Global maximal estimates for solutions to the Schrödinger equation

by

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Abstract. Global maximal estimates are considered for solutions to an initial value problem for the Schrödinger equation.

1. Introduction. Let \( f \) belong to the Schwartz space \( S(\mathbb{R}^n) \) and set

\[
S_t f(x) = u(x, t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \xi} e^{i t |\xi|^\alpha} \hat{f}(\xi) \, d\xi, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R},
\]

where \( \alpha > 1 \). Here \( \hat{f} \) denotes the Fourier transform of \( f \), defined by

\[
\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i \xi \cdot x} f(x) \, dx.
\]

It is then clear that \( u(x, 0) = f(x) \) and in the case \( \alpha = 2 \), \( u \) is a solution to the Schrödinger equation \( i\partial u / \partial t = \Delta u \).

We shall here consider the maximal functions

\[
S^* f(x) = \sup_{0 < t < 1} |S_t f(x)|, \quad x \in \mathbb{R}^n,
\]

and

\[
S^{**} f(x) = \sup_{t > 0} |S_t f(x)|, \quad x \in \mathbb{R}^n.
\]

We also introduce Sobolev spaces \( H_s \) by setting

\[H_s = \{ f \in S' : \| f \|_{H_s} < \infty \}, \quad s \in \mathbb{R},\]

where

\[
\| f \|_{H_s} = \left( \int_{\mathbb{R}^n} (1 + |\xi|^s)^s |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2}.
\]

Local estimates for \( S^* f \) and \( S^{**} f \) have been studied in several papers, see e.g. J. Bourgain [1], L. Carleson [3], B. E. J. Dahlberg and C. E. Kenig [5],
Proof of the theorem. We carry out the proof of the sufficiency for general $n$. Let $t(x)$ denote a measurable function in $\mathbb{R}^n$ with $0 < t(x) < 1$ and set

$$Tf(x) = \int_{\mathbb{R}^n} e^{it \cdot \xi} e^{it(x)}|\xi|^n \hat{f}(\xi) \, d\xi, \quad x \in \mathbb{R}^n, \ f \in \mathcal{S}(\mathbb{R}^n).$$

We want to prove that

$$\|Tf\|_2 \leq C\|f\|_{H_s} \tag{3}$$

if $s > an/4$.

It is well known that there exists a function $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with support in $\{\xi : 1/2 < |\xi| < 2\}$ such that $\sum_{k=-\infty}^{\infty} \varphi(2^{-k}\xi) = 1, \ \xi \neq 0$. We set

$$\varphi_0(\xi) = 1 - \sum_{k=1}^{\infty} \varphi(2^{-k}\xi)$$

so that $\varphi_0 \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Setting

$$\Phi(\xi) = \varphi_0(\xi) + \sum_{k=1}^{\infty} \varphi(2^{-k}\xi)2^{-ks}$$

one also has

$$c(1 + |\xi|^2)^{-s/2} \leq \Phi(\xi) \leq C(1 + |\xi|^2)^{-s/2}.$$

Defining an operator $R$ by

$$Rf(x) = \int_{\mathbb{R}^n} e^{it \cdot \xi} e^{it(x)}|\xi|^n \Phi(\xi) \hat{f}(\xi) \, d\xi,$$

we then have

$$Rf(x) = \int_{\mathbb{R}^n} e^{it \cdot \xi} e^{it(x)}|\xi|^n \varphi_0(\xi) \hat{f}(\xi) \, d\xi + \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} e^{it \cdot \xi} e^{it(x)}|\xi|^n \varphi(2^{-k}\xi) \hat{f}(\xi) \, d\xi 2^{-ks}$$

$$= P_0f(x) + \sum_{k=1}^{\infty} P_kf(x).$$

To prove (3) it is sufficient to prove that

$$\|Rf\|_2 \leq C\|f\|_2 \tag{4}$$

This is a consequence of the fact that

$$Tf(x) = \int_{\mathbb{R}^n} e^{it \cdot \xi} e^{it(x)}|\xi|^n \Phi(\xi) \hat{f}(\xi) \, d\xi = R\left(F^{-1}\left(\frac{\hat{f}}{\tilde{\Phi}}\right)\right)(x),$$

from which it follows that (4) implies

$$\|Tf\|_2 \leq C\|\frac{\hat{f}}{\tilde{\Phi}}\|_2 \leq C\|f\|_{H_s}.$$
We then set

\[ R_0 f(x) = \int e^{ix \cdot \xi} p_0(x, \xi) \hat{f}(\xi) \, d\xi, \]

where

\[ p_0(x, \xi) = e^{i\psi(x)} |\xi|^n \varphi_0(\xi), \]

and

\[ R_N f(x) = \int e^{ix \cdot \xi} p_N(x, \xi) \hat{f}(\xi) \, d\xi, \quad N \geq 1, \]

where

\[ p_N(x, \xi) = e^{i\psi(x)} |\xi|^n \varphi(\xi/N) N^{-n}. \]

Hence \( P_0 = R_0 \) and \( P_N = R_N \), \( k = 1, 2, 3, \ldots \) Then choose a real-valued function \( \varphi \in C_0^\infty(\mathbb{R}^n) \) such that \( \varphi(x) = 1 \) if \( |x| \leq 1 \), and \( \varphi(x) = 0 \) if \( |x| \geq 2 \), and set \( \psi = 1 - \varphi \). We shall need the symbols

\[ p_{N,M}(x, \xi) = \varphi(x/M) p_N(x, \xi), \quad M > 1, \]

and

\[ p_{N,M,\varepsilon}(x, \xi) = \varphi(\xi/\varepsilon) p_{N,M}(x, \xi), \quad 0 < \varepsilon < 1. \]

The corresponding operators \( R_{N,M} \) and \( R_{N,M,\varepsilon} \) are then trivially bounded on \( L^2 \).

The adjoint of \( R_{N,M,\varepsilon} \) is given by the formula

\[ R_{N,M,\varepsilon}^* g(x) = \int\int e^{i(x-y) \cdot \xi} p_{N,M,\varepsilon}(y, \xi)g(y) \, dy \, d\xi \]

and it follows that

\[ \lim_{\varepsilon \to 0} R_{N,M,\varepsilon}^* g(x) = R_{N,M}^* g(x), \quad g \in \mathcal{S}. \]

From the computation on p. 708 in [11] we conclude that

\[ \int |R_{N,M,\varepsilon}^* g(x)|^2 \, dx \]

\[ = (2\pi)^n \int \int \left( \int e^{i(x-y) \cdot \xi} p_{N,M,\varepsilon}(y, \xi)g(y) \, dy \right) \hat{g}(\xi) \, d\xi \, dy \, dz \]

\[ = (2\pi)^n \int \int \left( \int e^{i(x-y) \cdot \xi} g(y/M) \mathbb{E}(y/M) \varphi(\xi/\varepsilon) p_N(x, \xi) \, d\xi \right) \hat{g}(\xi) \, d\xi \, dy \, dz \]

\[ \leq \liminf_{\varepsilon \to 0} \int |R_{N,M,\varepsilon}^* g(x)|^2 \, dx \]

\[ = (2\pi)^n \int \int \left( \int e^{i(x-y) \cdot \xi} p_N(y, \xi) p_N(x, \xi) \, d\xi \right) \hat{g}(\xi) \, d\xi \, dy \, dz \]

\[ \times g(y/M) \mathbb{E}(y/M) \hat{g}(\xi) \, dy \, dz \]

and invoking Fatou's lemma one obtains

\[ \int |R_{N,M}^* g(x)|^2 \, dx \]

\[ \leq C \int \left( \int e^{i(x-y) \cdot \xi} \mathbb{E}(y) \varphi(\xi) \hat{g}(\xi) \, d\xi \right) \hat{g}(\xi) \, d\xi \, dy \, dz \]

\[ \times |g(y)| |g(z)| \, dy \, dz, \quad N \geq 1. \]

In the case \( N = 0 \) we have a similar inequality with \( \varphi(\xi) \mathbb{E}(\xi) N^{-n} \) replaced by \( \varphi_0(\xi) \) in the right hand side.

Now set

\[ I_N(x, \omega) = \int e^{i(x-y) \cdot \omega} \varphi(\xi) \mathbb{E}(\xi) N^{-n} \, dx \quad x \in \mathbb{R}^n, -1 < \omega < 1, N \geq 1, \]

\[ I_0(x, \omega) = \int e^{i(x-y) \cdot \omega} \varphi_0(\xi) \mathbb{E}(\xi) \, dx \quad x \in \mathbb{R}^n, -1 < \omega < 1, \]

and

\[ J_N(x) = \sup_{|\omega| < 1} |I_N(x, \omega)|, \quad x \in \mathbb{R}^n. \]

We shall prove that \( J_N \in L^1(\mathbb{R}^n) \) for \( N = 0 \) and \( N \geq 1 \), and that

\[ \|J_N\|_1 \leq CN^{-t}, \quad N \geq 1, \]

where \( \delta > 0 \). One therefore obtains

\[ \int |R_{N,M}^* g(x)|^2 \, dx \leq C \int \left( \int J_N(x-y) \mathbb{E}(y) \hat{g}(x) \, dy \right) \, dx \]

\[ \leq C \|J_N*\|_1 \|g\|_2 \leq C\|J_N\|_1 \|g\|_2. \]

It follows that

\[ \|R_{0,M}^* g\|_2 \leq C\|g\|_2 \]

and

\[ \|R_{N,M}^* g\|_2 \leq CN^{-\delta/2} \|g\|_2, \quad N \geq 1. \]

We can here replace \( R_{N,M}^* \) by \( R_{N,M} \) and letting \( M \to \infty \) we obtain

\[ \|R_0 g\|_2 \leq C \|g\|_2 \]

and

\[ \|R_N g\|_2 \leq CN^{-\delta/2} \|g\|_2, \quad N \geq 1. \]

Therefore \( \|P_0\| \leq C \) and \( \|P_k\| \leq C 2^{-k\delta/3}, \) \( k \geq 1 \). Hence \( R \) is bounded on \( L^2 \) since

\[ \|R\| \leq \sum_{k=0}^{\infty} \|P_k\|. \]

It remains to study \( J_N \). Setting \( \alpha = \varphi^2 \) and \( \alpha_0 = \varphi_0^2 \) and performing a change of variable we obtain

\[ I_0(x, \omega) = \int e^{i(x-y) \cdot \omega} \mathbb{E}(\xi) \alpha_0(\xi) \, d\xi \]

and

\[ I_N(x, \omega) = \int e^{i(Na-x) \cdot \omega} \mathbb{E}(\xi) \alpha(\xi) \, d\xi N^{-n} \quad N \geq 1. \]

We shall first prove that

\[ (6) \quad |I_0(x, \omega)| \leq C(1 + |x|)^{-n-1} \]
for $|\omega| < 1$. We set

$$I_0(x, \omega) = \int e^{ix \cdot \xi} (e^{\omega |\xi|^a} - 1) a_0(\xi) \, d\xi$$

and

$$I_{0, \varepsilon}(x, \omega) = \int e^{ix \cdot \xi} (e^{\omega |\xi|^a} - 1) \psi(\xi/\varepsilon) a_0(\xi) \, d\xi, \quad 0 < \varepsilon < 1.$$  

It is then clear that it is sufficient to prove (6) with $I_0$ replaced by $I_{0, \varepsilon}$. Since $|I_0(x, \omega)| \leq C$ we may assume that $|x| > 1$. Performing $n + 1$ integrations by parts one obtains

$$I_{0, \varepsilon}(x, \omega) \leq C|x|^{-n-1} \sum_{|\mu| + |\beta| + |\gamma| = n+1} I_{\mu, \beta, \gamma},$$

where

$$I_{\mu, \beta, \gamma} = \int |D^\mu (e^{ix \cdot \xi} - 1)| |D^\beta (\psi(\xi/\varepsilon))| |D^\gamma a_0(\xi)| \, d\xi.$$  

Since

$$|D^\mu (e^{ix \cdot \xi} - 1)| \leq C|\xi|^{a - |\mu|}, \quad |\xi| \leq 1,$$

and

$$|D^\beta (\psi(\xi/\varepsilon))| \leq C|\xi|^{-|\beta|},$$

we obtain

$$I_{\mu, \beta, \gamma} \leq C \int |\xi|^{a - n - 1} |d\xi| \leq C$$

for $\beta = 0$, and

$$I_{\mu, \beta, \gamma} \leq C \int_{|\xi| \leq 1} |\xi|^{a - |\mu| - |\beta|} \, d\xi \leq C \varepsilon |\xi|^{a - n - 1} \leq C$$

for $|\beta| \geq 1$. Hence (6) is proved and it follows that $I_0 \in L^1$.

To study $J_N$ for $N \geq 1$ we shall use the following two lemmas (see P. Sjölin [10] and the references in that paper).

**Lemma 1.** Let $\Omega$ denote an open set in $\mathbb{R}^n$ and let $\varphi \in C_c^\infty(\Omega)$. Assume that $\psi \in C^\infty(\Omega), \psi$ is real-valued and that $|\det(\partial^2 \psi/\partial x_i \partial x_j)| \geq c > 0$ in $\Omega$. Then

$$\left| \int_{\Omega} e^{i \xi \cdot x} \varphi(x) \, dx \right| \leq C(1 + |\xi|)^{-n/2}, \quad \xi \in \mathbb{R}^n, \xi \in \mathbb{R}.$$  

**Lemma 2.** Let $I$ denote an open interval in $\mathbb{R}$, let $g \in C_c^\infty(I)$, $F \in C^\infty(I)$ and assume that $F$ is real-valued and $F' \neq 0$. If $k$ is a positive integer then

$$\int_I e^{iF(x)} g(x) \, dx = \int_I e^{iF(x)} h_k(x) \, dx, \quad g \in C_c^\infty(I).$$

where $h_k$ is a linear combination of functions of the form

$$g^{(s)}(F')^{-k-r} \prod_{q=1}^{\sigma} F(q),$$

with $0 \leq s \leq k$, $0 \leq r \leq k$ and $2 \leq \sigma \leq k + 1$.

It follows from Lemma 1 that we always have

$$|I_N(x, \omega)| \leq C(N^n |\omega|)^{-n/2} N^{-n-2s},$$

If $N^n |\omega| < c_0 N |x|$, i.e. $|\omega| < c_0 |x|/N^{a-1}$, we also have according to Lemma 2,

$$|I_N(x, \omega)| \leq C(1 + N^K |x|^K)^{-1} N^{-n-2s},$$

where $K$ is large.

We shall estimate $J_N$ and begin with the case $0 < |x| < 1/\sqrt{N}$. It is clear that $|I_N(x, \omega)| \leq CN^{-2s}$ and it follows that

$$\int_{|\omega| \leq 1/\sqrt{N}} J_N(x) \, dx \leq CN^{-2s}.$$  

We then consider the case $1/\sqrt{N} < |x| \leq N^{a-1}/c_0$. If $|\omega| < c_0 |x|/N^{a-1}$ we use the estimate (8) and if $|\omega| \geq c_0 |x|/N^{a-1}$ we apply the inequality (7), which implies that

$$|I_N(x, \omega)| \leq C(N|x|)^{-n/2} N^{-n-2s}.$$  

Hence

$$J_N(x) \leq CN^{n/2-2s} |x|^{-n/2}, \quad N^{-1} < |x| \leq N^{a-1}/c_0.$$

We conclude that

$$\int_{N^{-1} < |x| \leq N^{a-1}/c_0} J_N(x) \, dx \leq CN^{n/2-2s} \int_{|\omega| \leq 1/\sqrt{N}} |x|^{-n/2} \, dx \leq CN^{n/2-2s} \int_0^{N^{a-1}/c_0} r^{-n/2} \, dr = CN^{n/2-2s}.$$  

In the case $|x| > N^{a-1}/c_0$ it follows from (8) that

$$J_N(x) \leq CN^{-n-2s} N^{-K} |x|^{-K},$$

and hence

$$\int_{|x| > N^{a-1}/c_0} J_N(x) \, dx \leq CN^{-L},$$

where $L$ is large.

Combining (9), (10) and (11) one obtains

$$\|J_N\| \leq CN^{n/2-2s}, \quad N \geq 1,$$
and hence (5) holds with \( \delta = 2s - an/2 \). The sufficiency in the theorem now follows since \( \delta > 0 \) if \( s > an/4 \).

We next prove that if \( n = 1 \) and (1) holds then \( s \geq a/4 \). First let \( \varphi \in C_0^\infty(\mathbb{R}) \) with \( \text{supp } \varphi \subset (-1, 1) \) and choose \( f \) such that

\[
\hat{f}(\xi) = \varphi(N^{a/2-1}\xi + N^{a/2}).
\]

It is then easy to see that \( \hat{f} \) vanishes outside the interval \([-N-N^{1-a/2}, -N+N^{1-a/2}] \) and it follows that

\[
\|f\|_{H_s} \leq CN^{s+1/2-a/4}.
\]

Setting \( \pi_n = (2\pi)^{-n} \) we have

\[
S_n f(x) = \pi_1 \int e^{ix\eta} e^{i\eta |n|} \varphi(N^{a/2-1}\eta + N^{a/2}) \, d\eta
\]

and performing a change of variable we obtain

\[
S_n f(x) = \pi_1 N^{-a/2} \int e^{i\xi(n/2 - \xi)} \varphi(N^{a/2}(1 - N^{a/2})\xi) \, d\xi
\]

\[
= \pi_1 N^{-a/2} \int e^{it\xi} \varphi(t\xi) \, dt.
\]

Using the Taylor expansion

\[
(1 + y)^a = 1 + c_1 y + c_2 y^2 + O(|y|^3),
\]

where \( c_1 = a \) and \( c_2 \) denotes a positive constant, we obtain

\[
P(\xi) = N^{a/2|n|} \xi - Nx
\]

\[
+ tN^a(1 - c_1 N^{-a/2}\xi + c_2 N^{-a}\xi^2 + O(N^{-3a/2}))
\]

\[
= tN^a - Nx + N^{a/2} \xi - c_1 tN^{a/2} \xi + c_2 t^2 \xi^2 + O(tN^{-a/2}).
\]

We now choose \( t \) such that

\[
N^{1-a/2} \xi = c_1 t N^{a/2}, \quad \text{ i.e. } \quad t = \frac{x}{c_1 N^{a-1}}.
\]

It follows that

\[
|S_n f(x)| = \pi_1 N^{-a/2} \left| \int e^{itG(\xi)} \varphi(\xi) \, d\xi \right|
\]

where

\[
G(\xi) = c_2 t^2 \xi^2 + O(tN^{-a/2}) = \frac{c_2}{c_1} \frac{x}{N^{a-1}} \xi^2 + O\left( \frac{x}{N^{a-1}} N^{-a/2} \right).
\]

Also choose \( x \) such that \((1 - \varepsilon)N^{a-1} \leq x \leq N^{a-1} \), where \( \varepsilon > 0 \) is a small number. One then has

\[
G(\xi) = \frac{c_2}{c_1} \xi^2 + O(\varepsilon) + O(N^{-a/2}) = \frac{c_2}{c_1} \xi^2 + O(\varepsilon)
\]

if \( N \) is large. Setting \( \delta = c_2/c_1 \) and choosing \( \varphi \) such that

\[
\int e^{i\delta \xi} \varphi(\xi) \, d\xi = 0 \neq 0,
\]

we obtain

\[
\int e^{iG(\xi)} \varphi(\xi) \, d\xi = \int e^{it\xi^2} e^{i\delta t(\xi)} \varphi(\xi) \, d\xi
\]

\[
= \int e^{it\xi^2} \varphi(\xi) \, d\xi + \int e^{it\xi^2} (e^{i\delta(\xi)} - 1) \varphi(\xi) \, d\xi
\]

\[
= c_0 + O(\varepsilon)
\]

and it follows that

\[
\left| \int e^{iG(\xi)} \varphi(\xi) \, d\xi \right| \geq c > 0,
\]

if \( \varepsilon \) is chosen sufficiently small. Hence \( |S_n f(x)| \geq c N^{1-a/2} \) and we conclude that \( S_n f(x) \geq c N^{1-a/2} \) for \((1 - \varepsilon)N^{a-1} \leq x \leq N^{a-1} \).

It follows that

\[
\left( \int |S_n f(x)|^2 \, dx \right)^{1/2} \geq c N^{1-a/2}(N^{a-1})^{1/2} = c N^{1/2}.
\]

Invoking (1) and (12) we then obtain \( N^{1/2} \leq CN^{s+1/2-a/4} \), and letting \( N \to \infty \) we necessarily have \( s \geq a/4 \). This completes the proof of the theorem.

References

Note on semigroups generated by positive Rockland operators on graded homogeneous groups

by

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Abstract. Let $L$ be a positive Rockland operator of homogeneous degree $d$ on a graded homogeneous group $G$ and let $p_t$ be the convolution kernels of the semigroup generated by $L$. We prove that if $\tau(x)$ is a Riemannian distance of $x$ from the unit element, then there are constants $c > 0$ and $C$ such that $|p_t(x)| \leq C \exp(-c\tau(x)^{d/(d-1)})$. Moreover, if $G$ is not stratified, more precise estimates of $p_t$ at infinity are given.

1. Introduction. Let $L$ be a positive Rockland operator on a homogeneous group $G$ (cf. [FS]) and let $d$ be the homogeneous degree of $L$ (cf. Section 2).

The operator $L$ satisfies the following subelliptic estimates proved by B. Helffer and J. Nourrigat [HN]: for every multi-index $I$ there are constants $C$ and $k$ such that

$$\|X_I f\|_{L^2(G)} \leq C(\|L^h f\|_{L^2(G)} + \|f\|_{L^2(G)}), \quad f \in C_c^\infty(G).$$

Theorem 4.25 of [FS] asserts that the closure $-\bar{L}$ of the essentially selfadjoint operator $-L$ is the infinitesimal generator of a semigroup of linear operators on $L^2(G)$ which has the form

$$T_t f = f * p_t, \quad t > 0,$$

where the $p_t$ belong to the Schwartz space $S(G)$.

The homogeneity of $L$ implies

$$p_t(x) = t^{-Q/d} p_1(\delta_{t^{-1/d}} x),$$

where $Q$ is the homogeneous dimension of $G$ and $\delta_t$ is the family of dilations associated with $G$ (cf. Section 2).

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