there are constants C_0 and $c > 0$ such that

$$|p_t(x)| \le C_0 t^{-Q/d} \exp(-c|x|^{d/(d-1)}/t^{1/(d-1)}), \tag{1.7}$$

where $|\cdot|$ is a homogeneous norm on G.

Semigroups of linear operators generated by differential operators of the form (1.6) in the setting of arbitrary Lie groups have been investigated in [He1].

The purpose of the present paper is to study the behaviour at infinity of the kernels p_t of the semigroup generated by an arbitrary positive Rockland operator. We prove that the estimate (1.7) also holds in the general case. Moreover, we obtain estimates of p_1 in various directions as $x \to \infty$. These seem to be optimal.

2. Preliminaries. Let G be a graded nilpotent Lie algebra of step q, that is,

$$G = \bigoplus_{j=1}^{s} V_j\,, \tag{2.1}$$

and $[V_j, V_i] \subset V_{i+j}$ for every $1 \le j, i \le s$. We assume that $\dim V_1 \ge 1$.

A *dilation structure* on a graded Lie algebra G is a one-parameter group $\{\delta_t\}_{t>0}$ of automorphisms of G determined by

$$\delta_t X = t^{\alpha_j} X \quad \text{for } X \in V_j, \tag{2.2}$$

where $1 = \alpha_1 < \alpha_2 < \ldots < \alpha_s$ are rational numbers called the *exponents of homogeneity*.

If we consider G as a nilpotent Lie group with multiplication given by the Campbell–Hausdorff formula

$$xy = x + y + \frac{1}{2}[x, y] + \ldots,$$

then $\{\delta_t\}$ forms a group of automorphisms on the group G, and the nilpotent Lie group G equipped with the dilations $\{\delta_t\}$ is said to be a *graded homogeneous group*.

A Lie algebra G is called *stratified* if G is graded and V_1 generates G as an algebra.

The *homogeneous dimension* of G is the number Q determined by

$$\int_G f(\delta_t x)\,dx = t^{-Q} \int_G f(x)\,dx,$$

where dx is a right-invariant Haar measure on G. It is evident that

$$Q = \sum_{j=1}^{s} \alpha_j \cdot \dim V_j.$$

A left-invariant differential operator L on G is called a *Rockland operator* if L is homogeneous of some degree $d > 0$, that is,

$$L(f\circ\delta_t) = t^d (Lf)\circ\delta_t \quad \text{for } f \in C^\infty(G), \tag{2.3}$$

and for every non-trivial irreducible unitary representation π of G the operator $\pi(L)$ is injective on C^∞ vectors.

We choose and fix a *homogeneous norm* on G, that is, a continuous, positive, symmetric function $x \mapsto |x|$ which is smooth away from 0, vanishes only for $x = 0$, and satisfies $|\delta_t x| = t|x|$. Henceforth we will assume that our homogeneous norm is subadditive, that is, $|xy| \le |x| + |y|$ (cf. e.g. [HS]).

Note that if $\{\mathbf{e}_{k,(j)}\}_{j=1,\ldots,s;\ k=1,\ldots,\dim V_j}$ is a homogeneous basis of G (i.e. $\mathbf{e}_{k,(j)} \in V_j$), then there is a constant $C > 0$ such that

$$C^{-1}|x| \le \sum_{j=1}^{s} \sum_{k=1}^{\dim V_j} |a_{k,(j)}|^{1/\alpha_j} \le C|x|, \quad \text{where} \quad x = \sum_{j=1}^{s}\sum_{k=1}^{\dim V_j} a_{k,(j)} \mathbf{e}_{k,(j)}.$$

If $\mathbf{e}_{k,(j)}$ is a fixed homogeneous basis of G, then define left-invariant vector fields $X_{k,(j)}$ by

$$X_{k,(j)} f(x) = \frac{d}{dt}\Big|_{t=0} f(xt\mathbf{e}_{k,(j)}).$$

If $I = (i_{1,(1)}, \ldots, i_{1,(2)}, \ldots, i_{1,(s)}, \ldots)$ is a multi-index, then put

$$X^I = X_{1,(1)}^{i_{1,(1)}} \ldots X_{1,(2)}^{i_{1,(2)}} \ldots X_{1,(s)}^{i_{1,(s)}} \ldots$$

The number $|I| = \sum_{j=1}^{s} \sum_{k=1}^{\dim V_j} \alpha_j \cdot i_{k,(j)}$ is called the *homogeneous length* of I and determines the homogeneous degree of the operator X^I.

3. Riemannian distance on graded homogeneous groups. This section is devoted to describing the behaviour at infinity of a right-invariant Riemannian distance on a graded homogeneous group G in some special coordinates. The facts presented here are known. We shall use the approach of A. Hulanicki and J. W. Jenkins [HJ] which was developed by many authors afterwards (cf. [NRS]).

Let $Y_1, \ldots, Y_n$ be a homogeneous basis of G and $d_1, \ldots, d_n$ be their exponents of homogeneity. We assume that $1 = d_1 \leq \ldots \leq d_n$. Let $\widetilde{G}$ be the free nilpotent Lie algebra of step q with free generators $\widetilde{Y}_1, \ldots, \widetilde{Y}_n$. We can consider $\widetilde{G}$ as a graded Lie algebra:

$$\widetilde{G} = \bigoplus_{j=1}^{l} \widetilde{V}_j, \tag{3.1}$$

where $\widetilde{V}_1 = \mathrm{lin}\{\widetilde{Y}_1, \ldots, \widetilde{Y}_n\}$, $\widetilde{V}_2 = \mathrm{lin}\{[\widetilde{Y}_i, \widetilde{Y}_j] : 1 \leq j \leq n,\ 1 \leq i \leq n\}, \ldots$ Obviously, $\widetilde{G}$ is stratified.

Let $\widetilde{\delta}_t$ denote the natural dilations on $\widetilde{G}$ defined by

$$\widetilde{\delta}_t \widetilde{X} = t^j \widetilde{X} \quad \text{for } \widetilde{X} \in \widetilde{V}_j.$$

There is the canonical homomorphism $\kappa : \widetilde{G} \to G$ such that $\kappa(\widetilde{Y}_j) = Y_j$. Let $\widetilde{H}$ be the kernel of κ. Following Hulanicki and Jenkins [HJ] define

$$\widetilde{V}_j{}^0 = \Big\{\widetilde{X} \in \widetilde{V}_j : \widetilde{X} + \widetilde{Y} \in \widetilde{H} \text{ for some } \widetilde{Y} \in \bigoplus_{i>j} \widetilde{V}_i\Big\}, \quad j = 1, \ldots, l. \tag{3.2}$$

Let $\widetilde{W}_j^0$ be linear complements of $\widetilde{V}_j{}^0$ in $\widetilde{V}_j$ such that each $\widetilde{W}_j^0$ has a basis consisting of vectors $\mathbf{e}_k^{(j)}$ of the form

$$\mathbf{e}_k^{(j)} = [\widetilde{Y}_{l_1}, [\widetilde{Y}_{l_2} \ldots [\widetilde{Y}_{l_{j-1}}, \widetilde{Y}_{l_j}] \ldots]]. \tag{3.3}$$

Let $\widetilde{\mathbf{k}} = \bigoplus_{j=1}^{l} \widetilde{W}_j^0$. Then

$$\widetilde{G} = \widetilde{\mathbf{k}} \oplus \widetilde{H}, \tag{3.4}$$

and the algebra G can be identified with $\widetilde{H} \backslash \widetilde{G} \simeq \widetilde{\mathbf{k}}$.

In the remaining part of the paper we fix the basis $\{\mathbf{e}_k^{(j)}\}$ of $\widetilde{\mathbf{k}}$ of the form (3.3). This basis will be treated as a basis of G. Note that the homogeneity of the vector $\mathbf{e}_k^{(j)}$ as an element of G is $d_k^{(j)} = d_{l_1} + d_{l_2} + \ldots + d_{l_j}$ while its homogeneity as a vector of $\widetilde{G}$ is j. Clearly, $d_k^{(j)} \geq j$. One should not confuse the basis $\mathbf{e}_{k,(j)}$ from Section 2 with the basis $\mathbf{e}_k^{(j)}$.

Note that there exists a constant C such that if

$$\widetilde{\mathbf{k}} \ni x = \sum_{j=1}^{l} \sum_{k=1}^{\dim \widetilde{W}_j^0} a_k^{(j)} \mathbf{e}_k^{(j)},$$

then

$$C^{-1}|x| \leq \sum_{j=1}^{l} \sum_{k=1}^{\dim \widetilde{W}_j^0} |a_k^{(j)}|^{1/d_k^{(j)}} \leq C|x| \tag{3.5}$$

and

$$C^{-1}|x|^{\sim} \leq \sum_{j=1}^{l} \sum_{k=1}^{\dim \widetilde{W}_j^0} |a_k^{(j)}|^{1/j} \leq C|x|^{\sim}, \tag{3.6}$$

where $|\cdot|^{\sim}$ is a fixed homogeneous norm on $\widetilde{G}$.

Consider $\widetilde{G}$ as a graded homogeneous group and $\widetilde{H}$ as its normal subgroup. Note that $\widetilde{H}$ need not be preserved under the dilations $\widetilde{\delta}_t$ of $\widetilde{G}$. Obviously G is isomorphic to $\widetilde{H} \backslash \widetilde{G}$.

Let $\widetilde{\tau}$ be a right-invariant Riemannian distance on $\widetilde{G}$. Since $\widetilde{G}$ is stratified, it follows e.g. from [J, Theorem 4] and [H, Section 1] that for every compact neighbourhood $\widetilde{K}$ of 0 there is a constant C such that

$$C^{-1}\widetilde{\tau}(x, y) \leq |xy^{-1}|^{\sim} \leq C\widetilde{\tau}(x, y) \quad \text{for } xy^{-1} \notin \widetilde{K}. \tag{3.7}$$

Now, define a right-invariant distance function ϱ on $G = \widetilde{H} \backslash \widetilde{G}$ by

$$\varrho(\dot{x}, \dot{y}) = \inf\{\widetilde{\tau}(hx, y) : h \in \widetilde{H}\}. \tag{3.8}$$

Let τ denote a fixed right-invariant Riemannian distance on G. Obviously, for every compact neighbourhood K of 0 in G there is a constant $C > 0$ such that

$$C^{-1}\tau(x, y) \leq \varrho(x, y) \leq C\tau(x, y) \quad \text{for } xy^{-1} \notin K. \tag{3.9}$$

For $r > 0$ let $\widetilde{B}(r) = \{x \in \widetilde{G} : |x|^{\sim} < r\}$ be the homogeneous ball of radius r. Proposition 2.1 of [HJ] asserts that for every $\alpha > 0$ there are constants a and b such that

$$\widetilde{B}(r) \subset (\widetilde{B}(ar) \cap \widetilde{\mathbf{k}}) \oplus (\widetilde{B}(ar) \cap \widetilde{H}) \subset \widetilde{B}(br) \quad \text{for } r \geq \alpha. \tag{3.10}$$

This combined with (3.7), (3.6), and (3.8) implies that for any compact neighbourhood K of 0 in G there is a constant $C > 0$ such that if $G = \widetilde{\mathbf{k}} \ni x = \sum_{j=1}^{l} \sum_{k=1}^{\dim \widetilde{W}_j^0} a_k^{(j)} \mathbf{e}_k^{(j)}$, then

$$C^{-1}\varrho(x, 0) \leq \sum_{j=1}^{l} \sum_{k=1}^{\dim \widetilde{W}_j^0} |a_k^{(j)}|^{1/j} \leq C\varrho(x, 0) \quad \text{whenever } x \notin K. \tag{3.11}$$

Finally, from (3.11) and (3.9), we see that for any compact neighbourhood K

of 0 in G there is a constant C_1 such that for $x = \sum_{j=1}^{l}\sum_{k=1}^{\dim \widetilde{W}_j^0} a_k^{(j)}\mathbf{e}_k^{(j)} \notin K$,

$$C_1^{-1}\tau(x,0) \le \sum_{j=1}^{l}\sum_{k=1}^{\dim \widetilde{W}_j^0} |a_k^{(j)}|^{1/j} \le C\tau(x,0). \tag{3.12}$$

We say that a Borel symmetric function γ on G is *subadditive* if

(a) $\gamma \ge 0$ and γ is bounded on compact subsets of G,
(b) $\gamma(xy) \le \gamma(x) + \gamma(y)$.

A Borel symmetric function w on G is said to be *submultiplicative* if

(a′) $w \ge 1$ and w is bounded on compact subsets of G,
(b′) $w(xy) \le w(x)w(y)$ for $x, y \in G$.

Clearly, if w' satisfies (a′) and (b′) with a constant $C \ge 1$, that is, $w'(xy) \le Cw'(x)w'(y)$, then the function $w = Cw'$ is submultiplicative.

Moreover, if w is submultiplicative then so is $w^{[t]}(x) = w(\delta_t x)$ for $t > 0$.

If $|\cdot|$ is a subadditive homogeneous norm on G, then the function $x \mapsto \exp(C|x|)$ is submultiplicative for any positive constant C.

The second example of a submultiplicative function is $x \mapsto \exp(C\tau(x))$, where $\tau(x) = \tau(x,0)$, $C > 0$.

The following proposition has been proved in [H].

PROPOSITION 1. *For any subadditive function γ on G there is a constant $C > 0$ such that $\gamma(x) \le C\tau(x) + C$.* ■

4. Semigroups on weighted spaces. For real $m > 0$ and a function f on G define

$$(D_m f)(x) = m^{-Q} f(\delta_{m^{-1}}x).$$

Obviously, $D_m(f * g) = (D_m f) * (D_m g)$.

For a submultiplicative function w we shall denote by $L^2(w)$ the Hilbert space of functions on G with the norm

$$\|f\|_w^2 = \|f\|_{L^2(w)}^2 = \int_G |f(x)|^2 w(x)\,dx.$$

Clearly,

$$D_m : L^2(w^{[m]}) \to L^2(w) \quad \text{and} \quad \|D_m f\|_{L^2(w)} = m^{-Q/2}\|f\|_{L^2(w^{[m]})}.$$

The following theorem has been actually proved in [D] (Proposition (5.11), Theorem (1.3)):

THEOREM 2. *For every submultiplicative w the closure of $-L$ considered on $C_c^\infty(G)$ in the norm $\|\cdot\|_w$ is the infinitesimal generator of a holomorphic semigroup $\{T_z\}_{\mathrm{Re}\, z>0}$ of operators on $L^2(w)$.* ■

As a consequence of Theorem 2, we get

LEMMA 3. *For every submultiplicative w there are $\lambda > 0$, $\theta > 0$ and $C_2 > 0$ such that the closure of $-(L + \lambda I)$ considered on $C_c^\infty(G)$ in the norm $\|\cdot\|_w$ is the infinitesimal generator of a uniformly bounded holomorphic semigroup $\{S_z\}$ of operators on $L^2(w)$ in the sector $\Delta_\theta = \{z : |\mathrm{Arg}\, z| < \theta\}$ and*

$$\|S_z f\|_w \le C_2\|f\|_w \quad \textit{for } z \in \Delta_\theta. \ ■$$

From Lemma 3 one can deduce

LEMMA 4. *For every $m > 0$ the operator $-(L + \lambda m^d I)$ is the infinitesimal generator of a uniformly bounded holomorphic semigroup $\{S_z^{(m)}\}$ of operators on $L^2(w^{[m]})$ in the sector Δ_θ and*

$$\|S_z^{(m)} f\|_{w^{[m]}} \le C_2\|f\|_{w^{[m]}} \quad \textit{for } z \in \Delta_\theta,$$

with the same C_2 and θ as in Lemma 3.

Proof. Set $S_z^{(m)} = D_{m^{-1}} S_{m^d z} D_m$ and note that the $S_z^{(m)}$ form a uniformly bounded holomorphic semigroup whose infinitesimal generator is $-(L + \lambda m^d I)$. ■

Obviously,

$$S_z^{(m)} = e^{-z\lambda m^d} T_z. \tag{4.1}$$

The Cauchy integral formula, Lemma 4, and (4.1) imply that there is a constant C_3 such that for every natural number $k \ge 0$ and every real $m > 0$

$$\|L^k T_t\|_{L^2(w^{[m]}) \to L^2(w^{[m]})} \le C_3^k k! e^{C_3 m^d} \quad \text{for } 1/8 < t < 2. \tag{4.2}$$

The next lemma is a weighted version of subelliptic estimates for L (cf. [D, Section 4]).

LEMMA 5. *For every multi-index I and every submultiplicative w there exist a constant C_4 and a natural number k such that*

$$\|X^I f\|_{L^2(w)}^2 \le C_4(\|L^k f\|_{L^2(w)}^2 + \|f\|_{L^2(w)}^2) \quad \textit{for } f \in C_c^\infty(G). \ ■ \tag{4.3}$$

Since L^k is also a Rockland operator, Theorem 2 implies

COROLLARY 6. *The estimates (4.3) hold for f in the domain $\mathcal{D}(L_w^k)$ of L_w^k, where L_w is the infinitesimal generator of the semigroup (1.2) considered on $L^2(w)$.* ■

Using the operators D_m and Corollary 6, we obtain

LEMMA 7. *For every multi-index I and for every submultiplicative w there exist a positive integer k and a constant C_5 such that for every $m > 0$,*

$$\|X^I f\|^2_{L^2(w^{[m]})} \le C_5 m^{2|I|}(m^{-2dk}\|L^k f\|^2_{L^2(w^{[m]})} + \|f\|^2_{L^2(w^{[m]})}) \quad \text{for } f \in \mathcal{D}(L^k_{w^{[m]}}). \blacksquare$$

We are now in a position to prove the following

PROPOSITION 8. *For every submultiplicative function η there exists a constant C_6 such that for every multi-index I there is a constant C_7 such that*

$$\sup_{x\in G}\{|X^I p_1(x)|\eta^{[m]}(x)\} \le C_7 \exp(C_6 m^d). \tag{4.4}$$

Proof. Let w be the submultiplicative function defined by $w(x) = \eta(x)^2$. Since $p_t \in \bigcap_{k,m}\mathcal{D}(L^k_{w^{[m]}})$ (cf. (1.4) and Theorem 2), by Lemma 7 and (4.2), we get

$$\begin{aligned}\|X^I p_{1/2}\|^2_{L^2(w^{[m]})} &= \|X^I T_{1/2-\varepsilon}p_\varepsilon\|^2_{L^2(w^{[m]})}\\ &\le C_5 m^{2|I|}(m^{-2dk}\|L^k T_{1/2-\varepsilon}p_\varepsilon\|^2_{L^2(w^{[m]})}\\ &\quad + \|T_{1/2-\varepsilon}p_\varepsilon\|^2_{L^2(w^{[m]})})\\ &\le C\exp(3C_3 m^d)\|p_\varepsilon\|^2_{L^2(w^{[m]})} \quad \text{for } \varepsilon < 1/4,\ m \ge 1.\end{aligned}$$

In virtue of (1.3), we obtain

$$\|p_\varepsilon\|^2_{L^2(w^{[m]})} = \varepsilon^{-Q/d}\int_G |p_1(x)|^2 w(\delta_{m\varepsilon^{1/d}}x)\,dx.$$

Putting $\varepsilon = m^{-d}$ ($m > 2$), we have

$$\|X^I p_{1/2}\|_{L^2(w^{[m]})} \le Cm^{Q/2}\exp(2C_3 m^d)\|p_1\|_{L^2(w)} \le C'\exp(3C_3 m^d). \tag{4.5}$$

Since

$$|X^I p_1(x)|\eta^{[m]}(x) \le \int_G |p_{1/2}(xy^{-1})X^I p_{1/2}(y)|\eta^{[m]}(xy^{-1})\eta^{[m]}(y)\,dy,$$

the Schwarz inequality and (4.5) imply (4.4) with $m > 2$.

Since the function $x \mapsto \sup_{0<m\le 2}\{\eta^{[m]}(x)\}$ is submultiplicative, the estimate (1.5) ends our proof. $\blacksquare$

COROLLARY 9. *For any subadditive γ there is a constant C_6 such that for every multi-index J there is a constant C_J such that*

$$|X^J p_1(x)| \le C_J e^{-\psi(x)}, \quad \text{where} \quad \psi(x) = \sup_{m>0}\{\gamma(\delta_m x) - C_6 m^d\}. \tag{4.6}$$

Proof. Setting $\eta(x) = e^{\gamma(x)}$ and applying Proposition 8, we get (4.6). $\blacksquare$

5. Pointwise estimates. Now we can make several choices of γ in Corollary 9 and obtain estimates on the corresponding functions ψ, and, consequently, on the kernels $X^J p_1$. However, because $\gamma(x) \le C(\tau(x)+1)$ (cf. Proposition 1), the function

$$\psi_\tau(x) = \sup_{m>0}\{\tau(\delta_m x) - C_6 m^d\} \tag{5.1}$$

is essentially the largest (that is, $\psi_\gamma(x) \le C\psi_\tau(x) + C$ for some constant $C > 0$).

Obtaining precise estimates on ψ_τ requires some effort, so we begin with simpler estimates which involve the homogeneous norm.

Let $\gamma_h(x) = |x|$. Then

$$\psi_h(x) = \sup_{m>0}\{m|x| - C_6 m^d\} = c|x|^{d/(d-1)}, \tag{5.2}$$

with $c = \frac{d-1}{d}(C_6 d)^{1/(1-d)} > 0$.

Corollary 9 combined with (5.2) leads to

THEOREM 10. *Let p_t be the kernels associated with the semigroup generated by a positive Rockland operator L. Let d be the homogeneous degree of L. Then there is a constant $c > 0$ such that for every multi-index J there exists a constant C_J such that*

$$|X^J p_1(x)| \le C_J \exp(-c|x|^{d/(d-1)}). \ \blacksquare \tag{5.3}$$

Remark 11. Using dilations (cf. (1.3)) and the fact that $\partial p_t = -Lp_t$, we get the following estimates which generalize (1.7) to semigroups associated with arbitrary Rockland operators:

$$|\partial_t^s X^J p_t(x)| \le C_{J,s} t^{-s-Q/d-|J|/d}\exp(-c|x|^{d/(d-1)}/t^{1/(d-1)}).$$

The same argument can be used to extend the estimates for $X_J p_1$ in Theorems 12 and 13 below to $\partial_t^s X_J p_t$. $\blacksquare$

Now we turn to studying the function ψ_τ. First we give an easy estimate from below, then a more complex one which is essentially optimal.

Since the δ_m are linear automorphisms of G (cf. (2.2)), there exists a constant $c > 0$ such that $\tau(\delta_m x) \ge cm\tau(x)$ for $m \ge 1$, which implies

$$\psi_\tau(x) \ge \sup_{m\ge 1}\{cm\tau(x) - C_6 m^d\}. \tag{5.4}$$

By (5.4), there is a constant $c_1 > 0$ such that for $\tau(x)$ sufficiently large

$$\psi_\tau(x) \ge c_1\tau(x)^{d/(d-1)}. \tag{5.5}$$

The estimate (5.5) and Corollary 9 give

THEOREM 12. *There exists a constant $c_1 > 0$ such that for every multi-index J there is a constant C_J such that*

$$|X^J p_1(x)| \le C_J \exp(-c_1\tau(x)^{d/(d-1)}). \ \blacksquare \tag{5.6}$$

In order to show precise estimates on ψ_τ we will proceed more carefully. Fix for a moment a coordinate $\mathbf{e}_s^{(i)}$. By the definition of ψ_τ and (3.12), for a compact neighbourhood K of 0 there is a constant $c>0$ such that

$$\psi_\tau(x) \geq \sup_{m\geq 1}\{cm^{d_s^{(i)}/i}|a_s^{(i)}|^{1/i} - C_6 m^d\} \quad \text{for } x=\sum_{j=1}^{l}\sum_{k=1}^{\dim \widetilde{W}_j^0} a_k^{(j)}\mathbf{e}_k^{(j)} \notin K.$$

A simple calculation shows that there exists a constant $c>0$ such that

$$(5.7)\qquad 1+\psi_\tau(x) \geq c|a_s^{(i)}|^{d/(id-d_s^{(i)})}, \qquad x=\sum_{j=1}^{l}\sum_{k=1}^{\dim \widetilde{W}_j^0} a_k^{(j)}\mathbf{e}_k^{(j)} \in G.$$

Taking the arithmetical mean we can assert that

$$(5.8)\qquad 1+\psi_\tau(x) \geq c\sum_{j=1}^{l}\sum_{k=1}^{\dim \widetilde{W}_j^0} |a_k^{(j)}|^{d/(jd-d_k^{(j)})},$$

with a constant $c>0$.

Now we show estimates from above for ψ_τ which are of the same type as (5.8). By (3.12), we have

$$\begin{aligned}\psi_\tau(x) &= \sup_{m>0}\{\tau(\delta_m x) - C_6 m^d\}\\ &\leq C_8 + \sup_{m>0}\Big\{\sum_{j=1}^{l}\sum_{k=1}^{\dim \widetilde{W}_j^0} C_9|m^{d_k^{(j)}}a_k^{(j)}|^{1/j} - C_6 m^d\Big\}\\ &\leq C_8 + \sum_{j=1}^{l}\sum_{k=1}^{\dim \widetilde{W}_j^0}\sup_{m>0}\{C_9|m^{d_k^{(j)}}a_k^{(j)}|^{1/j} - C_6 m^d\}\\ &\leq C_8 + C_{10}\sum_{j=1}^{l}\sum_{k=1}^{\dim \widetilde{W}_j^0}|a_k^{(j)}|^{d/(jd-d_k^{(j)})}.\end{aligned}$$

As an immediate consequence of (5.8) and Corollary 9 we get

THEOREM 13. *There is a constant $c>0$ such that for every multi-index J there exists a constant C_J such that*

$$(5.9)\qquad |X^J p_1(x)| \leq C_J \exp\Big(-c\sum_{j=1}^{l}\sum_{k=1}^{\dim \widetilde{W}_j^0}|a_k^{(j)}|^{d/(jd-d_k^{(j)})}\Big),$$

where $x=\sum_{j=1}^{l}\sum_{k=1}^{\dim \widetilde{W}_j^0} a_k^{(j)}\mathbf{e}_k^{(j)}\in G$. ■

We already know that $\psi_h(x)\leq C\psi_\tau(x)+C$ (cf. (5.1)). Hence Theorem 13 is at least as strong as Theorem 10. Moreover, by Proposition 1, the estimates in Theorem 12 are at least as strong as the estimates in Theorem 10. The question is: When do we get an improvement? To this end we write the estimates (5.3) and (5.6) in the coordinates $\mathbf{e}_k^{(j)}$. Let us state them in the following two remarks.

Remark 14. Applying (3.5) we can rewrite the estimate (5.3) as

$$(5.10)\qquad |X^J p_1(x)| \leq C_J \exp\Big(-c\sum_{j=1}^{l}\sum_{k=1}^{\dim \widetilde{W}_j^0}|a_k^{(j)}|^{d/(d_k^{(j)}d-d_k^{(j)})}\Big),$$

for x as before, with a constant $c>0$ independent of J. ■

Remark 15. By (3.12), in our coordinates the estimate (5.6) can be expressed as

$$(5.11)\qquad |X^J p_1(x)| \leq C_J \exp\Big(-c\sum_{j=1}^{l}\sum_{k=1}^{\dim \widetilde{W}_j^0}|a_k^{(j)}|^{d/(jd-j)}\Big),$$

where c is a strictly positive constant. ■

Note that

$$(5.12)\qquad d/(d_k^{(j)}d-d_k^{(j)}) \leq d/(jd-j)$$

and

$$(5.13)\qquad d/(jd-j) \leq d/(jd-d_k^{(j)}).$$

One can prove that G is not stratified if and only if there exists a coordinate $\mathbf{e}_k^{(j)}$ such that $d_k^{(j)}>j$, and, consequently, the inequalities (5.12) and (5.13) are strict. So, we conclude that if G is not stratified, then the estimate (5.11) strictly improves (5.10). Analogously, in this case, the estimate (5.9) strictly improves (5.10) and (5.11).

We expect that the estimates (5.9) are in some sense optimal. Our expectations are based on the fact that if $\mathbf{e}_s^{(i)}\notin[G,G]$, then for M large enough

$$(5.14)\qquad \int_G |p_1(x)|\exp(M|a_s^{(i)}|^{d/(id-d_s^{(i)})})\,dx=\infty,$$

$$\text{where}\quad x=\sum a_k^{(j)}\mathbf{e}_k^{(j)},$$

whereas for small $M>0$ the integral (5.14) is obviously convergent (cf. (5.9)). This follows by a Fourier transform computation.

It seems likely that (5.14) holds for each direction $\mathbf{e}_k^{(j)}$.

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On the characterization of Hardy–Besov spaces on the dyadic group and its applications

by

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Dedicated to Professor C. Watari
on the occasion of his sixtieth birthday

Abstract. C. Watari [12] obtained a simple characterization of Lipschitz classes $\mathrm{Lip}^{(p)}\alpha(W)$ ($1 \le p \le \infty$, $\alpha > 0$) on the dyadic group using the L^p-modulus of continuity and the best approximation by Walsh polynomials. Onneweer and Weiyi [4] characterized homogeneous Besov spaces $B_{p,q}^{\alpha}$ on locally compact Vilenkin groups, but there are still some gaps to be filled up. Our purpose is to give the characterization of Besov spaces $B_{p,q}^{\alpha}$ by oscillations, atoms and others on the dyadic groups. As applications, we show a strong capacity inequality of the type of the Maz'ya inequality, a weak type estimate for maximal Cesàro means and a sufficient condition of absolute convergence of Walsh–Fourier series.

0. Introduction and notation. The dyadic group, 2^{ω}, is viewed classically as the set of all sequences of 0's and 1's with addition (mod 2) defined pointwise, and is supplied with the usual product topology. Our results are stated in the situation that 2^{ω} is the additive subgroup of the ring of integers in the 2-series field K of formal Laurent series in one variable over $GF(2)$ (see [9]). Such a field K is a particular instance of a local field; that is, a locally compact, totally disconnected, non-discrete, complete field. The results of this paper have extensions to any local field.

We need to set some basic notation. It is taken from [9] where the fundamentals are detailed. For the additive subgroup K^+ of the 2-series field K, we may choose a Haar measure dx. Let $d(\alpha x) = |\alpha| dx$ and call $|\alpha|$ the *valuation* of α. Let $|0| = 0$. The mapping $x \mapsto |x|$ has the following properties: $|x| = 0 \Leftrightarrow x = 0$, $|xy| = |x| \cdot |y|$, $|x + y| \le \max(|x|, |y|)$.

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