

$$\int_x^\infty y^{-2} dy \int_0^x (y-x)e^{-y} dy \leq C$$

which clearly holds for all $x > 0$. So the weak-type boundedness holds.

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On group extensions of 2-fold simple ergodic actions

by

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Abstract. Compact group extensions of 2-fold simple actions of locally compact second countable amenable groups are considered. It is shown what the elements of the centralizer of such a system look like. It is also proved that each factor of such a system is determined by a compact subgroup in the centralizer of a normal factor.

1. Introduction. In this paper we describe the centralizer and the structure of factors for ergodic group extensions of a 2-fold simple action of a locally compact second countable amenable group on a standard Borel space.

Our method is an adaptation of the methods developed by Lemańczyk and Mentzen in [5], [7], [4] for \mathbb{Z} -actions and consists in a description of the ergodic joinings of these actions. We show that ergodic joinings are relatively independent extensions of certain isomorphisms between normal natural factors. For that we will need the ergodic theorem for our general case (for proofs we refer to [1] and Krengel's book [3]). For ergodic \mathbb{Z} -actions the form of the elements of the centralizer for group extensions of a discrete spectrum transformation was found by D. Newton [8] in the abelian case and by M. K. Mentzen [7] in general case. Here we generalize Mentzen's result to arbitrary locally compact second countable group actions. We also generalize the main result of [5], [7] describing factors in terms of compact subgroups in the centralizers of normal natural factors (for related results see [2], [4], [9]).

2. Definitions and theorems. Let \mathfrak{R} be a locally compact second countable group and (X, \mathcal{B}, μ) be a standard Borel space. We will say that \mathfrak{R} acts on (X, \mathcal{B}, μ) if there exists a Borel map from $X \times \mathfrak{R}$ to X (we denote it by $(x, t) \mapsto xt$) such that

- (i) $x(t_1 t_2) = (x t_1) t_2$ for all $t_1, t_2 \in \mathfrak{R}$ and a.a. $x \in X$,
- (ii) $x e = x$ for a.a. $x \in X$.

Moreover, for all $t \in \mathfrak{R}$, $x \mapsto xt$ is a measure-preserving, invertible map from X onto itself (such maps will be called *automorphisms*). The action of \mathfrak{R} on (X, \mathcal{B}, μ) will be denoted by T .

We say that T is *ergodic* if all T -invariant Borel sets have measure 0 or 1. We call T *weakly mixing* if $T \times T$ with the product measure is ergodic. Let $T_i, i = 1, \dots, n$, be ergodic actions on X_i , respectively. Let λ be a $T_1 \times \dots \times T_n$ -invariant measure. If for each $i = 1, \dots, n$ and for each $A_i \in \mathcal{B}_i$,

$$\lambda(X_1 \times \dots \times X_{i-1} \times A_i \times X_{i+1} \times \dots \times X_n) = \mu(A_i)$$

then λ is called an *n-joining* of T_1, \dots, T_n . The set of all n -joinings of T_1, \dots, T_n will be denoted by $J(T_1, \dots, T_n)$ and the set of all ergodic n -joinings by $J^e(T_1, \dots, T_n)$. If $T_1 = \dots = T_n = T$ we will write $J_n(T)$ and $J_n^e(T)$ and refer to the elements as *n-self-joinings*.

Take the ergodic decomposition

$$\lambda = \int_{E(T_1, \dots, T_n)} m d\xi(m)$$

of $\lambda \in J(T_1, \dots, T_n)$, where $E(T_1, \dots, T_n)$ is the set of all ergodic measures on $\mathcal{B}_1 \otimes \dots \otimes \mathcal{B}_n$. Then $\xi(J^e(T_1, \dots, T_n)) = 1$ (see [2]). Therefore $J^e(T_1, \dots, T_n)$ is nonempty since $\mu_1 \times \dots \times \mu_n \in J(T_1, \dots, T_n)$. A measurable, measure-preserving map from (X, \mathcal{B}, μ) onto itself will be called an *endomorphism* of (X, \mathcal{B}, μ) . By the *centralizer*, $C_1(T)$, of T we mean the set of all endomorphisms of (X, \mathcal{B}, μ) commuting with the action T . The set of all automorphisms commuting with T is denoted by $C(T)$. The set $C_1(T)$ is endowed with the weak topology (where $S_n \rightarrow S$ iff $\mu(S_n^{-1}A \Delta S^{-1}A) \rightarrow 0$), for each Borel set A making $C_1(T)$ a complete separable metric space.

If $C_1(T) = C(T)$ then we say that T is *coalescent*. Being coalescent is equivalent to being a canonical factor of itself (see [8]).

If $S \in C(T)$ then the measure $\mu_S(A \times B) = \mu(A \cap S^{-1}B)$ is a self-joining. Such joinings will be called *graph-joinings*.

Following [2] we will say that the action T is *2-fold simple* if every ergodic 2-self-joining is either product measure or a graph-joining. It is not difficult to see that such an action is always coalescent (see [2]).

Let G be a compact metric group (not necessarily abelian) equipped with the normalized Haar measure $\nu = \nu_G$ on the family \mathcal{D} of Borel subsets of G . Let $\tilde{\mathcal{B}} = \mathcal{B} \otimes \mathcal{D}$. There exists a natural action of G on $X \times G$ given by

$$(x, g)h = (x, gh).$$

Define $\tilde{\mu} = \mu \times \nu$. Suppose that $\varphi : X \times \mathfrak{R} \rightarrow G$ is a Borel map. Moreover, assume that φ satisfies the *cocycle property*

$$\varphi(x, t_1 t_2) = \varphi(x t_1, t_2) \varphi(x, t_1).$$

For this cocycle we define the action T_φ of the group \mathfrak{R} on $X \times G$ by

$$(x, g)t = (xt, \varphi(x, t)g).$$

It is called a *group extension*, or, indicating the group, a *G-extension* of T . Each G -extension of T commutes with the natural action of G .

Each closed subgroup $F \subset G$ determines the sub- σ -algebra $\tilde{\mathcal{B}}_F$ of $\tilde{\mathcal{B}}$ of all sets satisfying $Af = A$ for all $f \in F$. The action $T_{\varphi, F}$ of \mathfrak{R} on $(X \times G/F, \tilde{\mathcal{B}}_F, \tilde{\mu})$ given by

$$(x, gF)t = (xt, \varphi(x, t)gF)$$

is called a *natural factor* of T_φ . Each action isomorphic to $T_{\varphi, F}$ is called an *isometric extension* of T . If F is normal in G then we will call $T_{\varphi, F}$ a *normal natural factor* of T .

We use the following ergodic theorem in our general situation, referring to [1] and [3] for proofs.

THEOREM. *Assume that the group \mathfrak{R} is amenable (see [3] for the definition) and that its action T on X is ergodic. Then we can find a Følner sequence $\mathbf{Y} = \{Y_n\}$ of subsets of \mathfrak{R} such that*

$$\lim_{n \rightarrow \infty} A_n^{\mathbf{Y}} f(x) = \lim_{n \rightarrow \infty} \frac{1}{\delta(Y_n)} \int_{Y_n} f(xt) d\delta(t) = \int_X f(x) d\mu(x)$$

for any $f \in L^p(X, \mathcal{B}, \mu)$ ($p \geq 1$) and a.a. $x \in X$, where δ is Haar measure on the Borel subsets of \mathfrak{R} .

3. Structure of joinings. In this section and in Section 4, we generalize the main results of [5] and [7]. As a rule, we omit the proofs which are natural extensions of the proofs from those papers.

We first consider the case where the cocycle φ is not necessarily ergodic (this means that the action T_φ is not necessarily ergodic). Assume that T (as before) is an ergodic 2-fold simple action of a locally compact second countable amenable group \mathfrak{R} on a standard Borel space (X, \mathcal{B}, μ) .

THEOREM 1. *Each ergodic isometric extension of T is a natural factor of some ergodic group extension of T .*

Assume that λ is a T_φ -ergodic component of T_φ . Before passing to the proof we prove some lemmas. The ergodic theorem yields that for any $f \in L^p(\lambda)$ there exists a T_φ -invariant $Y \subset X \times G$ such that $\lambda(Y) = 1$ and

$$\lim_{n \rightarrow \infty} A_n^{\mathbf{Y}} f(x, g) = \lim_{n \rightarrow \infty} \frac{1}{\delta(Y_n)} \int_{Y_n} f((x, g)t) d\delta(t) = \int_{X \times G} f(x) d\tilde{\lambda}(x, g)$$

for $(x, g) \in Y$. Let H be the stabilizer of λ (consisting of all $h \in H$ with $\lambda h = \lambda$).

LEMMA 1. *The subgroup H is closed in G and if $(x, g), (x, h) \in Y$ then $hH = gH$.*

Proof. The first part is obvious so we only show the second one. We know that $\lim_{n \rightarrow \infty} A_n^Y f(x, g) = \int_{X \times G} f d\lambda$ and

$$\begin{aligned} \lim_{n \rightarrow \infty} A_n^Y f(x, g) &= \lim_{n \rightarrow \infty} A_n^Y f(x, hh^{-1}g) = \lim_{n \rightarrow \infty} (A_n^Y f) \circ h^{-1}g(x, h) \\ &= \lim_{n \rightarrow \infty} (A_n^Y f \circ h^{-1}g)(x, h) \\ &= \int_{X \times G} f \circ h^{-1}g d\lambda = \int_{X \times G} f d(\lambda h^{-1}g) \end{aligned}$$

since the action T_φ commutes with the natural action of G . This means that $\lambda h^{-1}g = \lambda$ so $h^{-1}g \in H$. The proof is complete. ■

Let

$$\lambda = \int_X \lambda_x d\mu(x)$$

be a decomposition of λ over the factor (T, X, \mathcal{B}, μ) .

LEMMA 2 ([5]). *For a.a. $x \in X$ there exists $g = g(x) \in G$ such that*

$$\lambda_x = \delta_x \times g\nu_H,$$

where δ_x is Dirac measure. ■

Let the map $\tau : X \rightarrow G/H$ be given by

$$\tau(x) = g(x)H,$$

where $(x, g(x)) \in Y$. Lemma 1 yields that τ is well defined. By Lemma 2, τ is measurable. From T_φ -invariance it follows that $\tau(xt) = \varphi(x, t)\tau(x)$ for all $t \in \mathfrak{R}$. Let $U_H : G/H \rightarrow G$ be a measurable cross-section of the canonical projection $G \rightarrow G/H$. Define $q : X \rightarrow G$ and $\psi : X \times \mathfrak{R} \rightarrow G$ by

$$q(x) = U_H(\tau(x)) \quad \text{and} \quad \psi(x, t) = q(xt)^{-1}\varphi(x, t)q(x).$$

LEMMA 3. *The action $(T_\varphi, X \times G, \lambda)$ is isomorphic to $(T_\psi, X \times H, \mu \times \nu_H)$.*

Proof. It is clear that $q(x) \in \tau(x)$ for a.a. $x \in X$. We also have

$$q(x)H = U_H(\tau(x))H = \tau(x), \quad q(xt)H = U_H(\tau(xt))H = \varphi(x, t)\tau(x).$$

Therefore for a.a. $x \in X$ and all $t \in \mathfrak{R}$,

$$\begin{aligned} \psi(x, t) &= q(xt)^{-1}\varphi(x, t)q(x) \\ &= q(xt)^{-1}\varphi(x, t)\tau(x) = q(xt)^{-1}q(xt)H = H. \end{aligned}$$

This forces that ψ is a cocycle with values in H . The isomorphism between $(T_\varphi, X \times G, \lambda)$ and $(T_\psi, X \times H, \mu \times \nu_H)$ is given by $J : X \times H \rightarrow X \times G$, $J(x, h) = (x, q(x)h)$.

Now we verify that J commutes with the group actions. If $t \in \mathfrak{R}$, then

$$\begin{aligned} J((x, h)t) &= J(xt, \psi(x, t)h) = (xt, q(xt)\psi(x, t)h) \\ &= (xt, q(xt)q(xt)^{-1}\varphi(x, t)q(x)h) = (xt, \varphi(x, t)q(x)h) \\ &= (x, q(x)h)t = (J(x, h))t. \end{aligned}$$

Hence by Lemma 2, for $A \in \mathcal{B}$ and $B \in \mathcal{D}_H$ we have

$$\begin{aligned} \lambda(J(A \times B)) &= \int_X \delta_x \times g(x)\nu_H \left(\bigcup_{x \in A} \{x\} \times q(x)B \right) d\mu(x) \\ &= \mu(A)\nu_H(g(x)^{-1}q(x)B) = \mu \times \nu_H(A \times B). \end{aligned}$$

The last equality follows from the fact that $g(x)^{-1}q(x) \in H$.

Therefore J is an isomorphism between (T_φ, λ) and $(T_\psi, \mu \times \nu_H)$ so the proof of Lemma 3 is complete. ■

Proof of Theorem 1. Assume that $(T_{\varphi, F}, \tilde{\mu})$ is an ergodic natural factor of $(T_\varphi, \tilde{\mu})$ and λ is as above. Lemma 3 yields that there exists a closed subgroup $H \subset G$ and a cocycle $\psi : X \rightarrow H$ such that (T_φ, λ) and $(T_\psi, \mu \times \nu_H)$ are isomorphic. To prove our theorem we will show that the action $(T_{\varphi, F}, X \times G, \tilde{\mu})$ is isomorphic to $(T_{\psi, H/H \cap F}, X \times H/H \cap F, \mu \times \nu_H)$. The construction of an isomorphism is carried out in several steps.

• First, notice that the map $W : X \times G/F \rightarrow G$ given by

$$W(x, gF) = (x, W_{F \cap H}(gF \cap \tau(x))),$$

where $W_{F \cap H}$ is a measurable cross-section of $p : G \rightarrow G/F \cap H$, is well defined. Indeed, this becomes clear when we notice that

$$\tilde{\mu}(\{x, g\} \in X \times G : gF \cap \tau(x) \neq \emptyset) = 1.$$

• Let $\pi : H \rightarrow H/F \cap H$ be the natural homomorphism. Define our isomorphism by

$$R = (\text{Id} \times \pi) \circ J^{-1} \circ W.$$

Let $\{x\} \times h(F \cap H) \in X \times H/F \cap H$. The element $\{x\} \times h(F \cap H)$ is the inverse image of the set $\{x\} \times q(x)hF$ by J . However, $(x, q(x)hF) \in X \times G/F$ is just the inverse image of $\{x\} \times q(x)hF$ by W . Therefore R is one-to-one.

• The same calculations as in the proof of (B) on page 24 of [7] show that

$$\tilde{\mu} = \mu \times \nu_H \circ R.$$

• Now we show that R commutes with the actions $T_{\varphi, F}$ and $T_{\psi, F \cap H}$ of \mathfrak{R} on $X \times G/F$ and $X \times H/F \cap H$, respectively. Indeed, fix $t \in \mathfrak{R}$. Then

$$\begin{aligned} R((x, gF)t) &= R(xt, \varphi(x, t)gF) \\ &= (xt, q(xt)^{-1}\varphi(x, t)W_{F \cap H}(gF \cap \tau(x))(F \cap H)). \end{aligned}$$

On the other hand,

$$\begin{aligned} (R(x, gF))t &= (x, q(x)^{-1}W_{F \cap H}(gF \cap \tau(x))(F \cap H))t \\ &= (xt, \psi(x, t)q(x)^{-1}W_{F \cap H}(gF \cap \tau(x))(F \cap H)) \\ &= (xt, q(xt)^{-1}\varphi(x, t)q(x)q(x)^{-1}W_{F \cap H}(gF \cap \tau(x))(F \cap H)) \\ &= (xt, q(xt)^{-1}\varphi(x, t)W_{F \cap H}(gF \cap \tau(x))(F \cap H)). \end{aligned}$$

Notice that $\varphi(x, t)W_{F \cap H}(gF \cap \tau(x))(F \cap H) \in H$. Now the proof of Theorem 1 is complete. ■

Let G_i , $i = 1, 2$, be compact metric groups equipped with normalized Haar measures ν_i . Assume that $\varphi_i : X \rightarrow G_i$ are cocycles. We will describe all ergodic joinings of the ergodic extensions $T_{\varphi_1}, T_{\varphi_2}$ whose projection on $X \times X$ is a graph-joining of T with itself. More precisely, we will show that any such $\lambda \in J(T_{\varphi_1}, T_{\varphi_2})$ is a relatively independent extension of some isomorphism between the normal factors of T_{φ_i} , $i = 1, 2$, respectively.

THEOREM 2. *There exist closed normal subgroups $H_1 \subset G_1$, $H_2 \subset G_2$, a continuous group isomorphism $\zeta : G_1/H_1 \rightarrow G_2/H_2$, an $S \in C(T)$ and a measurable function $f : X \rightarrow G_2/H_2$ such that for all $A \in \mathcal{B} \otimes \mathcal{D}_1$ and $B \in \mathcal{B} \otimes \mathcal{D}_2$,*

$$\begin{aligned} &\lambda(A \times B) \\ &= \int_{X \times G_1/H_1} E(A | H_1)(x, gH_1) \cdot E(B | H_1)(S(x), f(x)\zeta(gH_1)) d(\mu \times \nu_1)(x, gH_1), \end{aligned}$$

where $E(A | H_i)$ denotes the conditional expectation of the characteristic function of A with respect to $\tilde{\mathcal{B}}_{H_i}$.

To prove Theorem 2 we first show a few lemmas.

LEMMA 4. *Assume that $\lambda \in J_2^{\circ}(T)$. Then $\lambda \in \{\mu_S : S \in C(T)\}$ iff for each $A \in \mathcal{B}$, there exist $B_1, B_2 \in \mathcal{B}$ such that $\lambda(X \times A \triangle B_1 \times X) = \lambda(A \times X \triangle X \times B_2) = 0$. ■*

Since the proof is almost the same as the one in [4] we omit it.

Take $\lambda \in J^{\circ}(T_{\varphi_1}, T_{\varphi_2})$. Now we construct two subgroups $H_i \subset G_i$, $i = 1, 2$, and some isomorphism between the factors $T_{\varphi_1, H_1}, T_{\varphi_2, H_2}$. Then we show that λ is a relatively independent extension of this isomorphism.

Let $\tilde{\pi} : X \times G_1 \times X \times G_2 \rightarrow X \times X$ be given by $\tilde{\pi}(x_1, g_1, x_2, g_2) = (x_1, x_2)$. Then $\lambda \circ \tilde{\pi}^{-1} \in J_2^{\circ}(T)$. Therefore $\lambda \circ \tilde{\pi}^{-1} = \mu_S$ for some $S \in C(T)$. Thus

$$\begin{aligned} \lambda\left(\bigcup_{x \in X} \{x\} \times G_1 \times \{Sx\} \times G_2\right) &= \mu_S \circ \tilde{\pi}\left(\bigcup_{x \in X} \{x\} \times G_1 \times \{Sx\} \times G_2\right) \\ &= \mu_S\left(\bigcup_{x \in X} \{x\} \times \{Sx\}\right) = 1. \end{aligned}$$

Therefore, via the map $(x, g_1, Sx, g_2) \mapsto (x, g_1, g_2)$, we can identify the following actions of the group \mathfrak{R} :

$$(T_{\varphi_1} \times T_{\varphi_2}, X \times G_1 \times X \times G_2, \lambda) \sim (T_{\varphi_1 \times \varphi_2 \circ S}, X \times G_1 \times G_1, \tilde{\lambda}),$$

where $\tilde{\lambda}$ is given by

$$\tilde{\lambda}(A \times B \times C) = \lambda(A \times B \times SA \times C),$$

and $(\varphi_2 \circ S)(x, t) = \varphi_2(Sx, t)$. Let $H \subset G_1 \times G_2$ be the stabilizer of $\tilde{\lambda}$. From Lemma 2 it follows that

$$\tilde{\lambda} = \int_X \delta_x \times (g_1, g_2)(x) \nu_H d\mu(x).$$

Let

$$H_1 = \{g_1 \in G_1 : (g_1, e_2) \in H\}, \quad H_2 = \{g_2 \in G_2 : (e_1, g_2) \in H\},$$

where e_i is the unit element of G_i , $i = 1, 2$. Let $\Pi_i : G_1 \times G_2 \rightarrow G_i$ be given by $\Pi_i(g_1, g_2) = g_i$, $i = 1, 2$.

LEMMA 5. $\Pi_i(H) = G_i$, $i = 1, 2$. ■

As a consequence of the above lemma we have

LEMMA 6. *The subgroup H_i is normal in G_i . ■*

It is easy to show the following lemma.

LEMMA 7.

- (i) $(g_1, g_2) \in H, (g_1, g'_2) \in H \Rightarrow g_2'^{-1}g_2 \in H_2$;
- (ii) $(g_1, g_2) \in H, (g'_1, g_2) \in H \Rightarrow g_1'^{-1}g_1 \in H_1$;
- (iii) $(g_1, g_2) \in H \Leftrightarrow g_1H_1 \times g_2H_2 \subset H$. ■

Define $\zeta : G_1/H_1 \rightarrow G_2/H_2$ by

$$\zeta(g_1H_1) = \Pi_2((g_1H_1 \times G_2) \cap H).$$

LEMMA 8. *The map ζ is a continuous group isomorphism. ■*

The last two lemmas yield

LEMMA 9. *We have*

$$H = \bigcup_{g \in G_1} gH_1 \times \zeta(gH_1)$$

and if $(h_1, h_2) \in H$ then $h_2\zeta(h_1^{-1}H_1) = H_2$. ■

Proof of Theorem 2. Let $\gamma : (G_1 \times G_2)/H \rightarrow G_2/H_2$ be given by

$$\gamma((g_1, g_2)H) = g_2\zeta(g_1^{-1}H_1).$$

From Lemma 9 we know that γ is well defined. Now we define $f : X \rightarrow G_2/H_2$ by

$$f(x) = \gamma(\tau(x)).$$

The required isomorphism between the normal natural factors T_{φ_1, H_1} and T_{φ_2, H_2} is defined by

$$\bar{S}(x, gH_1) = S_{f, \zeta}(x, gH_1) = (Sx, f(x)\zeta(gH_1)),$$

where S is an element of $C(T)$ for which $\lambda \circ \bar{\pi}^{-1} = \mu_S$.

Let $t \in \mathfrak{R}$. Then

$$\begin{aligned} S_{f, \zeta}((x, gH_1)t) &= S_{f, \zeta}(xt, \varphi_1(x, t)gH_1) \\ &= (S(xt), f(xt)\zeta(\varphi_1(x, t)gH_1)) \\ &= (S(xt), f(xt)\zeta(\varphi_1(x, t)H_1)\zeta(gH_1)) \\ &= (S(xt), \gamma(\tau(xt))\zeta(\varphi_1(x, t)H_1)\zeta(gH_1)) \\ &= (S(xt), \gamma((\varphi_1(x, t), \varphi_2(Sx, t))\tau(x))\zeta(\varphi_1(x, t)H_1)\zeta(gH_1)) \\ &= (S(xt), \gamma((\varphi_1(x, t)g_1(x), \varphi_2(Sx, t)g_2(x))H)\zeta(\varphi_1(x, t)H_1)\zeta(gH_1)) \\ &= (S(xt), \varphi_2(Sx, t)g_2(x)\zeta(g_1(x)^{-1}\varphi_1(x, t)^{-1}H_1)\zeta(\varphi_1(x, t)H_1)\zeta(gH_1)) \\ &= (S(xt), \varphi_2(Sx, t)g_2(x)\zeta(g_1(x)^{-1}H_1)\zeta(gH_1)) \\ &= (S(xt), \varphi_2(Sx, t)\gamma(\tau(x))\zeta(gH_1)) \\ &= (S(xt), \varphi_2(Sx, t)f(x)\zeta(gH_1)). \end{aligned}$$

On the other hand,

$$\begin{aligned} (S_{f, \zeta}(x, gH_1))t &= (Sx, f(x)\zeta(gH_1))t \\ &= ((Sx)t, \varphi_2(Sx, t)f(x)\zeta(gH_1)) \\ &= (S(xt), \varphi_2(Sx, t)f(x)\zeta(gH_1)). \end{aligned}$$

We have shown that $S_{f, \zeta}$ is an isomorphism.

From the definition of $\tilde{\lambda}$ we have

$$\tilde{\lambda} = \int_X \delta_x \times (g_1(x), g_2(x))\nu_H d\mu(x),$$

where $(g_1(x), g_2(x)) \in H = \tau(x)$ and $S_{f, \zeta}(x, g_1(x)H_1) = (Sx, g_2(x)H_2)$. This follows from the fact that

$$(g_1, g_2)H = \bigcup_{g \in G_1} gH_1 \times g_2\zeta(g_1^{-1}H_1)\zeta(gH_1),$$

and if $(g_1, g_2)H = \tau(x)$ then $f(x)\zeta(g_1H_1) = g_2H_2$.

So we have

$$\begin{aligned} \tilde{\lambda} &= \int_X \delta_x \\ &\quad \times \left(\int_{G_1/H_1} \nu_{H_1} \circ gH_1 \times \nu_{H_2} \circ (g_2(x)\zeta(g_1(x)^{-1}H_1)\zeta(gH_1)) d\nu_1(gH_1) \right) d\mu(x) \\ &= \int_X \delta_x \times \left(\int_{G_1/H_1} \nu_{H_1} \circ gH_1 \times \nu_{H_2} \circ (f(x)\zeta(gH_1)) d\nu_1(gH_1) \right) d\mu(x). \end{aligned}$$

Now we can calculate the decomposition of λ :

$$\begin{aligned} \lambda &= \int_X \delta_x \times \delta_{Sx} \\ &\quad \times \left(\int_{G_1/H_1} \nu_{H_1} \circ gH_1 \times \nu_{H_2} \circ (g_2(x)\zeta(g_1(x)^{-1}H_1)\zeta(gH_1)) d\nu_1(gH_1) \right) d\mu(x) \\ &= \int_{X \times G_1/H_1} (\mu \times \nu_{H_1})_{(x, gH_1)} \times (\mu \times \nu_{H_2})_{S_{f, \zeta}(x, gH_1)} d(\mu \times \nu_1)(x, gH_1) \\ &= \int_{X \times G_1/H_1} E(\cdot | H_1)(x, gH_1) \cdot E(\cdot | H_2)(S_{f, \zeta}(x, gH_1)) d(\mu \times \nu_1)(x, gH_1). \end{aligned}$$

Theorem 2 is proved. ■

THEOREM 3. Assume that T_{φ_1} and T_{φ_2} are ergodic. If T_{φ_2} is a factor of T_{φ_1} via the map \bar{S} then there exist $S \in C(T)$, a measurable map $f : X \rightarrow G_2$ and a continuous group epimorphism $\zeta : G_1 \rightarrow G_2$ such that

$$\bar{S}(x, g) = S_{f, \zeta}(x, g) = (Sx, f(x)\zeta(g)).$$

Proof. Assume that $\lambda = (\mu \times \nu_1)_{\bar{S}}$, so $\lambda \in J^e(T_{\varphi_1}, T_{\varphi_2})$. From Theorem 2 it follows that we can find normal subgroups $H_i \subset G_i$, $i = 1, 2$, a measurable $f : X \rightarrow G_2/H_2$ and a continuous group isomorphism $\varrho : G_1/H_1 \rightarrow G_2/H_2$ such that

$$\lambda = \int_{X \times G_1/H_1} E(\cdot | H_1)(x, gH_1) \cdot E(\cdot | H_2)(S_{f, \varrho}(x, gH_1)) d(\mu \times \nu_1)(x, gH_1).$$

But $H_2 = \{e_2\}$ because $\lambda = (\mu \times \nu_1)_{\bar{S}}$. Let $p : G_1 \rightarrow G_1/H_1$ be the natural homomorphism. Put $\zeta = \varrho \circ p$. Then

$$\begin{aligned} \lambda(A \times B) &= \int_{X \times G_1} E(A | H_1)(x, g) \cdot \chi_B \circ S_{f, \zeta}(x, g) d(\mu \times \nu_1)(x, g) \\ &= \int_{S_{f, \zeta}^{-1}(B)} E(A | H_1)(\text{Id} \times p)(x, g) d(\mu \times \nu_1)(x, g) \end{aligned}$$

$$\begin{aligned}
 &= \int_{S_{f,\zeta}^{-1}(B)} \chi_A(x, g) d(\mu \times \nu_1)(x, g) \\
 &= (\mu \times \nu_1)(A \cap S_{f,\zeta}^{-1}(B)),
 \end{aligned}$$

where $A \in \mathcal{B} \otimes \mathcal{D}_1$ and $B \in \mathcal{B} \otimes \mathcal{D}_2$ are arbitrary. We have thus shown that $\bar{S} = S_{f,\zeta}$. ■

When we take $G_1 = G_2 = G$ and $\varphi_1 = \varphi_2 = \varphi$ in Theorem 3 then we get the following theorem.

THEOREM 4. *If $\bar{S} \in C(T_\varphi)$ then there exist $S \in C(T)$, a measurable $f : X \rightarrow G$ and a continuous group epimorphism $\zeta : G \rightarrow G$ such that $\bar{S} = S_{f,\zeta}$.* ■

4. Structure of factors. Let $(T_\varphi, X \times G, \tilde{\mathcal{B}}, \tilde{\mu})$ be a weakly mixing group extension of an ergodic 2-fold simple action (T, X, \mathcal{B}, μ) . Let $\mathcal{C} \subset \tilde{\mathcal{B}}$ be a nontrivial factor of T_φ . Define

$$\lambda = \tilde{\mu} \times_{\mathcal{C}} \tilde{\mu} = \int_{X_{\mathcal{C}}} E(\cdot | \mathcal{C}) \cdot E(\cdot | \mathcal{C}) d\tilde{\mu}.$$

Then λ is a self-joining of T_φ .

We will need some lemmas.

LEMMA 10. *If $A \in \tilde{\mathcal{B}}$ then*

$$A \in \mathcal{C} \Leftrightarrow \lambda(A \times A^c \cup A^c \times A) = 0.$$

Proof. This follows from the definition of λ . ■

The measure λ is not necessarily ergodic so take its ergodic decomposition

$$\lambda = \int_{J_2^o(T_\varphi)} m d\xi(m).$$

Define $E = \{m \in J_2^o(T_\varphi) : \forall A \in \mathcal{C} m(A \times A^c \cup A^c \times A) = 0\}$. It is easy to show the following.

LEMMA 11. $\xi(E) = 1$. ■

Assume that $m \in J_2^o(T_\varphi)$ and $m \circ \tilde{\pi}^{-1} = \mu \times \mu$. Take $(g_1, g_2) \in G \times G$. Then $m \circ (g_1, g_2)$ is ergodic and its projection on $X \times X$ is the product measure. Set

$$\kappa = \int_{G \times G} m \circ (g_1, g_2) d\nu(g_1) d\nu(g_2).$$

We have $\kappa \circ (g_1, g_2) = \kappa$ and the projection of κ on $X \times X$ is a product measure. We have thus obtained the equality

$$\kappa = \mu \times \mu \times \nu \times \nu = \tilde{\mu} \times \tilde{\mu}.$$

So κ is ergodic and $m \circ (g_1, g_2) = \tilde{\mu} \times \tilde{\mu}$, in particular $m = \tilde{\mu} \times \tilde{\mu}$. Thus m cannot be in E , since \mathcal{C} is nontrivial. If $m \in E$ then Theorem 2 yields that

$$m = \int_{X \times G_1/H_1} E(\cdot | H_1)(x, gH_1) \cdot E(\cdot | H_2)(S_{f,\zeta}(x, gH_1)) d\tilde{\mu}(x, gH_1),$$

where S, f, ζ are as above.

LEMMA 12. $m \in E \Leftrightarrow \mathcal{C} \subset \tilde{\mathcal{B}}_{H_1, H_2}$ and for each $A \in \mathcal{C}$, $S_{f,\zeta}^{-1}(A) = A$.

Let F be the largest closed normal subgroup of G such that \mathcal{C} is a factor of $T_{\varphi, F}$. Consider the following subset of the centralizer of $T_{\varphi, F}$:

$$F(\mathcal{C}) = \{U \in C_1(T_\varphi) : \forall A \in \mathcal{C} U^{-1}(A) = A\}.$$

We know, from the definition of F , that $F(\mathcal{C}) \subset C(T_{\varphi, F})$. Assume that $\tilde{\mu}$ is a quotient measure on the quotient space $\tilde{\mathcal{B}}_F$. Let $\lambda = \tilde{\mu} \times \tilde{\mu}$. Then as a consequence of the last two lemmas we have

LEMMA 13. *For each $m \in E \cap J_2^o(T_{\varphi, F})$ there exists $\bar{S} \in C(T_{\varphi, F})$ satisfying $m = \tilde{\mu}_{\bar{S}}$.* ■

Now, we are in a position to pass to the main theorem of this part of the paper.

THEOREM 5. $\mathcal{C} = \{A \in \tilde{\mathcal{B}}_F : \forall U \in F(\mathcal{C}) U^{-1}(A) = A\}$. Moreover, $F(\mathcal{C})$ is a compact subgroup of $C(T_{\varphi, F})$.

Proof. From Lemmas 11–13 we know that the measure $\tilde{\mu} \times_{\mathcal{C}} \tilde{\mu}$ on $(X \times G/F) \times (X \times G/F)$ has a decomposition which consists solely of graph measures of the form $\tilde{\mu}_{\bar{S}}$ where $\bar{S} \in C(T_{\varphi, F})$. Now we can apply Theorem 1.8.2 of [2]. The proof of Theorem 5 is finished. ■

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Continuous linear right inverses for convolution operators in spaces of real analytic functions

by

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Abstract. We determine the convolution operators $T_\mu := \mu*$ on the real analytic functions in one variable which admit a continuous linear right inverse. The characterization is given by means of a slowly decreasing condition of Ehrenpreis type and a restriction of hyperbolic type on the location of zeros of the Fourier transform $\widehat{\mu}(z)$.

The existence of continuous linear right inverses for convolution operators $T_\mu := \mu*$ has been studied in many classes of (generalized) functions on \mathbb{R} : The first result for $C^\infty(\mathbb{R})$ was obtained by Ehrenpreis [5] and the problem was solved for nonquasianalytic ultradifferentiable functions and ultradistributions by Meise and Vogt [15] and Braun, Meise and Vogt [3]. The characterization was given through estimates on the location of zeros of the Fourier transform $\widehat{\mu}$ of the (ultra)distribution μ , similar to that for hyperbolic convolution operators (Ehrenpreis [5]). For convolution operators on holomorphic functions defined on convex open sets $\Omega \subset \mathbb{C}$ the corresponding question was solved in Taylor [25], Schwerdtfeger [24] and Meise [12] for $\Omega = \mathbb{C}$, and in Momm [19, 21, 22] for general convex $\Omega \neq \mathbb{C}$ (see also Korobeĭnik and Melikhov [8]), again leading to a restriction on the location of zeros of $\widehat{\mu}$, connected with the angular derivative on the boundary $\partial\Omega$ of the Riemann mapping function for Ω .

In the present paper, continuous linear right inverses for convolution operators on real analytic functions on open or compact intervals will be studied. Neither necessary conditions nor nontrivial positive examples seem to be known in this case.

Let $I \subset \mathbb{R}$ be an open interval and let $A(I)$ be the space of real analytic functions on I with its canonical topology. Fix $\mu \in A(\mathbb{R})'$ and assume $\text{supp } \mu = \{0\}$ if $I \neq \mathbb{R}$. Then μ defines a continuous linear convolution operator

$$T_\mu : A(I) \rightarrow A(I), \quad T_\mu(f)(x) := \langle \mu, f(x - y) \rangle.$$