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Tauberian theorems for Cesàro summable double sequences

by

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Abstract. Let $(s_{jk} : j,k = 0,1,\ldots)$ be a double sequence of real numbers which is summable $(C,1,1)$ to a finite limit. We give necessary and sufficient conditions under which (s_{jk}) converges in Pringsheim's sense. These conditions are satisfied if (s_{jk}) is slowly decreasing in certain senses defined in this paper. Among other things we deduce the following Tauberian theorem of Landau and Hardy type: If (s_{jk}) is summable $(C,1,1)$ to a finite limit and there exist constants $n_1 > 0$ and H such that

$$jk(s_{jk} - s_{j-1,k} - s_{j-1,k} + s_{j-1,k-1}) \ge -H,$$
$$j(s_{jk} - s_{j-1,k}) \ge -H \quad \text{and} \quad k(s_{jk} - s_{j,k-1}) \ge -H$$

whenever $j,k > n_1$, then (s_{jk}) converges. We always mean convergence in Pringsheim's sense. Our method is suitable to obtain analogous Tauberian results for double sequences of complex numbers or for those in an ordered linear space over the real numbers.

1. Preliminary results for single sequences.

Let $(s_k : k = 0,1,\ldots)$ be a single sequence of real numbers. A classical one-sided Tauberian theorem of Landau [2] asserts that if (s_k) is summable $(C,1)$ to a finite number s and there exists a constant H such that

$$k(s_k - s_{k-1}) \ge -H \quad (k = 1,2,\ldots), \tag{1.1}$$

then (s_k) converges to s.

Following Schmidt [5], we say that (s_k) is *slowly decreasing* if for each $\varepsilon > 0$ there exist $n_1 > 0$ and $\lambda > 1$ such that

$$s_k - s_n \ge -\varepsilon \quad \text{whenever} \quad n_1 < n < k \le \lambda n, \tag{1.2}$$

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or equivalently

$$\lim_{\lambda\downarrow 1}\liminf_{n\to\infty}\min_{n<k\le\lambda n}(s_k-s_n)\ge 0.$$

Clearly, (1.1) is a particular case of (1.2).

Hardy [1, Theorem 68] proved that if a sequence (s_k) is summable $(C,1)$ to a finite number s and (s_k) is slowly decreasing, then (s_k) converges to s.

In a recent paper [4], we proved the following necessary and sufficient Tauberian conditions, under which convergence follows from summability $(C,1)$:

$$\limsup_{\lambda\downarrow 1}\liminf_{n\to\infty}\frac{1}{\lambda_n-n}\sum_{n<k\le\lambda_n}(s_k-s_n)\ge 0 \tag{1.3}$$

and

$$\limsup_{\lambda\uparrow 1}\liminf_{n\to\infty}\frac{1}{n-\lambda_n}\sum_{\lambda_n<k\le n}(s_n-s_k)\ge 0, \tag{1.4}$$

where $\lambda_n:=[\lambda n]$ and $[\cdot]$ denotes the integral part. Both conditions are clearly satisfied if (s_k) is slowly decreasing.

The symmetric counterparts of conditions (1.3) and (1.4) are those when "lim sup" and "lim inf" are interchanged on the left-hand sides, while "$\ge$" is changed for "$\le$".

Next, let (s_k) be a sequence of complex numbers which is summable $(C,1)$ to a finite limit. As is known, Landau's theorem remains valid if condition (1.1) is replaced by

$$k|s_k-s_{k-1}|\le H\qquad(k=1,2,\ldots)$$

(see, e.g., [6, Vol. 1, p. 78]). In [4], we proved that (s_k) converges if and only if one of the following conditions is satisfied:

$$\liminf_{\lambda\downarrow 1}\limsup_{n\to\infty}\left|\frac{1}{\lambda_n-n}\sum_{n<k\le\lambda_n}(s_k-s_n)\right|=0,$$

$$\liminf_{\lambda\uparrow 1}\limsup_{n\to\infty}\left|\frac{1}{n-\lambda_n}\sum_{\lambda_n<k\le n}(s_n-s_k)\right|=0.$$

In the general setting of an ordered linear space $(X,\le)$ over the real numbers, the notions of convergence and slow decrease were introduced by Maddox [3]. He extended Hardy's theorem to this case. In [4], we also proved a more general Tauberian theorem for ordered spaces.

2. Definitions for double sequences. Our goal is to extend the above results from single to double sequences. We use the convergence notion in Pringsheim's sense. More exactly, we say that a double sequence $(s_{jk}:j,k=0,1,\ldots)$ converges to a finite number s if s_{jk} tends to s as both j and k tend to ∞, independently of one another. (See, e.g., [6, Vol. 2, pp. 302–303].)

We define the *means* $(C,1,1)$ of (s_{jk}) by

$$\sigma_{mn}:=\sigma_{mn}^{11}:=\frac{1}{(m+1)(n+1)}\sum_{j=0}^{m}\sum_{k=0}^{n}s_{jk}\qquad(m,n=0,1,\ldots).$$

The *means* $(C,1,0)$ and $(C,0,1)$ are defined respectively by

$$\sigma_{mn}^{10}:=\frac{1}{m+1}\sum_{j=0}^{m}s_{jn}\quad\text{and}\quad\sigma_{mn}^{01}:=\frac{1}{n+1}\sum_{k=0}^{n}s_{mk}.$$

We say that (s_{jk}) is *summable* $(C,1,1)$ to a finite limit s if the means σ_{mn}^{11} converge to s. The notion of summability $(C,1,0)$ or $(C,0,1)$ is defined analogously.

We say that (s_{jk}) is *slowly decreasing in sense* $(1,1)$ if for each $\varepsilon>0$ there exist $n_1>0$ and $\lambda>1$ such that

$$s_{jk}-s_{mk}-s_{jn}+s_{mn}\ge-\varepsilon$$
$$\text{whenever}\quad n_1<m<j\le\lambda_m\ \text{and}\ n_1<n<k\le\lambda_n.$$

An equivalent reformulation is the following:

$$\lim_{\lambda\downarrow 1}\liminf_{m,n\to\infty}\min_{m<j\le\lambda_m,n<k\le\lambda_n}(s_{jk}-s_{mk}-s_{jn}+s_{mn})\ge 0. \tag{2.1}$$

Obviously, (2.1) implies

$$\lim_{\lambda\uparrow 1}\liminf_{m,n\to\infty}\min_{\lambda_m<j\le m,\lambda_n<k\le n}(s_{mn}-s_{jn}-s_{mk}+s_{jk})\ge 0, \tag{2.2}$$

and vice versa.

We say that (s_{jk}) satisfies *Landau's condition in sense* $(1,1)$ if there exist constants $n_1>0$ and H such that

$$jk(s_{jk}-s_{j-1,k}-s_{j,k-1}+s_{j-1,k-1})\ge-H\quad\text{whenever}\quad j,k>n_1. \tag{2.3}$$

Clearly, (2.1) is a consequence of (2.3).

Furthermore, we say that (s_{jk}) is *slowly decreasing in sense* $(1,0)$ if

$$\lim_{\lambda\downarrow 1}\liminf_{m,n\to\infty}\min_{m<j\le\lambda_m}(s_{jn}-s_{mn})\ge 0; \tag{2.4}$$

and that (s_{jk}) satisfies *Landau's condition in sense* $(1,0)$ if there exist $n_1>0$ and H such that

$$j(s_{jn}-s_{j-1,n})\ge-H\quad\text{whenever}\quad j,n>n_1. \tag{2.5}$$

Again, (2.4) is an easy consequence of (2.5).

The slowly decreasing property as well as Landau's condition in sense $(0,1)$ are defined in an analogous manner. In particular, we say that (s_{jk})

satisfies *Landau's condition in sense* $(0,1)$ if

$$k(s_{mk}-s_{m,k-1}) \ge -H \quad \text{whenever} \quad m,k>n_1. \tag{2.6}$$

3. Main results. In Sections 3 and 4, we assume that (s_{jk}) is a double sequence of real numbers. We will prove the following one-sided Tauberian theorems.

THEOREM 1. *If (s_{jk}) is summable $(C,1,1)$ to a finite limit s, then the limit*

$$\lim_{j,k\to\infty} s_{jk}=s \quad \textit{exists} \tag{3.1}$$

if and only if

$$\limsup_{\lambda\downarrow 1}\liminf_{m,n\to\infty}\frac{1}{(\lambda_m-m)(\lambda_n-n)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}(s_{jk}-s_{mn})\ge 0 \tag{3.2}$$

and

$$\limsup_{\lambda\uparrow 1}\liminf_{m,n\to\infty}\frac{1}{(m-\lambda_m)(n-\lambda_n)}\sum_{j=\lambda_m+1}^{m}\sum_{k=\lambda_n+1}^{n}(s_{mn}-s_{jk})\ge 0; \tag{3.3}$$

in which case we necessarily have

$$\lim_{m,n\to\infty}\frac{1}{(\lambda_m-m)(\lambda_n-n)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}(s_{jk}-s_{mn})=0 \tag{3.4}$$

for all $\lambda>1$, and

$$\lim_{m,n\to\infty}\frac{1}{(m-\lambda_m)(n-\lambda_n)}\sum_{j=\lambda_m+1}^{m}\sum_{k=\lambda_n+1}^{n}(s_{mn}-s_{jk})=0 \tag{3.5}$$

for all $0<\lambda<1$.

A few comments are appropriate here.

(i) Conditions (3.2) and (3.3) can be reformulated as follows: For all $\varepsilon>0$ and $\lambda_1>1$ there exist $n_1>0$ and λ, $1<\lambda<\lambda_1$, such that for all $m,n>n_1$ we have

$$\frac{1}{(\lambda_m-m)(\lambda_n-n)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}(s_{jk}-s_{mn})\ge -\varepsilon \tag{3.6}$$

and possibly with another λ, $1<\lambda<\lambda_1$, we have

$$\frac{1}{(m-\lambda_m^{-1})(n-\lambda_n^{-1})}\sum_{j=\lambda_m^{-1}+1}^{m}\sum_{k=\lambda_n^{-1}+1}^{n}(s_{mn}-s_{jk}) \ge -\varepsilon, \quad \text{where} \quad \lambda_m^{-1}:=[\lambda^{-1}m]. \tag{3.7}$$

(ii) Conditions (3.2) and (3.3) are satisfied if (s_{jk}) is slowly decreasing in senses $(1,1)$, $(1,0)$ and $(0,1)$. This follows immediately from the representation

$$\begin{aligned}\frac{1}{(\lambda_m-m)(\lambda_n-n)}&\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}(s_{jk}-s_{mn})\\ &=\frac{1}{(\lambda_m-m)(\lambda_n-n)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}(s_{jk}-s_{mk}-s_{jn}+s_{mn})\\ &\quad+\frac{1}{\lambda_m-m}\sum_{j=m+1}^{\lambda_m}(s_{jn}-s_{mn})+\frac{1}{\lambda_n-n}\sum_{k=n+1}^{\lambda_n}(s_{mk}-s_{mn}).\end{aligned} \tag{3.8}$$

Thus, the following two corollaries of Theorem 1 are obvious.

COROLLARY 1. *If (s_{jk}) is summable $(C,1,1)$ to a finite limit and (s_{jk}) is slowly decreasing in senses $(1,1)$, $(1,0)$ and $(0,1)$, then (s_{jk}) converges.*

COROLLARY 2. *If (s_{jk}) is summable $(C,1,1)$ to a finite limit and conditions (2.3), (2.5) and (2.6) are satisfied, then (s_{jk}) converges.*

PROBLEM 1. We guess that there exists a double sequence (s_{jk}) which is summable $(C,1,1)$ to a finite limit, conditions (2.3) and (2.5) are satisfied, but (2.6) is not satisfied and (s_{jk}) fails to converge. We are unable to construct such an example.

(iii) The symmetric counterparts of conditions (3.2) and (3.3) are the following:

$$\liminf_{\lambda\downarrow 1}\limsup_{m,n\to\infty}\frac{1}{(\lambda_m-m)(\lambda_n-n)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}(s_{jk}-s_{mn})\le 0 \tag{3.9}$$

and

$$\liminf_{\lambda\uparrow 1}\limsup_{m,n\to\infty}\frac{1}{(m-\lambda_m)(n-\lambda_n)}\sum_{j=\lambda_m+1}^{m}\sum_{k=\lambda_n+1}^{n}(s_{mn}-s_{jk})\le 0. \tag{3.10}$$

Assume that (s_{jk}) is summable $(C,1,1)$ to a finite limit s. Analogously to Theorem 1, one can prove that condition (3.1) is satisfied if and only if (3.9) and (3.10) are satisfied. Consequently, if conditions (3.2), (3.3) are satisfied, then conditions (3.9), (3.10) are also satisfied, and vice versa.

PROBLEM 2. It seems likely that conditions (3.2) and (3.3) are not related to each other. Nevertheless, we are unable to construct an example of a double sequence (s_{jk}) which is summable $(C,1,1)$, condition (3.2) is satisfied, but (3.3) is violated, and consequently (s_{jk}) fails to converge.

THEOREM 2. *If* (s_{jk}) *is summable* $(C,1,0)$ *to a finite limit* s, *then* s_{jk} *converges to* s *if and only if*

$$\limsup_{\lambda\downarrow 1}\liminf_{m,n\to\infty}\frac{1}{\lambda_m-m}\sum_{j=m+1}^{\lambda_m}(s_{jn}-s_{mn})\ge 0 \tag{3.11}$$

and

$$\limsup_{\lambda\uparrow 1}\liminf_{m,n\to\infty}\frac{1}{m-\lambda_m}\sum_{j=\lambda_m+1}^{m}(s_{mn}-s_{jn})\ge 0; \tag{3.12}$$

in which case we necessarily have

$$\lim_{m,n\to\infty}\frac{1}{\lambda_m-m}\sum_{j=m+1}^{\lambda_m}(s_{jn}-s_{mn})=0 \tag{3.13}$$

for all $\lambda>1$, *and*

$$\lim_{m,n\to\infty}\frac{1}{m-\lambda_m}\sum_{j=\lambda_m+1}^{m}(s_{mn}-s_{jn})=0 \tag{3.14}$$

for all $0<\lambda<1$.

Comments analogous to those made after Theorem 1 are appropriate here, as well. We only state the following two corollaries of Theorem 2.

COROLLARY 3. *If* (s_{jk}) *is summable* $(C,1,0)$ *to a finite limit and* (s_{jk}) *is slowly decreasing in sense* $(1,0)$, *then* (s_{jk}) *converges.*

COROLLARY 4. *If* (s_{jk}) *is summable* $(C,1,0)$ *to a finite limit and condition* (2.5) *is satisfied, then* (s_{jk}) *converges.*

PROBLEM 3. Again, it seems likely that conditions (3.11) and (3.12) are not related to each other. But we are unable to present a counterexample (cf. Problem 2).

The symmetric counterparts of Theorem 2 and Corollaries 3, 4 are also valid, when we consider summability $(C,0,1)$ instead of $(C,1,0)$.

4. Proofs of Theorems 1 and 2. We begin with two representations of the difference $s_{mn}-\sigma_{mn}$, interesting in themselves.

LEMMA 1. (i) *If* $\lambda>1$, $\lambda_m>m$, *and* $\lambda_n>n$, *then*

$$\begin{aligned}s_{mn}-\sigma_{mn}&=\frac{(\lambda_m+1)(\lambda_n+1)}{(\lambda_m-m)(\lambda_n-n)}(\sigma_{\lambda_m,\lambda_n}-\sigma_{\lambda_m,n}-\sigma_{m,\lambda_n}+\sigma_{mn})\\&\quad+\frac{\lambda_m+1}{\lambda_m-m}(\sigma_{\lambda_m,n}-\sigma_{mn})+\frac{\lambda_n+1}{\lambda_n-n}(\sigma_{m,\lambda_n}-\sigma_{mn})\\&\quad-\frac{1}{(\lambda_m-m)(\lambda_n-n)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}(s_{jk}-s_{mn}).\end{aligned}\tag{4.1}$$

(ii) *If* $0<\lambda<1$, $\lambda_m<m$, *and* $\lambda_n<n$, *then*

$$\begin{aligned}s_{mn}-\sigma_{mn}&=\frac{(\lambda_m+1)(\lambda_n+1)}{(m-\lambda_m)(n-\lambda_n)}(\sigma_{mn}-\sigma_{\lambda_m,n}-\sigma_{m,\lambda_n}+\sigma_{\lambda_m,\lambda_n})\\&\quad+\frac{\lambda_m+1}{m-\lambda_m}(\sigma_{mn}-\sigma_{\lambda_m,n})+\frac{\lambda_n+1}{n-\lambda_n}(\sigma_{mn}-\sigma_{m,\lambda_n})\\&\quad+\frac{1}{(m-\lambda_m)(n-\lambda_n)}\sum_{j=\lambda_m+1}^{m}\sum_{k=\lambda_n+1}^{n}(s_{mn}-s_{jk}).\end{aligned}\tag{4.2}$$

Proof. (i) By definition,

$$\begin{aligned}&\sigma_{\lambda_m,\lambda_n}-\sigma_{\lambda_m,n}-\sigma_{m,\lambda_n}+\sigma_{mn}\\&=\sum_{j=0}^{m}\sum_{k=0}^{n}s_{jk}\Big(\frac{1}{(\lambda_m+1)(\lambda_n+1)}-\frac{1}{(\lambda_m+1)(n+1)}\\&\qquad-\frac{1}{(m+1)(\lambda_n+1)}+\frac{1}{(m+1)(n+1)}\Big)\\&\quad+\sum_{j=m+1}^{\lambda_m}\sum_{k=0}^{n}s_{jk}\Big(\frac{1}{(\lambda_m+1)(\lambda_n+1)}-\frac{1}{(\lambda_m+1)(n+1)}\Big)\\&\quad+\sum_{j=0}^{m}\sum_{k=n+1}^{\lambda_n}s_{jk}\Big(\frac{1}{(\lambda_m+1)(\lambda_n+1)}-\frac{1}{(m+1)(\lambda_n+1)}\Big)\\&\quad+\frac{1}{(\lambda_m+1)(\lambda_n+1)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}s_{jk}\\&=\frac{(\lambda_m-m)(\lambda_n-n)}{(\lambda_m+1)(\lambda_n+1)}\sigma_{mn}-\frac{\lambda_n-n}{(\lambda_m+1)(\lambda_n+1)(n+1)}\sum_{j=m+1}^{\lambda_m}\sum_{k=0}^{n}s_{jk}\\&\quad-\frac{\lambda_m-m}{(\lambda_m+1)(\lambda_n+1)(m+1)}\sum_{j=0}^{m}\sum_{k=n+1}^{\lambda_n}s_{jk}\\&\quad+\frac{1}{(\lambda_m+1)(\lambda_n+1)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_m}s_{jk}.\end{aligned}$$

Hence

$$(4.3)\qquad \frac{(\lambda_m+1)(\lambda_n+1)}{(\lambda_m-m)(\lambda_n-n)}(\sigma_{\lambda_m,\lambda_n}-\sigma_{\lambda_m,n}-\sigma_{m,\lambda_n}+\sigma_{mn})$$
$$=\sigma_{mn}-\frac{1}{(\lambda_m-m)(n+1)}\sum_{j=m+1}^{\lambda_m}\sum_{k=0}^{n}s_{jk}$$
$$-\frac{1}{(m+1)(\lambda_n-n)}\sum_{j=0}^{m}\sum_{k=n+1}^{\lambda_n}s_{jk}$$
$$+\frac{1}{(\lambda_m-m)(\lambda_n-n)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}s_{jk}.$$

In a similar way, we obtain

$$(4.4)\qquad \frac{\lambda_m+1}{\lambda_m-m}(\sigma_{\lambda_m,n}-\sigma_{mn})=-\sigma_{mn}+\frac{1}{(\lambda_m-m)(n+1)}\sum_{j=m+1}^{\lambda_m}\sum_{k=0}^{n}s_{jk}$$

and

$$(4.5)\qquad \frac{\lambda_n+1}{\lambda_n-n}(\sigma_{m,\lambda_n}-\sigma_{mn})=-\sigma_{mn}+\frac{1}{(m+1)(\lambda_n-n)}\sum_{j=0}^{m}\sum_{k=n+1}^{\lambda_n}s_{jk}.$$

Combining (4.3)–(4.5) yields

$$\frac{(\lambda_m+1)(\lambda_n+1)}{(\lambda_m-m)(\lambda_n-n)}(\sigma_{\lambda_m,\lambda_n}-\sigma_{\lambda_m,n}-\sigma_{m,\lambda_n}+\sigma_{mn})$$
$$+\frac{\lambda_m+1}{\lambda_m-m}(\sigma_{\lambda_m,n}-\sigma_{mn})+\frac{\lambda_n+1}{\lambda_n-n}(\sigma_{m,\lambda_n}-\sigma_{mn})$$
$$=-\sigma_{mn}+\frac{1}{(\lambda_m-m)(\lambda_n-n)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}s_{jk},$$

which is equivalent to (4.1).

(ii) The proof of (4.2) is similar.

Proof of Theorem 1. *Necessity*. If (s_{jk}) is summable $(C,1,1)$ to a finite limit s and (3.1) is satisfied, then

$$(4.6)\qquad \lim_{m,n\to\infty}(s_{mn}-\sigma_{mn})=0.$$

It is plain that for all $\lambda>1$ and n large enough,

$$\frac{\lambda}{\lambda-1}<\frac{\lambda_n+1}{\lambda_n-n}<\frac{2\lambda}{\lambda-1}.$$

Analogous inequalities hold for λ_m with m large enough. Consequently, for each fixed $\lambda>1$, we have

$$(4.7)\qquad \lim_{m,n\to\infty}\left\{\frac{(\lambda_m+1)(\lambda_n+1)}{(\lambda_m-m)(\lambda_n-n)}(\sigma_{\lambda_m,\lambda_n}-\sigma_{\lambda_m,n}-\sigma_{m,\lambda_n}+\sigma_{mn})\right.$$
$$\left.+\frac{\lambda_m+1}{\lambda_m-m}(\sigma_{\lambda_m,n}-\sigma_{mn})+\frac{\lambda_n+1}{\lambda_n-n}(\sigma_{m,\lambda_n}-\sigma_{mn})\right\}=0.$$

The same is true for each fixed $0<\lambda<1$. Now, (3.4) (resp. (3.5)) follows from (4.1) (resp. (4.2)), (4.6) and (4.7).

Sufficiency. Assume the fulfillment of (3.2) and (3.3). From (3.2) it follows that there exists a sequence $\lambda_\ell\downarrow 1$ such that

$$(4.8)\qquad \lim_{\ell\to\infty}\liminf_{m,n\to\infty}\frac{1}{(\lambda_{\ell m}-m)(\lambda_{\ell n}-n)}\sum_{j=m+1}^{\lambda_{\ell m}}\sum_{k=n+1}^{\lambda_{\ell n}}(s_{jk}-s_{mn})\ge 0,$$

where $\lambda_{\ell m}:=[m\lambda_\ell]$ and $\lambda_{\ell n}:=[n\lambda_\ell]$ for $\ell,m,n=1,2,\dots$ By (4.1),

$$\limsup_{m,n\to\infty}(s_{mn}-\sigma_{mn})$$
$$\le\lim_{\ell\to\infty}\limsup_{m,n\to\infty}\left\{\frac{(\lambda_{\ell m}+1)(\lambda_{\ell n}+1)}{(\lambda_{\ell m}-m)(\lambda_{\ell n}-n)}(\sigma_{\lambda_{\ell m},\lambda_{\ell n}}-\dots)+\dots\right\}$$
$$+\lim_{\ell\to\infty}\limsup_{m,n\to\infty}\left\{-\frac{1}{(\lambda_{\ell m}-m)(\lambda_{\ell n}-n)}\sum_{j=m+1}^{\lambda_{\ell m}}\sum_{k=n+1}^{\lambda_{\ell n}}(s_{jk}-s_{mn})\right\}.$$

Taking into account that (s_{jk}) is summable $(C,1,1)$ to a finite limit, and (4.7), (4.8), we hence get

$$(4.9)\qquad \limsup_{m,n\to\infty}(s_{mn}-\sigma_{mn})$$
$$\le-\lim_{\ell\to\infty}\liminf_{m,n\to\infty}\frac{1}{(\lambda_{\ell m}-m)(\lambda_{\ell n}-n)}\sum_{j=m+1}^{\lambda_{\ell m}}\sum_{k=n+1}^{\lambda_{\ell n}}(s_{jk}-s_{mn})\le 0.$$

From (3.3) it follows that for some sequence $\lambda_\ell\uparrow 1$ we have

$$\lim_{\ell\to\infty}\liminf_{m,n\to\infty}\frac{1}{(m-\lambda_{\ell m})(n-\lambda_{\ell n})}\sum_{j=\lambda_{\ell m}+1}^{m}\sum_{k=\lambda_{\ell n}+1}^{n}(s_{mn}-s_{jk})\ge 0;$$

whence, in a similar way, we obtain

$$(4.10)\qquad \liminf_{m,n\to\infty}(s_{mn}-\sigma_{mn})$$
$$\ge\lim_{\ell\to\infty}\liminf_{m,n\to\infty}\frac{1}{(m-\lambda_{\ell m})(n-\lambda_{\ell n})}\sum_{j=\lambda_{\ell m}+1}^{m}\sum_{k=\lambda_{\ell n}+1}^{n}(s_{mn}-s_{jk})\ge 0.$$

Combining (4.9) and (4.10) yields (4.6), which is equivalent to (3.1) due to the fact that (s_{jk}) is summable $(C,1,1)$ to s.

In order to prove Theorem 2, we start with the following two representations of the difference $s_{mn} - \sigma^{10}_{mn}$.

LEMMA 2. (i) *If* $\lambda > 1$, $\lambda_m > m$, *and* $n \ge 0$, *then*

$$(4.11)\quad s_{mn} - \sigma^{10}_{mn} = \frac{\lambda_m + 1}{\lambda_m - m}(\sigma^{10}_{\lambda_m,n} - \sigma^{10}_{mn}) - \frac{1}{\lambda_m - m}\sum_{j=m+1}^{\lambda_m}(s_{jn} - s_{mn}).$$

(ii) *If* $0 < \lambda < 1$, $\lambda_m < m$, *and* $n \ge 0$, *then*

$$(4.12)\quad s_{mn} - \sigma^{10}_{mn} = \frac{\lambda_m + 1}{m - \lambda_m}(\sigma^{10}_{mn} - \sigma^{10}_{\lambda_m,n}) + \frac{1}{m - \lambda_m}\sum_{j=\lambda_m+1}^{m}(s_{mn} - s_{jn}).$$

Proof. It is modelled after the proof of the corresponding lemma for single sequences in [4]. We do not enter into details.

Proof of Theorem 2. We only sketch it.

Necessity. If (s_{jk}) is summable $(C,1,0)$ to a finite limit s and (3.1) is satisfied, then

$$(4.13)\quad \lim_{m,n\to\infty}(s_{mn} - \sigma^{10}_{mn}) = 0.$$

On the other hand, for each fixed $\lambda > 0$, $\lambda \ne 1$, we have

$$\lim_{m,n\to\infty}\frac{\lambda_m + 1}{\lambda_m - m}(\sigma^{10}_{\lambda_m,n} - \sigma^{10}_{mn}) = 0.$$

Due to representations (4.11) and (4.12), hence (3.13) and (3.14) follow immediately.

Sufficiency. Assume the fulfillment of (3.11) and (3.12). By (3.11), there exists a sequence $\lambda_\ell \downarrow 1$ such that

$$(4.14)\quad \lim_{\ell\to\infty}\liminf_{m,n\to\infty}\frac{1}{\lambda_{\ell m} - m}\sum_{j=m+1}^{\lambda_{\ell m}}(s_{jn} - s_{mn}) \ge 0,$$

where again $\lambda_{\ell m} := [m\lambda_\ell]$, for $\ell, m = 1,2,\ldots$ By (4.11),

$$(4.15)\quad \limsup_{m,n\to\infty}(s_{mn} - \sigma^{10}_{mn})$$
$$\le \lim_{\ell\to\infty}\limsup_{m,n\to\infty}\frac{\lambda_{\ell m}+1}{\lambda_{\ell m} - m}(\sigma^{10}_{\lambda_{\ell m},n} - \sigma^{10}_{mn})$$
$$+ \lim_{\ell\to\infty}\limsup_{m,n\to\infty}\Big\{-\frac{1}{\lambda_{\ell m} - m}\sum_{j=m+1}^{\lambda_{\ell m}}(s_{jn} - s_{mn})\Big\}$$
$$= -\lim_{\ell\to\infty}\liminf_{m,n\to\infty}\frac{1}{\lambda_{\ell m} - m}\sum_{j=m+1}^{\lambda_{\ell m}}(s_{jk} - s_{mn}) \le 0,$$

due to (4.14) and the fact that (s_{jk}) is summable $(C,1,0)$ to a finite limit.

In a similar way, for an appropriate sequence $\lambda_\ell \uparrow 1$, we have

$$(4.16)\quad \liminf_{m,n\to\infty}(\sigma_{mn} - \sigma^{10}_{mn})$$
$$\ge \lim_{\ell\to\infty}\liminf_{m,n\to\infty}\frac{1}{m - \lambda_{\ell m}}\sum_{j=\lambda_{\ell m}+1}^{m}(s_{mn} - s_{jn}) \ge 0.$$

Combining (4.15) and (4.16) yields (4.13), which is equivalent to the convergence of (s_{jk}) to s.

5. Extensions. We formulate Tauberian theorems for sequences of complex numbers, and for those in an ordered linear space. In Theorems 3, 4 and Corollaries 5, 6, we assume that (s_{jk}) is a double sequence of complex numbers.

THEOREM 3. *If* (s_{jk}) *is summable* $(C,1,1)$ *to a finite limit, then* (s_{jk}) *converges if and only if either*

$$\liminf_{\lambda\downarrow 1}\limsup_{m,n\to\infty}\Big|\frac{1}{(\lambda_m - m)(\lambda_n - n)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}(s_{jk} - s_{mn})\Big| = 0$$

or

$$\liminf_{\lambda\uparrow 1}\limsup_{m,n\to\infty}\Big|\frac{1}{(m - \lambda_m)(n - \lambda_n)}\sum_{j=\lambda_m+1}^{m}\sum_{k=\lambda_n+1}^{n}(s_{mn} - s_{jk})\Big| = 0;$$

in which case we necessarily have (3.4) *for all* $\lambda > 1$, *and* (3.5) *for all* $0 < \lambda < 1$.

The proof of Theorem 3 also relies on representations (4.1) and (4.2), and closely follows that of Theorem 1. We omit it, but mention the following interesting

COROLLARY 5. *If* (s_{jk}) *is summable* $(C,1,1)$ *to a finite limit and there exist constants* $n_1 > 0$ *and* H *such that*

$$(5.1)\quad jk|s_{jk} - s_{j-1,k} - s_{j,k-1} + s_{j-1,k-1}| \le H \quad \text{whenever} \quad j,k > n_1,$$
$$(5.2)\quad j|s_{jn} - s_{j-1,n}| \le H \quad \text{whenever} \quad j,n > n_1,$$

and

$$(5.3)\quad k|s_{mk} - s_{m,k-1}| \le H \quad \text{whenever} \quad m,k > n_1,$$

then (s_{jk}) *converges*.

It is instructive to compare conditions (2.3), (2.5) and (2.6) with (5.1)–(5.3), respectively.

THEOREM 4. *If* (s_{jk}) *is summable* $(C,1,0)$ *to a finite limit, then* (s_{jk}) *converges if and only if either*

$$\liminf_{\lambda\downarrow 1}\limsup_{m,n\to\infty}\left|\frac{1}{\lambda_m - m}\sum_{j=m+1}^{\lambda_m}(s_{jn}-s_{mn})\right| = 0$$

or

$$\liminf_{\lambda\uparrow 1}\limsup_{m,n\to\infty}\left|\frac{1}{m-\lambda_m}\sum_{j=\lambda_m+1}^{m}(s_{mn}-s_{jn})\right| = 0;$$

in which case we necessarily have (3.13) *for all* $\lambda > 1$, *and* (3.14) *for all* $0<\lambda<1$.

The proof of Theorem 4 also relies on representations (4.11) and (4.12), and closely follows that of Theorem 2. We omit it.

COROLLARY 6. *If* (s_{jk}) *is summable* $(C,1,0)$ *to a finite limit and there exist constants* $n_1 > 0$ *and* H *such that condition* (5.2) *is satisfied, then* (s_{jk}) *converges.*

Finally, let $(X,\le)$ be an ordered linear space over the real numbers, in which we denote by o the zero element and by p a given nonnegative element. In the sequel, we assume that (s_{jk}) is a double sequence of elements in X.

We begin with a few definitions. We say that (s_{jk}) *converges* (*in Pringsheim's sense*) *to* s *relative to* $p\in X$ if for all $\varepsilon>0$ there exists $n_1>0$ such that

$$-\varepsilon p \le s_{mn}-s\le \varepsilon p \quad \text{whenever}\quad m,n>n_1.$$

We say that (s_{jk}) is *slowly decreasing in sense* $(1,1)$ *relative to* $p\in X$ if for all $\varepsilon>0$ there exist $n_1>0$ and $\lambda>1$ such that

$$s_{jk}-s_{mk}-s_{jn}+s_{mn}\ge -\varepsilon p \quad \text{whenever}\quad n_1<m<j\le\lambda_m \text{ and } n_1<n<k\le\lambda_n;$$

and *slowly decreasing in sense* $(1,0)$ if

$$s_{jn}-s_{mn}\ge-\varepsilon p \quad\text{whenever}\quad n_1<m<j\le\lambda_m \quad\text{and}\quad n_1<n;$$

and slow decrease in sense $(0,1)$ is defined analogously.

THEOREM 5. *If* (σ_{mn}) *converges to* $s\in X$ *relative to* $p\in X$, *then* (s_{jk}) *converges to* s *relative to* p *if and only if for all* $\varepsilon>0$ *and* $\lambda_1>1$ *there exist* $n_1>0$ *and* λ, $1<\lambda<\lambda_1$, *such that for all* $m,n>n_1$ *we have*

$$\text{(5.4)}\qquad \frac{1}{(\lambda_m-m)(\lambda_n-n)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}(s_{jk}-s_{mn})\ge-\varepsilon p,$$

and, possibly with another $1<\lambda<\lambda_1$, *we have*

$$\text{(5.5)}\qquad \frac{1}{(m-\lambda_m^{-1})(n-\lambda_n^{-1})}\sum_{j=\lambda_m^{-1}+1}^{m}\sum_{k=\lambda_n^{-1}+1}^{n}(s_{mn}-s_{jk})\ge-\varepsilon p,$$

where

$$\lambda_m^{-1}:=[\lambda^{-1}m]\quad\text{and}\quad\lambda_n^{-1}:=[\lambda^{-1}n];$$

in which case the double sequences occurring on the left-hand sides of (5.4) *and* (5.5) (*for all* $\lambda>1$) *necessarily converge to* o *relative to* p, *as* $m,n\to\infty$.

Comments analogous to (ii) and (iii) made after Theorem 1 are appropriate here, as well.

COROLLARY 7. *If* (σ_{mn}) *converges to* $s\in X$ *and* (s_{jk}) *is slowly decreasing in senses* $(1,1)$, $(1,0)$ *and* $(0,1)$ *relative to* $p\in X$, *then* (s_{jk}) *converges to* s *relative to* p.

We note that Corollary 7 is the extension of a theorem of Maddox [3] from single to double sequences.

THEOREM 6. *If* (σ_{mn}^{10}) *converges to* $s\in X$ *relative to* $p\in X$, *then* (s_{jk}) *converges to* s *relative to* p *if and only if for all* $\varepsilon>0$ *and* $\lambda_1>1$ *there exist* $n_1>0$ *and* λ, $1<\lambda<\lambda_1$, *such that for all* $m,n>n_1$ *we have*

$$\text{(5.6)}\qquad \frac{1}{\lambda_m-m}\sum_{j=m+1}^{\lambda_m}(s_{jn}-s_{mn})\ge-\varepsilon p,$$

and, possibly with another $1<\lambda<\lambda_1$, *we have*

$$\text{(5.7)}\qquad \frac{1}{m-\lambda_m^{-1}}\sum_{j=\lambda_m^{-1}+1}^{m}(s_{mn}-s_{jn})\ge-\varepsilon p,\quad\text{where}\quad \lambda_m^{-1}:=[\lambda^{-1}m];$$

in which case the double sequences occurring on the left-hand sides of (5.6) *and* (5.7) (*for all* $\lambda>1$) *necessarily converge to* o *relative to* p, *as* $m,n\to\infty$.

COROLLARY 8. *If* (σ_{mn}^{10}) *converges to* $s\in X$ *and* (s_{jk}) *is slowly decreasing in sense* $(1,0)$ *relative to* $p\in X$, *then* (s_{jk}) *converges to* s *relative to* p.

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When is there a discontinuous homomorphism from $L^1(G)$?

by

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Abstract. Let A be an A^*-algebra with enveloping C^*-algebra $C^*(A)$. We show that, under certain conditions, a homomorphism from $C^*(A)$ into a Banach algebra is continuous if and only if its restriction to A is continuous. We apply this result to the question in the title.

Introduction. One of the central questions in automatic continuity is the following: For which Banach algebras A is there a Banach algebra B and a discontinuous homomorphism $\theta : A \to B$?

A fundamental result obtained independently by H. G. Dales and J. Esterle (see [Dal] for a streamlined exposition) asserts that if X is a locally compact Hausdorff space and if the continuum hypothesis holds, then there is a discontinuous homomorphism from $C_0(X)$ into a Banach algebra if and only if X is infinite. Surprisingly, this result cannot be proved within the confinements of Zermelo–Fraenkel set theory and the axiom of choice ([D-W]). In this note, we shall not delve into these set theoretic intricacies and assume throughout that the continuum hypothesis holds.

In [A-D], E. Albrecht and Dales conjectured the following non-commutative version of the Dales–Esterle theorem:

CONJECTURE A. Let A be a C^*-algebra. Then there is a discontinuous homomorphism from A into a Banach algebra if and only if there is $n \in \mathbb{N}$ such that A has an infinite number of inequivalent, n-dimensional, irreducible *-representations.

Albrecht and Dales were able to prove the "if" part of their conjecture and to confirm the "only if" part for so-called AW^*M-algebras, a class of C^*-algebras containing all commutative C^*-algebras and all closed ideals of AW^*-algebras.

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