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Continuous linear right inverses for convolution operators in spaces of real analytic functions

by

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Abstract. We determine the convolution operators $T_\mu := \mu*$ on the real analytic functions in one variable which admit a continuous linear right inverse. The characterization is given by means of a slowly decreasing condition of Ehrenpreis type and a restriction of hyperbolic type on the location of zeros of the Fourier transform $\widehat{\mu}(z)$.

The existence of continuous linear right inverses for convolution operators $T_\mu := \mu*$ has been studied in many classes of (generalized) functions on $\mathbb{R}$: The first result for $C^\infty(\mathbb{R})$ was obtained by Ehrenpreis [5] and the problem was solved for nonquasianalytic ultradifferentiable functions and ultradistributions by Meise and Vogt [15] and Braun, Meise and Vogt [3]. The characterization was given through estimates on the location of zeros of the Fourier transform $\widehat{\mu}$ of the (ultra)distribution μ, similar to that for hyperbolic convolution operators (Ehrenpreis [5]). For convolution operators on holomorphic functions defined on convex open sets $\Omega \subset \mathbb{C}$ the corresponding question was solved in Taylor [25], Schwerdtfeger [24] and Meise [12] for $\Omega = \mathbb{C}$, and in Momm [19, 21, 22] for general convex $\Omega \neq \mathbb{C}$ (see also Korobeĭnik and Melikhov [8]), again leading to a restriction on the location of zeros of $\widehat{\mu}$, connected with the angular derivative on the boundary $\partial\Omega$ of the Riemann mapping function for Ω.

In the present paper, continuous linear right inverses for convolution operators on real analytic functions on open or compact intervals will be studied. Neither necessary conditions nor nontrivial positive examples seem to be known in this case.

Let $I \subset \mathbb{R}$ be an open interval and let $A(I)$ be the space of real analytic functions on I with its canonical topology. Fix $\mu \in A(\mathbb{R})'$ and assume $\operatorname{supp}\mu = \{0\}$ if $I \neq \mathbb{R}$. Then μ defines a continuous linear convolution operator

$$T_\mu : A(I) \to A(I), \quad T_\mu(f)(x) := \langle {}_y\mu, f(x-y)\rangle.$$

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We get the following characterization:

THEOREM. *Suppose* $\operatorname{supp}\mu = \{0\}$ *if* $I \neq \mathbb{R}$. *Then* T_μ *has a continuous linear right inverse if and only if there is a function* $r(x) = o(x)$ *on* $\mathbb{R}_+$ *such that*

$$(*) \qquad |\operatorname{Im} z| \leq r(|z|) \quad \text{for any } z \in \mathbb{C} \text{ with } \widehat{\mu}(z) = 0$$

and for any $x \in \mathbb{R}$ *there is* $t \in \mathbb{C}$ *such that*

$$(\mathrm{E}) \qquad |x - t| \leq r(x) \quad \text{and} \quad |\widehat{\mu}(t)| \geq \exp(-r(t)).$$

Under condition (E), convolution operators satisfying (*) are just the hyperbolic operators in the sense of hyperfunctions, i.e., the convolution operators admitting hyperfunction elementary solutions E_+ and E_- supported in $[C, \infty[$ (and $]-\infty, -C]$, respectively) for some $C \geq 0$ (Kawai [7, Section 6.2]; condition (E) is condition (S) in [7]).

A condition similar to (*) characterizes the convolution operators admitting a right inverse on (F)-spaces of nonquasianalytic ultradifferentiable functions (Meise and Vogt [15]). Condition (*) can be considered as the generalization of the corresponding condition 4.2.(4) in Braun, Meise and Vogt [3] to the quasianalytic case $\omega(x) = |x|$. Also, (*) can be considered as the limiting case of the characterization given by Momm [21, Example 4.4] for convolution operators on holomorphic functions defined on open polyhedral convex sets, if the polyhedral neighbourhoods of I "tend to I".

Conditions of type (E) were introduced by Ehrenpreis [4] and have been frequently used to characterize the surjectivity of convolution operators in many spaces of (generalized) functions. So (E) is also natural in our situation. (E) is always satisfied if $\operatorname{supp}\mu = \{0\}$ (see (1.8) below).

Notice that (*) and (E) are implied by the conditions obtained by Meise and Vogt [15] and Braun, Meise and Vogt [3]. So any convolution operator having a right inverse within the classes of functions studied in [15] and [3] also has a right inverse on the real analytic functions.

The paper is divided into four parts. The first section contains the necessary tools from Fourier theory, functional analysis and function theory, and especially a representation of the dual of the kernel of T_μ as in Meise [12, 13]. In Section 2 the necessity of (E) is proved by a variant of the corresponding proof in Momm [20], while the necessity of (*) is reduced to the result of Momm [21]. The sufficiency of (*) and (E) is proved in Section 3. The proof is based on the existence of a continuous linear right inverse for the Cauchy–Riemann operator on sets defined by (*), which is obtained by an application of the tame splitting theorem of Poppenberg and Vogt [23] (see Theorem 1.2). In the final section, convolution operators T_μ on real analytic functions on compact intervals J are considered (for $\operatorname{supp}\mu = \{0\}$). Again, the convolution operators admitting a continuous linear right inverse

in this case are characterized by (*). Similarly to the results of Momm [21, 22] the proof relies on the Riemann mapping theorem and the fact that any continuous linear map between power series spaces of finite type is linear tame for the canonical systems of seminorms (Vogt [26], Lemma 5.1).

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1. Preliminaries. In this section the main notations used in this paper are introduced and the necessary tools from functional analysis and function theory are given. Also, the right inverse problem for convolution operators are translated (by Fourier transformation) into a left inverse problem for multiplication operators on certain weighted spaces of entire functions and a suitable sequence space representation is given for the corresponding quotient spaces.

For an open set $U \subset \mathbb{C}$ let $A(U)$ denote the holomorphic functions on U. If K is compact in $\mathbb{C}$ then

$$A(K) := \operatorname*{lim\,ind}_{U \supset K} A(U)$$

is the space of holomorphic functions near K. If $I \subset \mathbb{R}$ is an open interval then

$$A(I) := \operatorname*{lim\,proj}_{J \Subset I} A(J)$$

is the space of real analytic functions on I. Here the projective limit is taken over all compact intervals J contained in I.

For $\mu \in A(\mathbb{C})'$ let

$$\mathcal{F}(\mu)(z) := \widehat{\mu}(z) := \langle {}_x\mu, e^{-ixz} \rangle, \qquad z \in \mathbb{C},$$

be the Fourier transform of μ.

For a compact convex set $K \subset \mathbb{C}$ let $H_K(z) := \sup\{\operatorname{Im}(\xi z) \mid \xi \in K\}$ be the support functional of K.

If Ω is a convex open set in $\mathbb{C}$, then $\mathcal{F}$ is a linear topological isomorphism from the strong dual $A(\Omega)'_b$ onto

$$\mathcal{H}_\Omega := \{f \in A(\mathbb{C}) \mid \exists K \Subset \Omega : |f(z)| \leq C \exp(H_K(z)) \text{ for any } z \in \mathbb{C}\}$$

(Hörmander [6, Section 4.5]). Also, if $I \subset \mathbb{R}$ is an open interval, $\mathcal{F}$ defines a linear topological isomorphism from $A(I)'_b$ onto

$$\mathcal{H}_I := \{f \in A(\mathbb{C}) \mid \exists J \Subset I\ \forall n \geq 1 : \sup_{z \in \mathbb{C}} |f(z)| \exp(-H_J(z) - |z|/n) < \infty\}$$

(Hörmander [6, Section 4.5], Meyer [18, Satz 5.9]). $\mathcal{H}_I$ is an (LF)-space, namely

$$\mathcal{H}_I := \operatorname*{lim\,ind}_{J \Subset I} \mathcal{H}^J,$$

where the (F)-space

$$\mathcal{H}^J := \{f \in A(\mathbb{C}) \mid \forall n \ge 1 : \sup_{z\in\mathbb{C}} |f(z)| \exp(-H_J(z) - |z|/n) < \infty\}$$

is via Fourier transformation isomorphic to $A(J)'_b$.

Here and in the following, I (and J) will always denote open (respectively, compact) intervals.

The results of Sections 3 and 4 rely on the theory of power series spaces of finite type and linear tame mappings. With values in a Hilbert space, power series spaces of finite type are defined as follows:

For a sequence α_k of positive real numbers tending to ∞ and a Hilbert space $(E, |\ |)$ let

$$\Lambda_0(\alpha_k) = \Big\{(c_k) \in E^{\mathbb{N}} \Bigm| |(c_k)|_n^2 := \sum_{k\ge 1} |c_k|^2 e^{-2\alpha_k/n} < \infty \text{ for any } n \ge 1\Big\}.$$

This choice of norms $|\ |_n$ for power series spaces of finite type is fixed for the remaining part of this paper.

Let $(E, |\ |_n)$ and $(F, \|\ \|_n)$ be (F)-spaces with fixed increasing systems of seminorms defining the topology. A linear mapping $T : (E, |\ |_n) \to (F, \|\ \|_n)$ is called *linear tame* if there is $a \ge 1$ such that for any $n \ge 1$ there is C_n such that

$$\|T(x)\|_n \le C_n |x|_{an} \quad \text{for any } x \in E.$$

The spaces $(E, |\ |_n)$ and $(F, \|\ \|_n)$ are called *linear tamely isomorphic* if there is a linear isomorphism T from E onto F such that T and T^{-1} are linear tame. T is then called a *linear tame isomorphism*. Two seminorm systems $\{|\ |_n \mid n \ge 1\}$ and $\{\|\ \|_n \mid n \ge 1\}$ on E are called *linear tamely equivalent* if $\mathrm{id} : (E, |\ |_n) \to (E, \|\ \|_n)$ is a linear tame isomorphism.

For $J_1 = [-1, 1]$ the space $\mathcal{H}^{J_1}$ can also be endowed with the following system of seminorms:

$$\text{(1.1)} \qquad |||f|||_n^2 = \int |f(z)|^2 \exp(-2\omega_n(z))\, dz,$$

where $\omega_n(z) = (|\alpha_n \operatorname{Re} z|^2 + |\delta_n \operatorname{Im} z|^2)^{1/2}$ with $\alpha_n = \sinh(1/n)$ and $\delta_n = \cosh(1/n)$.

1.1. LEMMA. *$(\mathcal{H}^{J_1}, |||\ |||_n)$ is linear tamely isomorphic to $(\Lambda_0(k), |\ |_n)$.*

Proof. This has to be proved only for the corresponding sup-norms

$$\text{(1.2)} \qquad \|f\|_n := \sup\{|f(z)| \exp(-\omega_n(z)) \mid z \in \mathbb{C}\},$$

since the systems $\{\|\ \|_n \mid n \ge 1\}$ and $\{|||\ |||_n \mid n \ge 1\}$ are linear tamely equivalent. Let $\mathbb{D} := \{z \in \mathbb{C} \mid |z| > 1\}$ and let

$$A_0(\mathbb{D}) := \{f \in A(\mathbb{D}) \mid \lim_{z\to\infty} f(z) = 0\}$$

be endowed with the norms

$$|g|_n := \sup\{|g(z)| \mid z \in S_n := \{z \in \mathbb{C} \mid |z| \ge e^{1/n}\}\}.$$

Then $(A_0(\mathbb{D}), |\ |_n)$ is linear tamely isomorphic to $(\Lambda_0(k), |\ |_n)$ by Laurent series expansion.

The mapping $\varphi(z) := (z + 1/z)/2$ is a biholomorphic mapping from $\mathbb{D}$ onto $\complement J_1$ (see Lang [9, Chap. VII, §4, Example 11]) and with α_n and δ_n as above we get

$$\varphi(S_n) = G_n := \{z \in \mathbb{C} \mid (\operatorname{Re} z/\delta_n)^2 + (\operatorname{Im} z/\alpha_n)^2 \ge 1\}.$$

So φ defines a linear tame isomorphism from $(A_0(\mathbb{D}), |\ |_n)$ onto $(A_0(\complement J_1), |\ |_n)$, where the latter norms are defined by taking suprema over G_n.

Now the Fourier transformation, namely the mapping

$$\Phi(g)(z) := \frac{1}{2\pi i} \int_{\partial G_n} g(\xi) e^{-i\xi z}\, d\xi \quad \text{for } z \in \mathbb{C} \text{ and } g \in A_0(\complement J_1)$$

defines a linear tame isomorphism of $(A_0(\complement J_1), |\ |_n)$ onto $(\mathcal{H}^{J_1}, \|\ \|_n)$ (see Hörmander [6, proof of Theorem 4.5.3 for $n = 1$] and notice that

$$\sup\{|e^{-i\xi z}| \mid \xi \in \partial G_n\} = \exp(\omega_n(z)) = \exp(H_{\complement G_n}(z)).$$

Notice that for $k > n$,

$$\text{(1.3)} \qquad \mathcal{H}^{J_1} \text{ is dense in } \mathcal{H}_k := \{f \in A(\mathbb{C}) \mid |||f|||_k < \infty\} \text{ with respect to } |||\ |||_n,$$

since $A_0(\complement J_1)$ is dense in $A_0(G_k)$ (use also the Fourier transform Φ from the proof of Lemma 1.1).

A sequence

$$(\Delta) \qquad 0 \to (E, |\ |_n) \xrightarrow{i} (F, \|\ \|_n) \xrightarrow{q} (G, |||\ |||_n) \to 0$$

of (F)-spaces with fixed increasing systems of seminorms defining the topologies is called *linear tamely exact* if

$$i : (E, |\ |_n) \to (i(E), \|\ \|_n) \quad \text{and} \quad \widetilde{q} : (F/i(E), \|\ \|_n^{\sim}) \to (G, |||\ |||_n)$$

are linear tame isomorphisms (here $\|\ \|_n^{\sim}$ is the quotient seminorm for $\|\ \|_n$). The results of Section 3 are based on the following splitting theorem of Poppenberg and Vogt [23]:

1.2. THEOREM. *Suppose that the sequence (Δ) is linear tamely exact, and $(E, |\ |_n)$ and $(G, |||\ |||_n)$ are linear tamely isomorphic to (Hilbert space valued) power series spaces of finite type. Then the sequence (Δ) is split, i.e., q has a continuous linear right inverse.*

For $\mu \in A(\mathbb{R})'$ let $G := \operatorname{conv}(\operatorname{supp} \mu)$. We define the convolution operator T_μ on $A(I - G)$ and $A(J - G)$ by $T_\mu(f)(x) := \langle_y \mu, f(x - y)\rangle$ for $f \in A(I - G)$

(respectively, for $f \in A(J-G)$). Then

$$T_\mu : A(I-G) \to A(I) \quad \text{and} \quad T_\mu : A(J-G) \to A(J)$$

are continuous linear operators. Moreover, $\mathcal{F} \circ {}^t T_\mu \circ \mathcal{F}^{-1}$ is the multiplication operator

$$M_{\mu_-} : \mathcal{H}_I \to \mathcal{H}_{I-G}, \quad M_{\mu_-}(f)(z) := \widehat{\mu}(-z) f(z) \quad \text{for } z \in \mathbb{C}.$$

By duality and Fourier transformation it is clear that $T_\mu : A(I-G) \to A(I)$ has a continuous linear right inverse $R : A(I) \to A(I-G)$:

- if and only if $M_{\mu_-} : \mathcal{H}_I \to \mathcal{H}_{I-G}$ has a continuous linear left inverse $L : \mathcal{H}_{I-G} \to \mathcal{H}_I$;
- if and only if $M_{\mu_-}\mathcal{H}_I$ is a complemented subspace of $\mathcal{H}_{I-G}$ (if π is a continuous projection onto $M_{\mu_-}\mathcal{H}_I$, $L(f) := \pi(f)/\widehat{\mu}(-\cdot)$ is continuous by the closed graph theorem for (LF)-spaces);
- if and only if the sequence

$$0 \to \mathcal{H}_I \xrightarrow{M_{\mu_-}} \mathcal{H}_{I-G} \xrightarrow{q} \mathcal{H}_{I-G}/M_{\mu_-}\mathcal{H}_I \to 0 \tag{1.4}$$

is split, i.e., the quotient map q has a continuous linear right inverse.

The analogous statements hold for the convolution operator $T_\mu : A(J-G) \to A(J)$ and the multiplication operator $M_{\mu_-} : \mathcal{H}^J \to \mathcal{H}^{J-G}$ for any compact interval J.

As a final preparatory step we need a canonical representation for the quotient in (1.4) if $\widehat{\mu}(-\cdot) =: F \in \mathcal{H}^{\{0\}}$. Following the basic papers of Berenstein and Taylor [1, 2] and Meise [12] such representations have been proved in many situations and we could refer to the corresponding literature after introducing some necessary notations. For the convenience of the reader, the main arguments are included. Basically, we need a minimum modulus theorem (see e.g. Langenbruch and Momm [10, Lemma 1.11]):

1.3. Lemma. *Let u be subharmonic in $|z| \le 4D$ with $u(0) = 0$. For each $0 < d < D$ there are $C(D/d)$ and $d < \tau < D$ with*

$$u(\xi) \ge -C(D/d) \sup\{u(z) \mid |z| = 4D\} \quad \textit{for any } |\xi| = \tau.$$

Let $F := \widehat{\mu}(-\cdot) \in \mathcal{H}^{\{0\}}$. Then there is a function r on $\mathbb{C}$ such that

$$r(z) = r(|z|) = o(|z|) \tag{1.5}$$

and such that

$$|F(z)| \le e^{r(z)} \quad \text{on } \mathbb{C}. \tag{1.6}$$

We can assume that

$$r(2z) \le 4r(z) \quad \text{for any } z \in \mathbb{C}. \tag{1.7}$$

Indeed, set $\varrho(z) = \sum_{k\ge 0} 4^{-k} r(2^k z)$. Then for any $\varepsilon > 0$ we get

$$r(z) \le \varrho(z) \le \sum_{k\ge 0} \varepsilon |z| 2^{-k} = 2\varepsilon |z| \quad \text{for large } |z|$$

by (1.5) and

$$\varrho(2z) = 4 \sum_{k\ge 1} 4^{-k} r(2^k z) \le 4\varrho(z).$$

We now have the following: There is a function R with (1.5) such that

$$\text{for any } z \in \mathbb{C} \text{ there is } t \in \mathbb{C} \text{ with } |z-t| \le R(t) \text{ and } |F(t)| \ge e^{-R(t)}. \tag{1.8}$$

To prove this, we assume that $F(0) = 1$ and use Lemma 1.3 for $n \ge 1$ and $z \in \mathbb{C}$ with $D = (1+1/n)|z|$, $d = |z|$ and $u = \ln|F|$ and obtain, for any $|z| < |\xi| = \tau_{n,z} < (1+1/n)|z|$,

$$|F(\xi)| \ge \exp(-C_n r(8|z|)) \ge \exp(-64C_n r(|z|)) \ge \exp(-64C_n r(|\xi|)) \tag{1.9}$$

by (1.7). We now choose $\widetilde{R}$ with (1.5) such that for some increasing t_n,

$$\widetilde{R}(t) \ge 64C_n r(t) \quad \text{for } t \ge t_n,$$

and use (1.9) for $t_n \le |z| < t_{n+1}$ to get (1.8) with $R(z) := \max\{\widetilde{R}(|z|), |z|/n\}$ for $t_n \le |z| < t_{n+1}$.

We can assume that the functions r in (1.6) and R in (1.8) coincide. Let

$$S(F,C) := \{z \in \mathbb{C} \mid |F(z)| < \exp(-Cr(z))\}$$

and

$$V_F := \{z \in \mathbb{C} \mid F(z) = 0\},$$

and let (S_k) be the components of $S(F,C)$ such that $S_k \cap V_F \ne \emptyset$. Fix $z_k \in S_k \cap V_F$ for any k. With $C(2)$ from Lemma 1.3 we get, for $C \ge 4(2C(2)+1)$,

$$S_k \subset \{\xi \mid |\xi - z_k| \le 3r(z_k)\} \quad \text{for large } |z_k|. \tag{1.10}$$

Choose t_k for z_k by (1.8) and use Lemma 1.3 for $d = |z_k - t_k|$ and $D = 2d$ and $u(z) = \ln(F(t_k+z)/F(t_k))$. Then $|z_k - t_k| < \tau = \tau_k$ and

$$|F(\xi)| \ge \exp(-4(2C(2)+1)r(\xi)) \quad \text{for } |\xi - t_k| = \tau_k \text{ and large } |z_k|$$

by (1.5), (1.7) and (1.8). This shows (1.10). Finally, we have

$$\operatorname{dist}(S_k \cap S(F,5C), \complement S_k) \ge \exp(-C_1 r(z_k)) \quad \text{for large } |z_k| \tag{1.11}$$

with $C_1 := 4(C+1)$. To prove this, we take $\xi \in S_k \cap S(F,5C)$ and $z \in \partial S_k$ with $|z - \xi| \le 1$ and estimate $|F(z) - F(\xi)| = |\int_\xi^z F'(t)\,dt|$ as usual from above and below.

From now on we fix $r(z)$ such that (1.5)–(1.8) hold for r, set $C := 4(2C(2)+1)$ as in (1.10) and define $S(F,C)$, S_k and z_k as above.

Let

$$A^\infty(S_k) := \{f \in A(S_k) \mid \|f\|_k := \sup\{|f(z)| \mid z \in S_k\} < \infty\}$$

and

$$E_k = A^\infty(S_k)/M_F A^\infty(S_k) \quad \text{with the quotient norm } |\ |_k.$$

Let

$$K_I(\mathbb{E}) := \Big\{(f_k) \in \prod E_k \,\Big|\, \exists J \Subset I\ \forall n \ge 1 : \sup_k |f_k|_k \exp(-H_J(z_k) - |z_k|/n) < \infty\Big\},$$

$$K^J(\mathbb{E}) := \Big\{(f_k) \in \prod E_k \,\Big|\, \forall n \ge 1 : \sup_k |f_k|_k \exp(-H_J(z_k) - |z_k|/n) < \infty\Big\}$$

and

$$K_\Omega(\mathbb{E}) := \Big\{(f_k) \in \prod E_k \,\Big|\, \exists K \Subset \Omega : \sup_k |f_k|_k \exp(-H_K(z_k)) < \infty\Big\}$$

for $\Omega \subset \mathbb{C}$ convex and open. Let

$$\varrho_k : A^\infty(S_k) \to E_k$$

be the quotient map and let ϱ be the "restriction operator"

$$\varrho : A(\mathbb{C}) \to \prod_k E_k, \qquad \varrho(f) := (\varrho_k(f|_{S_k})).$$

The following lemma is now proved similarly to the corresponding results in Berenstein and Taylor [1, 2], Meise [12] and Momm [21].

1.4. Lemma. *Let $F \in \mathcal{H}^{\{0\}}$. Then the sequences*

$$(1.12)\qquad 0 \to \mathcal{H}^J \xrightarrow{M_F} \mathcal{H}^J \xrightarrow{\varrho} K^J(\mathbb{E}) \to 0,$$

$$(1.12')\qquad 0 \to \mathcal{H}_I \xrightarrow{M_F} \mathcal{H}_I \xrightarrow{\varrho} K_I(\mathbb{E}) \to 0,$$

$$(1.12'')\qquad 0 \to \mathcal{H}_\Omega \xrightarrow{M_F} \mathcal{H}_\Omega \xrightarrow{\varrho} K_\Omega(\mathbb{E}) \to 0$$

are exact for ϱ (and r and C) defined as above and any open interval $I \subset \mathbb{R}$, any compact interval J and any convex open set $\Omega \subset \mathbb{C}$.

Proof. We give the proof for the convenience of the reader, since we need Lemma 1.4 to prove Lemma 2.4. Let $\mathcal{H}$ be any of the spaces $\mathcal{H}^J$, $\mathcal{H}_I$ or $\mathcal{H}_\Omega$. The range of M_F is contained in the kernel of ϱ by definition. On the other hand, if $\varrho(f) = 0$ for $f \in \mathcal{H}$, then $g := f/F$ is entire. g obviously satisfies the estimates of $\mathcal{H}$ on $\complement S(F, C)$, hence on $\partial S(F, C)$. Now the maximum principle, (1.10) and (1.7) show that this also holds on $S(F, C)$. So $g \in \mathcal{H}$.

To prove the surjectivity of ϱ, we choose $\varphi_k \in D(S_k)$ such that $\varphi_k = 1$ on $S_k \cap S(F, 5C)$ and such that

$$(1.13)\qquad \|\nabla \varphi_k\| \le C_2 \exp(C_1 r(z_k)).$$

This is possible by (1.10). Let $L \subset \mathbb{C}$ be convex and compact. For $([f_k]) \in K^L(\mathbb{E})$ we can choose $g_k \in A^\infty(S_k)$ such that $\varrho_k(g_k) = [f_k]$ and

$$(1.14)\qquad \|g_k\|_k = \sup\{|f_k(z)| \mid z \in S_k\} \le 2|f_k|_k \le A_n \exp(H_L(z_k) + R(z_k))$$

for some $R(z)$ with (1.5) and any k. Let

$$v := \sum (\overline{\partial}\varphi_k) g_k / F.$$

Then

$$(1.15)\qquad \int |v(z)|^2 \exp(-2H_L(z) - 2\widetilde{R}(z))\, dz < \infty$$

for some radial function $\widetilde{R}$ with (1.5) (use (1.13), (1.14), (1.10), (1.5) and (1.7)). We can assume that $\widetilde{R}$ is logarithmically convex, hence subharmonic. By Hörmander [6] there is a solution $\widetilde{f}$ of $\overline{\partial}(\widetilde{f}) = v$ also satisfying (1.15) (with a new $R(z)$). Then

$$f := \sum \varphi_k g_k - \widetilde{f} F$$

is entire and also satisfies (1.15) for some $\widetilde{R}$ with (1.5). So $f \in \mathcal{H}^L$ and $\varrho_k(f|_{S_k}) = [f_k]$ for any k, since $\widetilde{f}$ is holomorphic near the zeros of F in S_k. So ϱ is surjective in (1.12)–(1.12″).

The exactness of (1.12′) is proved in a more general situation by Meyer [17].

Notice that the mapping ϱ in Lemma 1.4 is independent of J, I and Ω. This is important for the proof of Lemma 2.4 below.

By Lemma 1.4 and de Wilde's open mapping theorem (Meise and Vogt [16, Satz 24.30]) we can (e.g.) identify the quotient $\mathcal{H}^J/M_F\mathcal{H}^J$ with $K^J(\mathbb{E})$. Such representations are often referred to as sequence space representations. The reason is that E_k is finite-dimensional, since F has only finitely many zeros in S_k by (1.10). So $K^J(\mathbb{E})$ can be considered as a sequence space (indexed by the zeros of F with multiplicities).

2. Necessity.

In this section, I (and J) are always open intervals (respectively, compact intervals) and $\mu \in A(\mathbb{R})'$ with $\operatorname{conv}(\operatorname{supp}\mu) =: G$.

We will prove that the existence of a continuous linear right inverse for $T_\mu : A(I - G) \to A(I)$ implies that μ satisfies

$$(*)\qquad |\operatorname{Im} z| = o(|z|) \quad \text{on } V_\mu := \{z \in \mathbb{C} \mid \widehat{\mu}(z) = 0\}$$

and the following slowly decreasing condition of Ehrenpreis type (Ehrenpreis [4]): There is a function $r(z)$ defined on $\mathbb{C}$ and $C > 0$ such that

$$(1.5)\qquad r(z) = r(|z|) = o(|z|)$$

and such that for any $x \in \mathbb{R}$ with $|x| \ge C$ there is $t \in \mathbb{C}$ with

$$(\mathrm{E})\qquad |t - x| \le r(x) \quad \text{and} \quad |\widehat{\mu}(t)| \ge \exp(-r(t)).$$

Instead of (E), we could equivalently also use (with $Cr(2t)$ instead of $r(t)$)

$$(\mathrm{E}') \qquad |t-x| \le r(x) \quad \text{and} \quad |\widehat{\mu}(t)| \ge \exp(H_G(t) - r(t)).$$

Since (E) is satisfied also for $\varrho \ge r$ we can assume that $r(z)$ satisfies (1.7).

The first variant of condition (E) was introduced by Ehrenpreis [4] to characterize the surjective convolution operators on $C^\infty(\mathbb{R}^N)$. Since then conditions of this type have been frequently used for similar purposes. In our situation (E) follows by an easy variant of the reasoning of Momm [20] (see also Meyer [18]), based on the following special case of Momm [20, Lemma]:

2.1. LEMMA. *For* $\varepsilon > 0$ *let* $p(z) = \varepsilon|\mathrm{Im}\, z| + r(|\mathrm{Re}\, z|)$, *where* $\ln(1+|x|) \le r(x)$ *is continuous and increasing on* $\mathbb{R}_+$ *and satisfies* (1.5) *and* (1.7). *For* $R > 0$ *let* $h : \mathbb{C} \to \mathbb{R}_+$ *be the continuous function which equals* $\varepsilon|\mathrm{Im}\, z|$ *for* $|z| > R$ *and is harmonic in* $|z| < R$. *Then* h *is subharmonic and there are* $\delta, C > 0$ (*independent of* R) *such that*

$$\varepsilon\delta R - C \le h(0) \le \max_{|z|\le R} h(z) \le R\varepsilon.$$

2.2. LEMMA. *If* T_μ *is surjective, then* $\widehat{\mu}$ *satisfies* (E).

Proof. If T_μ is surjective, then T_μ is open by de Wilde's open mapping theorem (Meise and Vogt [16, Satz 24.30]). So ${}^tT_\mu$ is injective and $M \subset A(I)'$ is equicontinuous if ${}^tT_\mu(M) \subset A(I-G)'$ is equicontinuous. Since the bounded sets and the equicontinuous sets coincide, by Fourier transformation we deduce that for any bounded set $B \subset \mathcal{H}_{I-G}$ the set $\{f \in \mathcal{H}_I \mid M_{\mu_-} f \in B\}$ is bounded in $\mathcal{H}_I$ (so far we followed the proof of Meyer [18, Lemma 4.13]). We may suppose that

$$(2.1) \qquad \{x \in \mathbb{R} \mid |x| \le \varepsilon\} \Subset I \quad \text{for some } \varepsilon > 0.$$

Choose $0 \le r_1(z)$ with (1.5) such that

$$(2.2) \qquad |\widehat{\mu}(z)| \le \exp(H_G(z) + r_1(z) - 2\ln(1+|z|^2)).$$

Let

$$\mathcal{H} := \{f \in A(\mathbb{C}) \mid \exists C > 0 : \\ |f(z)| \le C\exp(H_{-G}(z) + r_1(z) + \varepsilon|\mathrm{Im}\, z| - 2\ln(1+|z|^2)^2)\}.$$

Then $\mathcal{H}$ is a Banach space which is continuously embedded in $\mathcal{H}_{I-G}$ by (2.1). So for any bounded set $B \subset \mathcal{H}$ the set $\{f \in \mathcal{H}_I \mid M_{\mu_-} f \in B\}$ is bounded in $\mathcal{H}_I$, in other words, $M_{\mu_-}^{-1} : M_{\mu_-}\mathcal{H}_I \cap \mathcal{H} \to \mathcal{H}_I$ is continuous. Since $\mathcal{H}_I$ is complete, we can extend $M_{\mu_-}^{-1}$ to the closure of $M_{\mu_-}\mathcal{H}_I$ in $\mathcal{H}$. Since $\mathcal{H}_I$ is an (LF)-space, the factorization theorem of Grothendieck (Meise and Vogt [16, 24.33]) implies the existence of $J \Subset I$ such that for any $m \in \mathbb{N}$ there

are $C_m \ge 1$ such that for any $f \in \mathcal{H}_I$,

$$(2.3) \qquad \sup_m \sup_{z\in\mathbb{C}} |f(z)|\exp(-|z|/m - H_J(z) - C_m) \\ \le \sup_{z\in\mathbb{C}} |\widehat{\mu}(-z)f(z)|\exp(-H_{-G}(z) - \varepsilon|\mathrm{Im}\, z| - r_1(z)).$$

With δ from Lemma 2.1 we can choose $r(z)$ with (1.5) and (1.7) such that $r(z) \to \infty$ as $z \to \infty$ and

$$(2.4) \qquad \varepsilon\delta r(x)/2 \ge \inf_m (|x|/m + C_m).$$

For $x \in \mathbb{R}$ we now choose functions $f_x \in \mathcal{H}_I$ such that (2.3) fails (for large x) if on the right hand side only points z with $|z-x| \ge r(x)+1$ are considered. Using Lemma 2.1 (for $R = r(x)$) and Hörmander [6, Theorem 4.4.2] as in the proof of (i)⇒(ii) of Momm [20, Proposition 1], we get analytic functions f_x such that for suitable $A > 0$ and any (large) $x \in \mathbb{R}$,

$$(2.5) \qquad |f_x(z)| \le A(1+|z|^2)^2\exp(\varepsilon|\mathrm{Im}\, z|) \quad \text{for } |z-x| \ge r(x)+1,$$

$$(2.6) \qquad |f_x(z)| \le A\exp(4r(z)) \quad \text{for } |z-x| \le r(x)+1$$

and

$$(2.7) \qquad |f_x(x)| \ge \exp(\varepsilon\delta r(x) - A).$$

We evaluate (2.3) for f_x: $f_x \in \mathcal{H}_I$ by (2.5) and (2.1). By (2.7) and (2.4) the left hand side of (2.3) tends to ∞ as $x \to \infty$, while the function on the right hand side is bounded by A for $|x-z| \ge r(x)+1$ (by (2.2) and (2.5)). So we get for large x, by (2.6),

$$2A \le \sup\{|\widehat{\mu}(-z)f_x(z)|\exp(-H_{-G}(z) - \varepsilon|\mathrm{Im}\, z| - r_1(z)) \mid |z-x| \le r(x)+1\} \\ \le A\sup\{|\widehat{\mu}(-z)|\exp(4r(z) - H_{-G}(z)) \mid |z-x| \le r(x)+1\}.$$

So (E′) holds and hence also (E) holds.

The next step is a reduction argument of Korobeĭnik and Melikhov [8] (see also Momm [22, Lemma 8]):

2.3. LEMMA. *Suppose* V_μ *is infinite. If* T_μ *has a continuous linear right inverse, then every sequence* $(z_k) \subset V_\mu$ *of distinct points has a subsequence* (ξ_k) *such that* $(\xi_k) = V_F$ *for some* $F \in \mathcal{H}^{\{0\}}$ *and such that the multiplication operator*

$$M_F : \mathcal{H}_I \to \mathcal{H}_I, \qquad M_F(f) = Ff \quad \text{for } f \in \mathcal{H}_I,$$

has a continuous linear left inverse.

Proof. (ξ_k) can be chosen such that

$$F(z) := \prod_{k\ge1}(1 - z/\xi_k) \in \mathcal{H}^{\{0\}}.$$

Then

$$g(z) := \widehat{\mu}(-z)/F(z) \in \mathcal{H}^{-G}$$

(see Levin [11, Chap. III, Theorem 5]). If $L_\mu : \mathcal{H}_{I-G} \to \mathcal{H}_I$ is a continuous linear left inverse for M_{μ_-}, then

$$L_F : \mathcal{H}_I \to \mathcal{H}_I, \quad L_F(f) := L_\mu(gf) \quad \text{for } f \in \mathcal{H}_I,$$

is a continuous linear left inverse for M_F.

The proof of the necessity of $(*)$ is now achieved by reduction to the case of convolution operators on open convex sets in $\mathbb{C}$, which was solved by Momm [21, 22].

2.4. Lemma. *If T_μ has a continuous linear right inverse, then*

$$(*) \qquad |\operatorname{Im} z| = o(|z|) \quad \text{on } V_\mu.$$

Proof. We can assume that $0 \in I$. Suppose that $(*)$ is not true. Then there is a sequence $(z_k) \subset V_{\mu_-}$ of distinct points and $\delta > 0$ such that

$$(2.8) \qquad |\operatorname{Im} z_k| \ge \delta |z_k| \quad \text{for any } k \in \mathbb{N}.$$

We can assume that $[-\delta, \delta] \subset I$. Let F be chosen for (z_k) by means of Lemma 2.3 and let $\Omega_\varepsilon := \operatorname{conv}(I, iI_\varepsilon) \subset \mathbb{C}$ for $I_\varepsilon :=]-\varepsilon, \varepsilon[$.

For $K \Subset \Omega_\varepsilon$ there are $J \Subset I$ such that

$$(2.9) \qquad \delta |\operatorname{Im} z| \le H_J(z)$$

and $\eta < \varepsilon$ such that $K \subset K_1 := \operatorname{conv}(J, iI_\eta)$, i.e.,

$$(2.10) \qquad H_K(z) \le H_{K_1}(z) = \max(\eta |\operatorname{Re} z|, H_J(z)) = H_J(z) \quad \text{for } z \in V_F,$$

by (2.8) and (2.9) if $\varepsilon \le \delta^2$. This implies that

$$K_I(\mathbb{E}) = K_{\Omega_\varepsilon}(\mathbb{E}),$$

where these spaces are defined for F as explained in the first section (the definition of $S(F, C)$ and S_k was independent of I and Ω_ε!). Since $\mathcal{H}_I$ is continuously embedded in $\mathcal{H}_{\Omega_\varepsilon}$ and since ϱ does not depend on I or Ω_ε, the continuous linear right inverse $R : K_I(\mathbb{E}) \to \mathcal{H}_I$ for $\varrho : \mathcal{H}_I \to K_I(\mathbb{E})$, existing by assumption and Lemma 1.4, is also a continuous linear right inverse for $\varrho : \mathcal{H}_{\Omega_\varepsilon} \to K_{\Omega_\varepsilon}(\mathbb{E})$.

Momm [21, Example 4.4] now shows that, for some C_0,

$$|\operatorname{Im} z| < 2\varepsilon |\operatorname{Re} z| \quad \text{for any } z \in V_F \text{ with } |z| \ge C_0$$

(the Fourier transformation in Momm [21] is $\mathcal{F}(f)(iz)$ in our notation). Together with (2.8) this implies (for $\varepsilon \le \delta/2$) that F can only have a finite number of zeros, a contradiction.

3. Sufficiency. It was pointed out by Taylor [25] that the existence of a continuous linear right inverse for the Cauchy–Riemann operator $\overline{\partial}$ in suitable weighted spaces implies the existence of interpolation operators (and also left inverses for multiplication operators) in the corresponding weighted spaces of entire functions. In the present situation these right inverses exist only on sets defined by $(*)$. This existence result relies on the theory of power series spaces of finite type and linear tame mappings introduced in Section 1.

Let $S \subset \mathbb{C}$ be a closed set such that for some $r(z) = o(|z|)$,

$$(3.1) \qquad |\operatorname{Im} z| \le r(|z|) \quad \text{for any } z \in S.$$

Let

$$L^J(S) := \Big\{ f \in L^2_{\mathrm{loc}}(\mathbb{C}) \;\Big|\; \operatorname{supp} f \subset S \text{ and}$$
$$|f|_n := \Big(\int |f(z)|^2 \exp(-2H_J(z) - 2|z|/n)\, dz \Big)^{1/2} < \infty \text{ for any } n \ge 1 \Big\}$$

and

$$L^J := \{ f \in L^2_{\mathrm{loc}}(\mathbb{C}) \mid \overline{\partial} f \in L^J(S),$$
$$\|f\|_n := (|f|_n^2 + |\overline{\partial} f|_n^2)^{1/2} < \infty \text{ for any } n \ge 1 \}.$$

3.1. Lemma. *Suppose S satisfies* (3.1) *and* $\operatorname{int}(J) \neq \emptyset$. *Then the sequence*

$$(3.2) \qquad 0 \to \mathcal{H}^J \to L^J \xrightarrow{\overline{\partial}} L^J(S) \to 0$$

is exact and split.

Proof. We can assume that $J = [-1, 1]$. $\mathcal{H}^J$ is the kernel of $\overline{\partial}$, since the sup-norms and the L^2-norms define the same space of holomorphic functions. A second seminorm system defining the topologies of $L^J(S)$ (and L^J) is given by the norms used on $\mathcal{H}^J$ in Lemma 1.1, i.e., by

$$|||f|||_n^2 = \int |f(z)|^2 \exp(-2\omega_n(z))\, dz$$

with ω_n defined as in (1.1). The $\overline{\partial}$ operator is linear tame by definition for this seminorm system, and it is surjective by Hörmander's solution of the weighted $\overline{\partial}$-problem (Hörmander [6]); more precisely, for $k > n$ there is C such that for any $f \in L^J(S)$ with $|||f|||_k \le 1$ there is $g \in L^J$ such that

$$(3.3) \qquad \overline{\partial} g = f \quad \text{and} \quad |||g|||_n \le C$$

by Hörmander's result. Now $\mathcal{H}^J$ is dense in $\mathcal{H}_k := \{ f \in A(\mathbb{C}) \mid |||f|||_k < \infty \}$ with respect to $|||\ |||_n$ for $k > n$ by (1.3).

So the Mittag-Leffler argument shows that for $k > n$ there is C such that for any $f \in L^J(S)$ with $|||f|||_k \le 1$ there is $g \in L^J$ such that

$$\overline{\partial} g = f \quad \text{and} \quad |||g|||_n \le C.$$

So far we have shown that (3.2) is linear tamely exact if $\mathcal{H}^J$, L^J and $L^J(S)$ are endowed with the seminorms $|||\ |||_n$. Now $(\mathcal{H}^J, |||\ |||_n)$ is linear tamely isomorphic to a power series space of finite type by Lemma 1.1. Also, on $L^J(S)$ the seminorm systems $\{|||\ |||_n \mid n \ge 1\}$ and

$$\Big\{|f|_n := \Big(\int |f(z)|^2 \exp(-2(H_J(z) + |z|/n))\,dz\Big)^{1/2} \Big| n \ge 1\Big\}$$

are linear tamely equivalent, since we get from (3.1), for suitable C and C_1,

$$H_J(z) + |z|/(2Cn) = |\mathrm{Im}\, z| + |z|/(2Cn) \le C_1 + |z|/(Cn) \le C_1 + \omega_n(z) \quad \text{for any } z \in S$$

and

$$\omega_{Cn}(z) \le C_1 + |\mathrm{Re}\, z|/n \le H_J(z) + |z|/n + C_1 \quad \text{for any } z \in S.$$

We can assume that

$$S = \bigcup_{k\in\mathbb{N}} \xi_k + [0,1]^2 \quad \text{for some sequence } (\xi_k) \in \mathbb{Z}^2.$$

Then $(L^J(S), |\ |_n)$ is isomorphic to a ($L_2([0,1]^2)$-valued) power series space of finite type by a diagonal transformation. So we have checked all assumptions of Theorem 1.2 (i.e., the splitting theorem of Poppenberg and Vogt) and the sequence (3.3) is split.

We now assume that $\mu \in A(\mathbb{R})'$ satisfies (E) and

$$(*) \qquad |\mathrm{Im}\, z| \le r(z) \quad \text{for any } z \in V_\mu,$$

where $r(z)$ satisfies (1.5) and (1.7). We can assume that $r(z)$ in (E) and $(*)$ are the same. Let $F := \widehat{\mu}(-\cdot)$. For any $z \in V_F$ we can choose t for $\mathrm{Re}\, z$ by (E) such that

$$|t - \mathrm{Re}\, z| \le r(\mathrm{Re}\, z) \quad \text{and} \quad |\widehat{\mu}(-t)| > \exp(-r(t)).$$

We can assume that also

$$(3.4) \qquad |\widehat{\mu}(z)| \le \exp(H_G(z) + r(z)) \quad \text{for any } z \in \mathbb{C}.$$

By Lemma 1.3 (i.e., by the minimum modulus theorem) there is C such that for any t there is $1 < d_t < 2$ such that

$$(3.5) \qquad |\widehat{\mu}(-\xi)| \ge e^{-Cr(\xi)} \quad \text{for any } \xi \text{ with } |\xi - t| = 3d_t r(z).$$

Let (S_k) be the components of $S(F, C)$ (defined by $r(z)$ and C from (3.5) as in Section 1) such that $S \cap V_F \ne \emptyset$ and fix $z_k \in S_k \cap V_F$. Then we get by $(*)$ and (3.5), with suitable C_j,

$$(3.6) \qquad \bigcup_k S_k \subset \{z \in \mathbb{C} \mid |\mathrm{Im}\, z| \le C_0 r(z) + C_1\} =: \Sigma.$$

Any component S of $S(F, C_2)$ contained in S_k for some k satisfies

$$(3.7) \qquad \mathrm{dist}(S, \complement S(F, C)) \ge \exp(-C_3 r(z_k))$$

(compare (1.11)). So we can choose $\varphi_k \in C^\infty(\mathbb{C})$ such that

$$(3.8) \qquad \varphi_k = 1 \quad \text{on } S_k \cap S(F, C_2), \quad \mathrm{supp}\, \varphi_k \subset S_k$$

and

$$(3.9) \qquad \Big\|\nabla \sum_k \varphi_k \exp(-C_4 r(z))\Big\|_\infty < \infty.$$

To prove the existence of a continuous linear right inverse for T_μ, we also need the following assumption: If I is bounded from above (respectively, from below), then there is $r(z) = o(|z|)$ such that for any $x \in i\mathbb{R}_+$ (respectively, for any $x \in i\mathbb{R}_-$) there is $t \in \mathbb{C}$ such that

$$(3.10) \qquad |t - x| \le r(x) \quad \text{and} \quad |\widehat{\mu}(t)| \ge \exp(H_G(t) - r(t)).$$

So $\widehat{\mu}$ should grow like its indicator at suitable points near the positive or negative imaginary half-axis. In fact, it would be sufficient to have this estimate from below on some half-line in the upper (respectively, in the lower) half-plane. If $\mathrm{supp}\, \mu =: G = \{0\}$, then (3.10) is satisfied by (1.8). It is not known if (3.10) is a consequence of the surjectivity of $T_\mu : A(I - G) \to A(I)$ (and eventually of condition $(*)$).

3.2. LEMMA. *Suppose that $\mu \in A(\mathbb{R})'$ satisfies $(*)$, (E) and (3.10). Then $T_\mu : A(I - G) \to A(I)$ has a continuous linear right inverse.*

Proof. We have to show that $M_{\mu_-} : \mathcal{H}_I \to \mathcal{H}_{I-G}$ has a continuous linear left inverse. With the circles Γ_k chosen by (3.5), let

$$L_k(f)(z) := \frac{1}{2\pi i} \int_{\Gamma_k} \frac{f(\xi)}{\widehat{\mu}(-\xi)(\xi - z)}\, d\xi \quad \text{for } z \in S_k$$

and

$$L(f) := \Big(1 - \sum_k \varphi_k\Big) f/\widehat{\mu}(-\cdot) + \sum_k \varphi_k L_k(f)$$

for $f \in A(\mathbb{C})$. Obviously, $L_k(f)$ is holomorphic on S_k, L is linear and

$$(3.11) \qquad L(M_{\mu_-} f) = f.$$

Let $J \Subset I$ and $\mathrm{int}(J) \ne \emptyset$. If $f \in \mathcal{H}_{I-G}$, then

$$\overline{\partial} L(f) \in L^J(\Sigma)$$

by (3.6)–(3.9).

Let R be a right inverse for $\overline{\partial}$ chosen for $L^J(\Sigma)$ by Lemma 3.1. Then the operator

$$\Lambda := (\mathrm{id} - R \circ \overline{\partial}) \circ L : \mathcal{H}_{I-G} \to A(\mathbb{C})$$

is defined, linear and continuous, and it is a left inverse for M_{μ_-} by (3.11). $\Lambda(f)$ satisfies the growth conditions of $\mathcal{H}_I$ on the real line if $f \in \mathcal{H}_{I-G}$. This also holds on the imaginary axis for $R \circ \overline{\partial} \circ L(f)$. Now Lemma 1.3 implies the following well known estimate: There is $A > 0$ such that for any $z \in \mathbb{C}$ with $|z| \geq A$ there is $1/4 < d_z < 1/2$ such that

$$(3.12)\qquad |\widehat{\mu}(\xi)| \geq e^{-A|\xi|} \quad \text{for any } |\xi - z| = d_z|z|.$$

This estimate is sufficient for our purposes on the positive (respectively, negative) imaginary half-axis if I is unbounded from above (respectively, from below). On the other hand, Lemma 1.3 and (3.10) imply that there is $B > 0$ such that for any t from (3.10) with $|t| \geq B$ there is $1 < d_t < 2$ such that

$$(3.13)\qquad |\widehat{\mu}(-\xi)| \geq \exp(H_{-G}(\xi) - Br(\xi)) \quad \text{for any } |\xi - t| = d_t r(x).$$

The maximum principle, (3.12) and (3.13) now show that also $L(f)$ satisfies the growth conditions of $\mathcal{H}_I$ on the imaginary axis. By the Phragmén–Lindelöf theorem we conclude that $\Lambda : \mathcal{H}_{I-G} \to \mathcal{H}_I$ is linear and continuous. The theorem is proved.

4. Convolution operators in $A(J)$. The aim of this section is to prove the following theorem, which extends the characterization given in our main theorem to convolution operators on real analytic functions on compact intervals.

4.1. THEOREM. *Let $\mu \in A(\{0\})'$ and let J be a compact interval. Then $T_\mu : A(J) \to A(J)$ has a continuous linear right inverse if and only if μ satisfies* $(*)$.

The sufficiency of $(*)$ has been shown in the proof of Lemma 3.2 already, since (E) is satisfied for any $\mu \in A(\{0\})'$ by (1.8).

The necessity of $(*)$ comes for the same reason as the results of Momm [21] on convolution equations on open convex sets, namely, two natural systems of bounded sets must fit together in a linear tame way by the theory of power series spaces of finite type (Vogt [26, Lemma 5.1]).

The argument is as follows: Let again $F := \widehat{\mu}(-\cdot)$ and let (S_k) be the components S of $S(F, C)$ with $S \cap V_F \neq \emptyset$ and fix $z_k \in S_k \cap V_F$. Let

$$\Lambda^J := \{(c_k) \in \mathbb{C}^{\mathbb{N}} \mid \forall n \geq 1 : |(c_k)|_n := \sup_k |c_k| \exp(-|\mathrm{Im}\, z_k| - |z_k|/n) < \infty\}.$$

Let

$$|||(c_k)|||_n := \sup_k |c_k| \exp(-\omega_n(z_k))$$

with ω_n as in (1.1). Both norm systems define the topology of Λ^J.

By considering $c_k \in \mathbb{C}$ as a constant function on S_k, we define a continuous linear injection $T : \Lambda^J \to K^J(\mathbb{E})$. Let $\widetilde{T}$ be the continuous operator

$$\widetilde{T} : \mathcal{H}^J \to \Lambda^J, \qquad \widetilde{T}(f) := (f(z_k)).$$

If R is a right inverse for $\varrho : \mathcal{H}^J \to K^J(\mathbb{E})$ from Lemma 1.4 (existing by assumption), then

$$R \circ T : (\Lambda^J, |\ |_n) \to (\mathcal{H}^J, |||\ |||_n)$$

is continuous and linear, hence it is linear tame by Vogt [26, Lemma 5.1], since Λ^J is linear tamely isomorphic to a power series space of finite type by a diagonal transformation and $(\mathcal{H}^J, |||\ |||_n)$ is so by Lemma 1.1. Then

$$\widetilde{T} : (\mathcal{H}^J, |||\ |||_n) \to (\Lambda^J, \|\ \|_n)$$

is obviously linear tame and therefore the identity mapping

$$\mathrm{id} = \widetilde{T} \circ R \circ T : (\Lambda^J, |\ |_n) \to (\Lambda^J, |||\ |||_n)$$

is linear tame. Thus there is $a \geq 1$ such that

$$|||(c_k)|||_n \leq C_n |(c_k)|_{an}.$$

This estimate is applied to the canonical unit vectors e_k, which gives

$$\begin{aligned}(|\mathrm{Im}\, z_k| + |z_k|/(an))^2 &= |\mathrm{Im}\, z_k|^2 + 2|\mathrm{Im}\, z_k||z_k|/(an) + |z_k|^2/(an)^2 \\ &\leq \omega(z_k)^2 + C'_n \\ &\leq 4|\mathrm{Re}\, z_k|^2/n^2 + (1 + 1/n^2)^2 |\mathrm{Im}\, z_k|^2 + C'_n \\ &\leq 4|z_k|^2/n^2 + |\mathrm{Im}\, z_k|^2 + C'_n\end{aligned}$$

since

$$\delta_n = \cosh(1/n) \leq 1 + 1/n^2 \quad \text{and} \quad \alpha_n = \sinh(1/n) \leq 2/n.$$

This implies for any n,

$$|\mathrm{Im}\, z_k| \leq 2a|z_k|/n + C''_n \quad \text{for } |z_k| \geq 1.$$

So condition $(*)$ holds and Theorem 4.1 is proved.

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Tauberian theorems for Cesàro summable double sequences

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Abstract. Let $(s_{jk} : j,k = 0,1,\ldots)$ be a double sequence of real numbers which is summable $(C,1,1)$ to a finite limit. We give necessary and sufficient conditions under which (s_{jk}) converges in Pringsheim's sense. These conditions are satisfied if (s_{jk}) is slowly decreasing in certain senses defined in this paper. Among other things we deduce the following Tauberian theorem of Landau and Hardy type: If (s_{jk}) is summable $(C,1,1)$ to a finite limit and there exist constants $n_1 > 0$ and H such that

$$jk(s_{jk} - s_{j-1,k} - s_{j-1,k} + s_{j-1,k-1}) \ge -H,$$
$$j(s_{jk} - s_{j-1,k}) \ge -H \quad \text{and} \quad k(s_{jk} - s_{j,k-1}) \ge -H$$

whenever $j,k > n_1$, then (s_{jk}) converges. We always mean convergence in Pringsheim's sense. Our method is suitable to obtain analogous Tauberian results for double sequences of complex numbers or for those in an ordered linear space over the real numbers.

1. Preliminary results for single sequences. Let $(s_k : k = 0,1,\ldots)$ be a single sequence of real numbers. A classical one-sided Tauberian theorem of Landau [2] asserts that if (s_k) is summable $(C,1)$ to a finite number s and there exists a constant H such that

$$k(s_k - s_{k-1}) \ge -H \quad (k = 1,2,\ldots), \tag{1.1}$$

then (s_k) converges to s.

Following Schmidt [5], we say that (s_k) is *slowly decreasing* if for each $\varepsilon > 0$ there exist $n_1 > 0$ and $\lambda > 1$ such that

$$s_k - s_n \ge -\varepsilon \quad \text{whenever} \quad n_1 < n < k \le \lambda n, \tag{1.2}$$

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Key words and phrases: double sequence, convergence in Pringsheim's sense, summability $(C,1,1)$, $(C,1,0)$ and $(C,0,1)$, one-sided Tauberian condition of Landau and Hardy type, slow decrease, ordered linear space.

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