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Global maximal estimates for solutions to the Schrödinger equation

by

PER SJÖLIN (Stockholm)

Abstract. Global maximal estimates are considered for solutions to an initial value problem for the Schrödinger equation.

1. Introduction. Let f belong to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and set

$$S_t f(x) = u(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{it|\xi|^a} \widehat{f}(\xi)\,d\xi, \quad x \in \mathbb{R}^n,\ t \in \mathbb{R},$$

where $a > 1$. Here $\widehat{f}$ denotes the Fourier transform of f, defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\,dx.$$

It is then clear that $u(x,0) = f(x)$ and in the case $a = 2$, u is a solution to the Schrödinger equation $i\partial u/\partial t = \Delta u$.

We shall here consider the maximal functions

$$S^* f(x) = \sup_{0<t<1} |S_t f(x)|, \quad x \in \mathbb{R}^n,$$

and

$$S^{**} f(x) = \sup_{t>0} |S_t f(x)|, \quad x \in \mathbb{R}^n.$$

We also introduce Sobolev spaces H_s by setting

$$H_s = \{f \in \mathcal{S}' : \|f\|_{H_s} < \infty\}, \quad s \in \mathbb{R},$$

where

$$\|f\|_{H_s} = \Big(\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\widehat{f}(\xi)|^2\,d\xi\Big)^{1/2}.$$

Local estimates for $S^* f$ and $S^{**} f$ have been studied in several papers, see e.g. J. Bourgain [1], L. Carleson [3], B. E. J. Dahlberg and C. E. Kenig [5],

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C. E. Kenig and A. Ruiz [8], E. Prestini [9], P. Sjölin [11], [12], L. Vega [14] and C. E. Kenig, G. Ponce and L. Vega [6].

We shall here consider global estimates of the type

(1) $$\|S^*f\|_2 \le C\|f\|_{H_s}$$

and

(2) $$\|S^{**}f\|_2 \le C\|f\|_{H_s},$$

where the norm on the left hand side is the norm in $L^2(\mathbb{R}^n)$. It is known and easy to prove that (1) holds for $s > a/2$ (see A. Carbery [2] and M. Cowling [4]). C. E. Kenig, G. Ponce and L. Vega [7] have proved that if $n = 1$ and $a \ge 2$ then $s > a/4$ is a sufficient condition for (1). We shall here prove the following theorem.

THEOREM. *If $n = 1$ and $a > 1$ then $s > a/4$ is a sufficient condition for* (1) *and $s \ge a/4$ is a necessary condition for* (1).

We remark that the method in the proof of the sufficiency in the theorem also gives the result that $s > an/4$ is a sufficient condition for (1) for all n. This coincides with the result mentioned above in the case $n = 2$.

We shall also observe that the inequality (2) does not hold for any value of s.

We also remark that a small modification of the proof of the theorem yields the estimate

$$\|S^*f\|_2 \le C\Big(\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2(1+|\xi|^2)^{an/4}(\log(2+|\xi|))^{2\gamma}\,d\xi\Big)^{1/2}$$

for $\gamma > 1$, which is of interest for $n = 1$ and 2.

2. Proofs. We first prove the impossibility of the inequality (2). We choose f such that $\widehat{f} = \varphi$ where $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\operatorname{supp}\varphi \subset \{\xi : 1/2 < |\xi| < 2\}$, φ is radial and $\varphi(\xi) = 1$ for $3/4 \le |\xi| \le 3/2$. Then S_tf is radial and we set $x = (x_1, 0, 0, \ldots, 0)$, where $x_1 > 0$ is large. Also choose $t = x_1/a$. Then

$$S_tf(x) = c_n \int e^{ix_1(\xi_1+|\xi|^a/a)}\varphi(\xi)\,d\xi$$

and it follows from the method of stationary phase (see e.g. E. M. Stein [13], p. 319) that

$$|S_tf(x)| \ge cx_1^{-n/2},$$

where c denotes a positive constant. Hence

$$S^{**}f(x) \ge c|x|^{-n/2}$$

for $|x|$ large and it follows that $S^{**}f$ does not belong to $L^2(\mathbb{R}^n)$. We conclude that the estimate (2) holds for no value of s.

Proof of the theorem. We carry out the proof of the sufficiency for general n. Let $t(x)$ denote a measurable function in $\mathbb{R}^n$ with $0 < t(x) < 1$ and set

$$Tf(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi}e^{it(x)|\xi|^a}\widehat{f}(\xi)\,d\xi, \quad x \in \mathbb{R}^n,\ f \in \mathcal{S}(\mathbb{R}^n).$$

We want to prove that

(3) $$\|Tf\|_2 \le C\|f\|_{H_s}$$

if $s > an/4$.

It is well known that there exists a function $\varphi \in C_0^\infty(\mathbb{R}^n)$ with support in $\{\xi : 1/2 < |\xi| < 2\}$ such that $\sum_{k=-\infty}^{\infty}\varphi(2^{-k}\xi) = 1$, $\xi \ne 0$. We set

$$\varphi_0(\xi) = 1 - \sum_{k=1}^{\infty}\varphi(2^{-k}\xi)$$

so that $\varphi_0 \in C_0^\infty(\mathbb{R}^n)$. Setting

$$\Phi(\xi) = \varphi_0(\xi) + \sum_{k=1}^{\infty}\varphi(2^{-k}\xi)2^{-ks}$$

one also has

$$c(1+|\xi|^2)^{-s/2} \le \Phi(\xi) \le C(1+|\xi|^2)^{-s/2}.$$

Defining an operator R by

$$Rf(x) = \int e^{ix\cdot\xi}e^{it(x)|\xi|^a}\Phi(\xi)\widehat{f}(\xi)\,d\xi,$$

we then have

$$\begin{aligned} Rf(x) &= \int e^{ix\cdot\xi}e^{it(x)|\xi|^a}\varphi_0(\xi)\widehat{f}(\xi)\,d\xi \\ &\quad + \sum_{k=1}^{\infty}\int e^{ix\cdot\xi}e^{it(x)|\xi|^a}\varphi(2^{-k}\xi)\widehat{f}(\xi)\,d\xi\,2^{-ks} \\ &= P_0f(x) + \sum_{k=1}^{\infty}P_kf(x). \end{aligned}$$

To prove (3) it is sufficient to prove that

(4) $$\|Rf\|_2 \le C\|f\|_2.$$

This is a consequence of the fact that

$$Tf(x) = \int e^{ix\cdot\xi}e^{it(x)|\xi|^a}\Phi(\xi)\frac{\widehat{f}(\xi)}{\Phi(\xi)}\,d\xi = R\Big(\mathcal{F}^{-1}\Big(\frac{\widehat{f}}{\Phi}\Big)\Big)(x),$$

from which it follows that (4) implies

$$\|Tf\|_2 \le C\|\widehat{f}/\Phi\|_2 \le C\|f\|_{H_s}.$$

Here $\mathcal{F}$ denotes Fourier transformation.

We then set

$$R_0 f(x) = \int e^{ix\cdot\xi} p_0(x,\xi) \widehat{f}(\xi)\, d\xi,$$

where

$$p_0(x,\xi) = e^{it(x)|\xi|^a} \varphi_0(\xi),$$

and

$$R_N f(x) = \int e^{ix\cdot\xi} p_N(x,\xi) \widehat{f}(\xi)\, d\xi, \quad N \geq 1,$$

where

$$p_N(x,\xi) = e^{it(x)|\xi|^a} \varphi(\xi/N) N^{-s}.$$

Hence $P_0 = R_0$ and $P_k = R_{2^k}$, $k = 1,2,3,\ldots$ Then choose a real-valued function $\varrho \in C_0^\infty(\mathbb{R}^n)$ such that $\varrho(x) = 1$ if $|x| \leq 1$, and $\varrho(x) = 0$ if $|x| \geq 2$, and set $\psi = 1 - \varrho$. We shall need the symbols

$$p_{N,M}(x,\xi) = \varrho(x/M) p_N(x,\xi), \quad M > 1,$$

and

$$p_{N,M,\varepsilon}(x,\xi) = \psi(\xi/\varepsilon) p_{N,M}(x,\xi), \quad 0 < \varepsilon < 1.$$

The corresponding operators $R_{N,M}$ and $R_{N,M,\varepsilon}$ are then trivially bounded on L^2.

The adjoint of $R_{N,M,\varepsilon}$ is given by the formula

$$R^*_{N,M,\varepsilon} g(x) = \iint e^{i(x-y)\cdot\xi} \overline{p_{N,M,\varepsilon}(y,\xi)} g(y)\, dy\, d\xi$$

and it follows that

$$\lim_{\varepsilon\to 0} R^*_{N,M,\varepsilon} g(x) = R^*_{N,M} g(x), \quad g \in \mathcal{S}.$$

From the computation on p. 708 in [11] we conclude that

$$\begin{aligned}
&\int |R^*_{N,M,\varepsilon} g(x)|^2\, dx \\
&\quad = (2\pi)^n \iint \Big(\int e^{i(z-y)\cdot\xi} \overline{p_{N,M,\varepsilon}(y,\xi)} p_{N,M,\varepsilon}(z,\xi)\, d\xi \Big) g(y) \overline{g(z)}\, dy\, dz \\
&\quad = (2\pi)^n \iint \Big(\int e^{i(z-y)\cdot\xi} \varrho(y/M) \varrho(z/M) \psi^2(\xi/\varepsilon) \overline{p_N(y,\xi)} p_N(z,\xi)\, d\xi \Big) \\
&\qquad\qquad \times g(y) \overline{g(z)}\, dy\, dz
\end{aligned}$$

and invoking Fatou's lemma one obtains

$$\begin{aligned}
&\int |R^*_{N,M} g(x)|^2\, dx \\
&\qquad \leq \liminf_{\varepsilon\to 0} \int |R^*_{N,M,\varepsilon} g(x)|^2\, dx \\
&\qquad = (2\pi)^n \iint \Big(\int e^{i(z-y)\cdot\xi} \overline{p_N(y,\xi)} p_N(z,\xi)\, d\xi \Big) \\
&\qquad\qquad \times \varrho(y/M) \varrho(z/M) g(y) \overline{g(z)}\, dy\, dz
\end{aligned}$$

$$\begin{aligned}
&\leq C \iint \Big| \int e^{i(z-y)\cdot\xi} e^{-it(y)|\xi|^a} e^{it(z)|\xi|^a} \varphi^2(\xi/N)\, d\xi\, N^{-2s} \Big| \\
&\qquad \times |g(y)||g(z)|\, dy\, dz, \quad N \geq 1.
\end{aligned}$$

In the case $N = 0$ we have a similar inequality with $\varphi^2(\xi/N)N^{-2s}$ replaced by $\varphi_0^2(\xi)$ in the right hand side.

Now set

$$I_N(x,\omega) = \int e^{i(x\cdot\xi + \omega|\xi|^a)} \varphi^2(\xi/N)\, d\xi\, N^{-2s}, \quad x \in \mathbb{R}^n, -1 < \omega < 1, N \geq 1,$$

$$I_0(x,\omega) = \int e^{i(x\cdot\xi + \omega|\xi|^a)} \varphi_0^2(\xi)\, d\xi, \quad x \in \mathbb{R}^n,\ -1 < \omega < 1,$$

and

$$J_N(x) = \sup_{|\omega|<1} |I_N(x,\omega)|, \quad x \in \mathbb{R}^n.$$

We shall prove that $J_N \in L^1(\mathbb{R}^n)$ for $N = 0$ and $N \geq 1$, and that

$$\|J_N\|_1 \leq C N^{-\delta}, \quad N \geq 1, \tag{5}$$

where $\delta > 0$. One therefore obtains

$$\begin{aligned}
\int |R^*_{N,M} g(x)|^2\, dx &\leq C \iint J_N(z-y) |g(y)|\, |g(z)|\, dy\, dz \\
&\leq C \|J_N * |g|\, \|_2 \|g\|_2 \leq C \|J_N\|_1 \|g\|_2^2.
\end{aligned}$$

It follows that $\|R^*_{0,M} g\|_2 \leq C \|g\|_2$ and

$$\|R^*_{N,M} g\|_2 \leq C N^{-\delta/2} \|g\|_2, \quad N \geq 1.$$

We can here replace $R^*_{N,M}$ by $R_{N,M}$ and letting $M \to \infty$ we obtain

$$\|R_0 g\|_2 \leq C \|g\|_2$$

and

$$\|R_N g\|_2 \leq C N^{-\delta/2} \|g\|_2, \quad N \geq 1.$$

Therefore $\|P_0\| \leq C$ and $\|P_k\| \leq C 2^{-k\delta/2}$, $k \geq 1$. Hence R is bounded on L^2 since

$$\|R\| \leq \sum_{k=0}^{\infty} \|P_k\|.$$

It remains to study J_N. Setting $\alpha = \varphi^2$ and $\alpha_0 = \varphi_0^2$ and performing a change of variable we obtain

$$I_0(x,\omega) = \int e^{i(x\cdot\xi + \omega|\xi|^a)} \alpha_0(\xi)\, d\xi$$

and

$$I_N(x,\omega) = \int e^{i(Nx\cdot\xi + N^a \omega|\xi|^a)} \alpha(\xi)\, d\xi\, N^{n-2s}, \quad N \geq 1.$$

We shall first prove that

$$|I_0(x,\omega)| \leq C(1+|x|)^{-n-1} \tag{6}$$

for $|\omega| < 1$. We set

$$I_0'(x,\omega) = \int e^{ix\cdot\xi}(e^{i\omega|\xi|^a} - 1)\alpha_0(\xi)\,d\xi$$

and

$$I_{0,\varepsilon}'(x,\omega) = \int e^{ix\cdot\xi}(e^{i\omega|\xi|^a} - 1)\psi(\xi/\varepsilon)\alpha_0(\xi)\,d\xi, \qquad 0<\varepsilon<1.$$

It is then clear that it is sufficient to prove (6) with I_0 replaced by $I_{0,\varepsilon}'$. Since $|I_0(x,\omega)| \le C$ we may assume that $|x| > 1$. Performing $n+1$ integrations by parts one obtains

$$I_{0,\varepsilon}'(x,\omega) \le C|x|^{-n-1} \sum_{|\mu|+|\beta|+|\gamma|=n+1} I_{\mu,\beta,\gamma},$$

where

$$I_{\mu,\beta,\gamma} = \int |D^\mu(e^{i\omega|\xi|^a} - 1)|\,|D^\beta(\psi(\xi/\varepsilon))|\,|D^\gamma\alpha_0(\xi)|\,d\xi.$$

Since

$$|D^\mu(e^{i\omega|\xi|^a} - 1)| \le C|\xi|^{a-|\mu|}, \qquad |\xi|\le 1,$$

and

$$|D^\beta(\psi(\xi/\varepsilon))| \le C|\xi|^{-|\beta|},$$

we obtain

$$I_{\mu,\beta,\gamma} \le C\int_{|\xi|\le 1} |\xi|^{a-n-1}\,d\xi \le C$$

for $\beta = 0$, and

$$I_{\mu,\beta,\gamma} \le C\int_{\varepsilon\le|\xi|\le 2\varepsilon} |\xi|^{a-|\mu|}\varepsilon^{-|\beta|}\,d\xi \le C\varepsilon^n\varepsilon^{a-n-1} \le C$$

for $|\beta| \ge 1$. Hence (6) is proved and it follows that $J_0 \in L^1$.

To study J_N for $N \ge 1$ we shall use the following two lemmas (see P. Sjölin [10] and the references in that paper).

LEMMA 1. *Let Ω denote an open set in $\mathbb{R}^n$ and let $\varphi \in C_0^\infty(\Omega)$. Assume that $\psi \in C^\infty(\Omega)$, ψ is real-valued and that $|\det(\partial^2\psi/\partial x_i\partial x_k)| \ge c > 0$ in Ω. Then*

$$\Big|\int_\Omega e^{i(\xi\cdot x+\zeta\psi(x))}\varphi(x)\,dx\Big| \le C(1+|\zeta|)^{-n/2}, \qquad \xi\in\mathbb{R}^n,\ \zeta\in\mathbb{R}.$$

LEMMA 2. *Let I denote an open interval in $\mathbb{R}$, let $g \in C_0^\infty(I)$, $F \in C^\infty(I)$ and assume that F is real-valued and $F' \ne 0$. If k is a positive integer then*

$$\int_I e^{iF(x)}g(x)\,dx = \int_I e^{iF(x)}h_k(x)\,dx,$$

where h_k is a linear combination of functions of the form

$$g^{(s)}(F')^{-k-r}\prod_{q=1}^r F^{(j_q)}$$

with $0\le s\le k$, $0\le r\le k$ and $2\le j_q\le k+1$.

It follows from Lemma 1 that we always have

(7) $$|I_N(x,\omega)| \le C(N^a|\omega|)^{-n/2}N^{n-2s}.$$

If $N^a|\omega| < c_0N|x|$, i.e. $|\omega| < c_0|x|/N^{a-1}$, we also have according to Lemma 2,

(8) $$|I_N(x,\omega)| \le C(1+N^K|x|^K)^{-1}N^{n-2s},$$

where K is large.

We shall estimate J_N and begin with the case $0<|x|<1/N$. It is clear that $|I_N(x,\omega)| \le CN^{n-2s}$ and it follows that

(9) $$\int_{|x|\le 1/N} J_N(x)\,dx \le CN^{-2s}.$$

We then consider the case $1/N < |x| \le N^{a-1}/c_0$. If $|\omega| < c_0|x|/N^{a-1}$ we use the estimate (8) and if $|\omega| \ge c_0|x|/N^{a-1}$ we apply the inequality (7), which implies that

$$|I_N(x,\omega)| \le C(N|x|)^{-n/2}N^{n-2s}.$$

Hence

$$J_N(x) \le CN^{n/2-2s}|x|^{-n/2}, \qquad N^{-1}<|x|\le N^{a-1}/c_0.$$

We conclude that

(10) $$\int_{N^{-1}<|x|\le N^{a-1}/c_0} J_N(x)\,dx \le CN^{n/2-2s}\int_{|x|\le N^{a-1}/c_0}|x|^{-n/2}\,dx \le CN^{n/2-2s}\int_0^{N^{a-1}/c_0} r^{n/2-1}dr = CN^{an/2-2s}.$$

In the case $|x| > N^{a-1}/c_0$ it follows from (8) that

$$J_N(x) \le CN^{n-2s}N^{-K}|x|^{-K},$$

and hence

(11) $$\int_{|x|>N^{a-1}/c_0} J_N(x)\,dx \le CN^{-L},$$

where L is large.

Combining (9), (10) and (11) one obtains

$$\|J_N\|_1 \le CN^{an/2-2s}, \qquad N\ge 1,$$

and hence (5) holds with $\delta = 2s - an/2$. The sufficiency in the theorem now follows since $\delta > 0$ if $s > an/4$.

We next prove that if $n = 1$ and (1) holds then $s \geq a/4$. First let $\varphi \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\varphi \subset (-1,1)$ and choose f such that

$$\widehat{f}(\xi) = \varphi(N^{a/2-1}\xi + N^{a/2}).$$

It is then easy to see that $\widehat{f}$ vanishes outside the interval $[-N-N^{1-a/2}, -N+N^{1-a/2}]$ and it follows that

$$\|f\|_{H_s} \leq CN^{s+1/2-a/4}. \tag{12}$$

Setting $\pi_n = (2\pi)^{-n}$ we have

$$S_t f(x) = \pi_1 \int e^{ix\eta} e^{it|\eta|^a} \varphi(N^{a/2-1}\eta + N^{a/2})\, d\eta$$

and performing a change of variable we obtain

$$\begin{aligned} S_t f(x) &= \pi_1 \int e^{ix(N^{1-a/2}\xi - N)} e^{it|N^{1-a/2}\xi - N|^a} \varphi(\xi)\, d\xi\, N^{1-a/2} \\ &= \pi_1 N^{1-a/2} \int e^{i(N^{1-a/2}x\xi - Nx)} e^{itN^a(1-N^{-a/2}\xi)^a} \varphi(\xi)\, d\xi \\ &= \pi_1 N^{1-a/2} \int e^{iF(\xi)} \varphi(\xi)\, d\xi. \end{aligned}$$

Using the Taylor expansion

$$(1+y)^a = 1 + c_1 y + c_2 y^2 + \mathcal{O}(|y|^3),$$

where $c_1 = a$ and c_2 denotes a positive constant, we obtain

$$\begin{aligned} F(\xi) &= N^{1-a/2}x\xi - Nx \\ &\quad + tN^a(1 - c_1 N^{-a/2}\xi + c_2 N^{-a}\xi^2 + \mathcal{O}(N^{-3a/2})) \\ &= tN^a - Nx + N^{1-a/2}x\xi - c_1 tN^{a/2}\xi + c_2 t\xi^2 + \mathcal{O}(tN^{-a/2}). \end{aligned}$$

We now choose t such that

$$N^{1-a/2}x = c_1 t N^{a/2}, \quad \text{i.e.} \quad t = \frac{1}{c_1}\frac{x}{N^{a-1}}.$$

It follows that

$$|S_t f(x)| = \pi_1 N^{1-a/2} \Big| \int e^{iG(\xi)} \varphi(\xi)\, d\xi \Big|,$$

where

$$G(\xi) = c_2 t\xi^2 + \mathcal{O}(tN^{-a/2}) = \frac{c_2}{c_1}\frac{x}{N^{a-1}}\xi^2 + \mathcal{O}\Big(\frac{x}{N^{a-1}} N^{-a/2}\Big).$$

Also choose x such that $(1-\varepsilon)N^{a-1} \leq x \leq N^{a-1}$, where $\varepsilon > 0$ is a small number. One then has

$$G(\xi) = \frac{c_2}{c_1}\xi^2 + \mathcal{O}(\varepsilon) + \mathcal{O}(N^{-a/2}) = \frac{c_2}{c_1}\xi^2 + \mathcal{O}(\varepsilon)$$

if N is large. Setting $d = c_2/c_1$ and choosing φ such that

$$\int e^{i\,d\xi^2} \varphi(\xi)\, d\xi = c_0 \neq 0,$$

we obtain

$$\begin{aligned} \int e^{iG(\xi)} \varphi(\xi)\, d\xi &= \int e^{i\,d\xi^2} e^{i\mathcal{O}(\varepsilon)} \varphi(\xi)\, d\xi \\ &= \int e^{i\,d\xi^2} \varphi(\xi)\, d\xi + \int e^{i\,d\xi^2} (e^{i\mathcal{O}(\varepsilon)} - 1) \varphi(\xi)\, d\xi \\ &= c_0 + \mathcal{O}(\varepsilon) \end{aligned}$$

and it follows that

$$\Big| \int e^{iG(\xi)} \varphi(\xi)\, d\xi \Big| \geq c > 0,$$

if ε is chosen sufficiently small. Hence $|S_t f(x)| \geq cN^{1-a/2}$ and we conclude that $S^* f(x) \geq cN^{1-a/2}$ for $(1-\varepsilon)N^{a-1} \leq x \leq N^{a-1}$.

It follows that

$$\Big(\int |S^* f|^2\, dx \Big)^{1/2} \geq cN^{1-a/2} (N^{a-1})^{1/2} = cN^{1/2}.$$

Invoking (1) and (12) we then obtain $N^{1/2} \leq CN^{s+1/2-a/4}$, and letting $N \to \infty$ we necessarily have $s \geq a/4$. This completes the proof of the theorem.

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Note on semigroups generated by positive Rockland operators on graded homogeneous groups

by

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Abstract. Let L be a positive Rockland operator of homogeneous degree d on a graded homogeneous group G and let p_t be the convolution kernels of the semigroup generated by L. We prove that if $\tau(x)$ is a Riemannian distance of x from the unit element, then there are constants $c>0$ and C such that $|p_1(x)|\le C\exp(-c\tau(x)^{d/(d-1)})$. Moreover, if G is not stratified, more precise estimates of p_1 at infinity are given.

1. Introduction. Let L be a positive Rockland operator on a homogeneous group G (cf. [FS]) and let d be the homogeneous degree of L (cf. Section 2).

The operator L satisfies the following subelliptic estimates proved by B. Helffer and J. Nourrigat [HN]: for every multi-index I there are constants C and k such that

$$\|X^I f\|_{L^2(G)} \le C(\|L^k f\|_{L^2(G)} + \|f\|_{L^2(G)}),\quad f\in C_c^\infty(G). \tag{1.1}$$

Theorem (4.25) of [FS] asserts that the closure $-\bar{L}$ of the essentially selfadjoint operator $-L$ is the infinitesimal generator of a semigroup of linear operators on $L^2(G)$ which has the form

$$T_t f = f * p_t,\quad t>0, \tag{1.2}$$

where the p_t belong to the Schwartz space $\mathcal{S}(G)$.

The homogeneity of L implies

$$p_t(x) = t^{-Q/d} p_1(\delta_{t^{-1/d}} x), \tag{1.3}$$

where Q is the homogeneous dimension of G and δ_t is the family of dilations associated with G (cf. Section 2).

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