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# When is there a discontinuous homomorphism from $L^1(G)$?

by

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**Abstract.** Let $A$ be an $A^*$-algebra with enveloping $C^*$-algebra $C^*(A)$. We show that, under certain conditions, a homomorphism from $C^*(A)$ into a Banach algebra is continuous if and only if its restriction to $A$ is continuous. We apply this result to the question in the title.

**Introduction.** One of the central questions in automatic continuity is the following: For which Banach algebras $A$ is there a Banach algebra $B$ and a discontinuous homomorphism $\theta : A \to B$?

A fundamental result obtained independently by H. G. Dales and J. Esterle (see [Dal] for a streamlined exposition) asserts that if $X$ is a locally compact Hausdorff space and if the continuum hypothesis holds, then there is a discontinuous homomorphism from $C_0(X)$ into a Banach algebra if and only if $X$ is infinite. Surprisingly, this result cannot be proved within the confinements of Zermelo–Fraenkel set theory and the axiom of choice ([D-W]). In this note, we shall not delve into these set theoretic intricacies and assume throughout that the continuum hypothesis holds.

In [A-D], E. Albrecht and Dales conjectured the following non-commutative version of the Dales–Esterle theorem:

CONJECTURE A. Let $A$ be a $C^*$-algebra. Then there is a discontinuous homomorphism from $A$ into a Banach algebra if and only if there is $n \in \mathbb{N}$ such that $A$ has an infinite number of inequivalent, $n$-dimensional, irreducible $^*$-representations.

Albrecht and Dales were able to prove the "if" part of their conjecture and to confirm the "only if" part for so-called $AW^*M$-algebras, a class of $C^*$-algebras containing all commutative $C^*$-algebras and all closed ideals of $AW^*$-algebras.

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Let $G$ be a locally compact, abelian group. Then $L^1(G)$ is a Banach function algebra whose character space can be identified with $\widehat{G}$, the dual group of $G$. From the general normability theory of complex algebras ([Dal, Chapter 5]), it follows that there is a discontinuous homomorphism from $L^1(G)$ if and only if $\widehat{G}$ is infinite.

This fact and the Albrecht–Dales conjecture motivate the following conjecture, which was stated in [Run].

CONJECTURE B. Let $G$ be a locally compact group. Then there is a discontinuous homomorphism from $L^1(G)$ into a Banach algebra if and only if there is $n \in \mathbb{N}$ such that $G$ has an infinite number of inequivalent, $n$-dimensional, irreducible unitary representations.

In [Run], we verified the "if" part of this conjecture for groups $G$ having an open subgroup $H \in [FIA]^-$ such that $[G : H] < \infty$.

Let $G$ be a locally compact group. If both Conjecture A and Conjecture B are correct, then there is a discontinuous homomorphism from $L^1(G)$ if and only if there is one from $C^*(G)$. This leads to the following question: Given a discontinuous homomorphism $\theta$ from $C^*(G)$ into a Banach algebra, is $\theta|L^1(G)$ also discontinuous?

In this note, we investigate this question from a more abstract point of view: Let $A$ be an $A^*$-algebra with enveloping $C^*$-algebra $C^*(A)$, and let $\theta$ be a discontinuous homomorphism from $C^*(A)$ into a Banach algebra. Is $\theta|A$ discontinuous? We prove a fairly general theorem which asserts that for many $A^*$-algebras $A$ a homomorphism $\theta$ from $C^*(A)$ into a Banach algebra is continuous if and only if $\theta|A$ is continuous. We use this theorem to confirm the "if" part of Conjecture B for certain groups $G$. In particular, we obtain a generalization of [Run, Corollary 5.2].

**1. Homomorphisms from $A^*$-algebras.** A *$C^*$-norm* on a *-algebra $A$ is an algebra norm $|\cdot|$ on $A$ satisfying

$$|a^*a| = |a|^2 \qquad (a \in A).$$

A Banach *-algebra endowed with a $C^*$-norm is called an *$A^*$-algebra*. The most prominent examples of $A^*$-algebras are $C^*$- and group algebras. Every $A^*$-algebra $A$ has a largest $C^*$-norm. When writing $|\cdot|$ we always mean this largest $C^*$-norm; we denote the original Banach algebra norm on $A$ by $\|\cdot\|$. The completion of $A$ with respect to $|\cdot|$ is called the *enveloping $C^*$-algebra* of $A$ and denoted by $C^*(A)$. In case $A = L^1(G)$ for a locally compact group $G$, we have $C^*(A) = C^*(G)$.

Let $A$ be an $A^*$-algebra. We let $\mathrm{Prim}_*(A)$ denote the collection of the kernels of the (topologically) irreducible *-representations of $A$; $\mathrm{Prim}_*(A)$ is naturally endowed with the Jacobson topology. In case $A$ is a $C^*$-algebra, $\mathrm{Prim}_*(A)$ can be identified with $\mathrm{Prim}(A)$, the space of primitive ideals of $A$.



Recall that $A$ is said to be

- *hermitian* if $\sigma_A(a) \subset \mathbb{R}$ for all self-adjoint $a \in A$,
- **-regular* if the map $\mathrm{Prim}(C^*(A)) \ni P \mapsto P \cap A$ is a homeomorphism between $\mathrm{Prim}(C^*(A))$ and $\mathrm{Prim}_*(A)$,
- *locally regular* if there is a dense subset $S$ of the set of self-adjoint elements of $A$ such that for all $a \in S$ the Banach subalgebra of $A$ generated by $a$ is regular.

Remarks. 1. The question whether the map $\mathrm{Prim}(C^*(A)) \ni P \mapsto P \cap A$ is a homeomorphism seems to have been investigated for the first time in [B-L-Sch-V] (in the context of group algebras).

2. Local regularity was introduced by B. A. Barnes in [Bar 1]. It implies *-regularity ([Bar 1, Theorem 4.3]).

Suppose $F$ is a closed subset of $\mathrm{Prim}_*(A)$. We call $F$ a *set of synthesis* for $A$ if $\ker(F)$ is the only closed ideal of $A$ whose hull equals $F$.

For any subset $S$ of $C^*(A)$ we write $S^-$ for the closure of $S$ in $C^*(A)$ with respect to $|\cdot|$. If $S \subset A$ we denote the closure of $S$ with respect to $\|\cdot\|$ by $S^=$.

LEMMA 1.1. *Let $A$ be a hermitian, locally regular $A^*$-algebra, and let $I$ be a (not necessarily closed) ideal of $C^*(A)$ whose hull is a set of synthesis for $A$. Then $(I \cap A)^= = I^- \cap A$.*

Proof. First, note that $I^- \cap A$ is $\|\cdot\|$-closed. Further, it follows easily from the *-regularity of $A$ that $I^-$ and $I^- \cap A$ have the same hull, say $F$. By [H-K-K, Lemma 1.2], there is an ideal $J(F)$ of $A$ whose hull equals $F$ and which is contained in every ideal of $A$ with this hull. This means that $J(F) \subset I^- \cap A$, and since $F$ is a set of synthesis for $A$, we have $J(F)^= = I^- \cap A$. From the construction of $J(F)$ in the proof of [H-K-K, Lemma 1.2], it is evident that $J(F)$ is contained in the Pedersen ideal $\mathcal{P}(I^-)$ of $I^-$. Since by [Ped, Theorem 5.6.1], $\mathcal{P}(I^-) \subset I$, we have

$$J(F) \subset \mathcal{P}(I^-) \cap A \subset I \cap A \subset I^- \cap A.$$

Taking the closures with respect to $\|\cdot\|$, we obtain

$$I^- \cap A = J(F)^= \subset (I \cap A)^= \subset I^- \cap A,$$

i.e. $(I \cap A)^= = I^- \cap A$ as claimed. ■

For the main result of this section recall two concepts from automatic continuity.

Let $A$ and $B$ be Banach algebras, and let $\theta : A \to B$ be a homomorphism. Then

$$\mathcal{S}(\theta) := \{b \in B : \text{there is a sequence } \{a_n\}_{n=1}^\infty \text{ in } A \text{ such that } a_n \to 0 \text{ and } \theta(a_n) \to b\}$$

is called the *separating space* of $\theta$. It is easy to see that $\mathcal{S}(\theta)$ is a closed ideal of the closure of $\theta(A)$, and obviously, by the closed graph theorem, $\theta$ is continuous if and only if $\mathcal{S}(\theta) = \{0\}$. For a comprehensive account of the properties of $\mathcal{S}(\theta)$, see the monographs [Dal] and [Sin 2].

Equally important is the *continuity ideal* of $\theta$, defined as

$$\mathcal{I}(\theta) := \{a \in A : \theta(a)\mathcal{S}(\theta) = \mathcal{S}(\theta)\theta(a) = \{0\}\}.$$

It is easy to see that $\mathcal{I}(\theta)$ is indeed an ideal of $A$. The continuity ideal is of special importance if $A$ is a $C^*$-algebra ([Sin 1]). In particular, $\mathcal{I}(\theta)^-$ has finite codimension in this case.

THEOREM 1.2. *Let $A$ be a hermitian, locally regular $A^*$-algebra with the following properties:*

(I) *Every closed ideal of $A$ with finite codimension has a bounded left approximate identity.*

(II) *Every finite subset $F$ of* $\mathrm{Prim}_*(A)$ *such that each $P \in F$ has finite codimension is a set of synthesis for $A$.*

*Suppose $\theta$ is a homomorphism from $C^*(A)$ into a Banach algebra. Then the following are equivalent:*

(i) *$\theta$ is continuous.*

(ii) *$\theta|A$ is continuous.*

(iii) *$\mathcal{I}(\theta) \cap A$ is closed (with respect to $\|\cdot\|$).*

Proof. The implications from (i) to (ii) and from (ii) to (iii) are trivial.

Suppose $I := \mathcal{I}(\theta) \cap A$ is $\|\cdot\|$-closed. Obviously, $\mathcal{I}(\theta)^-$ is a left Banach $I$-module. From Lemma 1.1, we conclude that $I$ has finite codimension in $A$. Hence, by assumption, $I$ has a bounded left approximate identity, say $\{e_\alpha\}$. Note that

$$\begin{aligned} I^- &= (\mathcal{I}(\theta) \cap A)^- \\ &= (\mathcal{I}(\theta)^- \cap A)^- && \text{by Lemma 1.1} \\ &= \mathcal{I}(\theta)^- && \text{by [Bar 1, Theorem 4.2].} \end{aligned}$$

As a consequence, $\{e_\alpha\}$ is also a bounded approximate identity for the left Banach $I$-module $\mathcal{I}(\theta)^-$. Let $c_0(\mathcal{I}(\theta)^-)$ denote the left Banach $I$-module of all sequences $\{x_n\}_{n=1}^\infty$ in $\mathcal{I}(\theta)^-$ such that $|x_n| \to 0$. It is easy to see that $\{e_\alpha\}$ is a bounded approximate identity for $c_0(\mathcal{I}(\theta)^-)$ as well. Let $\{x_n\}_{n=1}^\infty$



be a sequence belonging to $c_0(\mathcal{I}(\theta)^-)$. The module version of Cohen's factorization theorem ([B-D, Theorem 11.10]) then asserts that there is $a \in I$ and a sequence $\{y_n\}_{n=1}^\infty$ in $\mathcal{I}(\theta)^-$ such that

$$x_n = ay_n \quad (n \in \mathbb{N}) \quad \text{and} \quad |y_n| \to 0.$$

Since $I \subset \mathcal{I}(\theta)$, this means that

$$\theta(x_n) = \theta(ay_n) \to 0,$$

i.e. $\theta|\mathcal{I}(\theta)^-$ is continuous. Since $\mathcal{I}(\theta)^-$ has finite codimension in $C^*(A)$, this implies the continuity of $\theta$. ∎

Remarks. 1. Assumption (I) of Theorem 1.2 is automatically satisfied when $A$ is amenable ([Hel, Proposition VII.2.31]). If $G$ is a locally compact group, then $L^1(G)$ satisfies (I) if and only if $G$ is amenable.

2. We do not know if in Theorem 1.2 the demand that $A$ be hermitian and locally regular cannot be relaxed. Weakening this hypothesis would require a substitute for [H-K-K, Lemma 1.2] in the proof of Lemma 1.1.

**2. Applications to $L^1(G)$.** We now wish to apply Theorem 1.2 to the special case of a group algebra.

Recall the definitions of some classes of locally compact groups (compare [Pal]):

[Moore]: *Moore groups.* Groups all of whose irreducible, unitary representations are finite-dimensional.

[MAP]: *Maximally almost periodic groups.* Groups $G$ such that the finite-dimensional, irreducible unitary representations of $G$ separate its points.

[PG]: *Groups with polynomial growth.* Groups $G$ such that for each compact neighborhood $K$ of 1 there is $k \in \mathbb{N}$ such that

$$|K^n| = O(n^k) \qquad (n \in \mathbb{N})$$

($|K^n|$ denoting Haar measure of $K^n$).

[Her]: *Hermitian groups.* Groups $G$ such that $L^1(G)$ is hermitian.

Our first result on homomorphisms from group algebras is a rather straightforward application of Theorem 1.2.

THEOREM 2.1. *Let $G \in$ [PG] $\cap$ [Her], and let $\theta$ be a homomorphism from $C^*(G)$ into a Banach algebra. Then $\theta$ is continuous if and only if $\theta|L^1(G)$ is continuous. In particular, if there is $n \in \mathbb{N}$ such that $G$ has an infinite number of inequivalent, $n$-dimensional, irreducible unitary representations, then there is a discontinuous homomorphism from $L^1(G)$ into a Banach algebra.*

Proof. Obviously, $L^1(G)$ is hermitian. Also, by [Bar, Theorem 4.1], $L^1(G)$ is locally regular. Further, $L^1(G)$ is amenable by [B-L-Sch-V], and finally, by [Bar 2, Theorem 12], every finite subset $F$ of $\mathrm{Prim}_*(L^1(G))$ such that each $P \in F$ has finite codimension is a set of synthesis for $L^1(G)$. All in all, $L^1(G)$ satisfies the assumptions of Theorem 1.2. This proves the first part of the present theorem.

Now, suppose there is $n \in \mathbb{N}$ such that $G$ has an infinite number of inequivalent, $n$-dimensional, irreducible unitary representations. Then $C^*(G)$ has an infinite number of inequivalent, $n$-dimensional, irreducible $^*$-representations. By [A-D, Theorem 2.5], there is a discontinuous homomorphism from $C^*(G)$ into a Banach algebra; its restriction to $L^1(G)$ is discontinuous by the first part of the theorem. ∎

Remarks. 1. The second part of Theorem 2.1 clearly subsumes [Run, Corollary 5.2].

2. For abelian $G$, the first part of Theorem 2.1 yields in particular that if there is a discontinuous homomorphism from $C_0(\widehat{G})$ into a Banach algebra, then there is one from $L^1(G)$. This rather special case was already proved in [Lau], several years before the question whether there are discontinuous homomorphisms from commutative $C^*$-algebras was settled.

Concluding, we wish to confirm the "if" part of Conjecture B for another class of locally compact groups.

If $G$ is a locally compact group, we use $G_1$ to denote the component of $G$ containing 1. It is easy to see that $G_1$ is a closed, normal subgroup of $G$. If $G/G_1$ is compact, we call $G$ *almost connected*.

THEOREM 2.2. *Let $G$ be an almost connected group, and suppose there is $n \in \mathbb{N}$ such that $G$ has an infinite number of inequivalent, $n$-dimensional, irreducible unitary representations. Then there is a discontinuous homomorphism from $L^1(G)$ into a Banach algebra.*

Proof. Let $N$ denote the intersection of the kernels of all $n$-dimensional, irreducible unitary representations of $G$. Clearly, $N$ is a closed, normal subgroup. It is easy to see that the class of almost connected groups is stable under taking quotients, i.e. $G/N$ is almost connected as well. Moreover, $G/N \in$ [MAP] by the definition of $N$. By [G-M, Theorem 2.18], this means that $G/N \in$ [Moore]. By Theorem 2.1 (or alternatively by [Run, Corollary 5.2]) there is a discontinuous homomorphism $\theta$ from $L^1(G/N)$ into a Banach algebra. The canonical map $\pi : L^1(G) \to L^1(G/N)$ being open, we conclude that $\theta \circ \pi$ is discontinuous. ∎

Remarks. 1. Let [X] be *any* class of locally compact groups such that

(a) [X] is stable under the formation of quotients, and
(b) $[\mathrm{X}] \cap [\mathrm{MAP}] \subset [\mathrm{PG}] \cap [\mathrm{Her}]$.



Then, if $G \in$ [X], and if there is $n \in \mathbb{N}$ such that $G$ has an infinite number of inequivalent, $n$-dimensional, irreducible unitary representations, the same argument as in the proof of Theorem 2.2 shows that there is a discontinuous homomorphism from $L^1(G)$.

2. Both Theorem 2.1 and Theorem 2.2 are certainly not optimal: Let $G$ be a locally compact group having an infinite number of inequivalent, 1-dimensional, irreducible unitary representations, i.e. characters. Then $N$ defined as in the proof of Theorem 2.2 is the closed commutator subgroup of $G$. Consequently, $G/N$ is abelian with $\widehat{G/N}$ infinite. Therefore, there is a discontinuous homomorphism from $L^1(G/N)$ and hence one from $L^1(G)$. For $r \in \mathbb{N}$, let $\mathbb{F}_r$ denote the free group of $r$ generators. The preceding argument shows that there is a discontinuous homomorphism from $\ell^1(\mathbb{F}_r)$ although $\mathbb{F}_r$ is neither almost connected, nor, for $r \geq 2$, does it belong to $[\mathrm{PG}] \cap [\mathrm{Her}]$.

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