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Outer factorization of operator valued weight functions on the torus

by

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Abstract. An exact criterion is derived for an operator valued weight function $W(e^{is}, e^{it})$ on the torus to have a factorization

$$W(e^{is}, e^{it}) = \Phi(e^{is}, e^{it})^* \Phi(e^{is}, e^{it}),$$

where the operator valued Fourier coefficients of Φ vanish outside of the Helson–Lowdenslager halfplane

$$\Lambda = \{(m,n) \in \mathbb{Z}^2 : m \geq 1\} \cup \{(0,n) : n \geq 0\},$$

and Φ is "outer" in a related sense. The criterion is expressed in terms of a regularity condition on the weighted space $L^2(W)$ of vector valued functions on the torus. A logarithmic integrability test is also provided. The factor Φ is explicitly constructed in terms of Toeplitz operators and other structures associated with W. The corresponding version of Szegö's infimum is given.

1. Introduction. Recall that a scalar valued weight function $W(e^{i\theta})$ on the unit circle has a factorization $W(e^{i\theta}) = |\Phi(e^{i\theta})|^2$, where Φ is an outer function, exactly when $\log \Phi$ is integrable. This, in turn, is equivalent to the "regularity" condition

$$\bigcap_{n=0}^{\infty} S^n \mathcal{F} = (0), \tag{1}$$

where $\mathcal{F}$ is the span of $\{e^{ik\theta} : k \geq 0\}$ in the weighted space $L^2(W)$, and S is multiplication by the independent variable $e^{i\theta}$.

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The present work deals with the analogous problem of factoring an operator valued weight function $W(e^{is}, e^{it})$ on the torus. Specifically, we seek a factorization $W(e^{is}, e^{it}) = \Phi(e^{is}, e^{it})^* \Phi(e^{is}, e^{it})$, where the operator coefficients of Φ vanish outside of the halfplane

$$(2) \qquad \Lambda = \{(m,n) \in \mathbb{Z}^2 : m \geq 1\} \cup \{(0,n) : n \geq 0\},$$

and Φ is outer in an appropriate, related sense. It is shown that the existence of this factorization is equivalent to a regularity condition similar to (1). Under this regularity assumption, the outer factor Φ is constructed in the following way. First, the weight function W is factored into A^*A as a function of its first argument, using a one-variable method from the theory of Toeplitz operators. The resulting A turns out to be a function of e^{is} with values which are multiplication operators in the variable e^{it}. This A might not be the outer function that we are looking for, however, because its coefficients need not vanish outside of Λ. But now the regularity condition and a second application of the Toeplitz theory provide a partial isometry valued function $\Theta(e^{it})$ such that $\Phi = \Theta A$ has the desired outer property, and $W = \Phi^*\Phi$.

Spaces of vector valued functions on the torus are introduced in Sections 2 and 3. The technical device from Toeplitz theory is adapted for the present purpose, along with the concept of "regularity." Section 4 introduces the notion of "Λ-outer" functions and narrows the main issue to bounded weight functions. Such weight functions determine Toeplitz operators, and in Section 5 a preliminary factorization is executed via the technical device prepared earlier. A second application of this device, in Section 6, effects the final adjustments in the factorization. The pieces are brought together in the main theorem, on the equivalence of factorization and regularity. Some uniqueness properties, an application to prediction theory, and a logarithmic integrability test for regularity are given. Historical matters are included in the final remarks.

In this paper, all Hilbert spaces are complex, all "operators" are bounded and linear, all "subspaces" are closed, and all "projections" are orthogonal. We omit the words "almost every" and "equivalence classes of" when this is not likely to cause confusion.

2. Lebesgue spaces and shift analysis. Let $\mathbb{T}$ be the unit circle in the complex plane $\mathbb{C}$, and write σ for normalized Lebesgue measure on $\mathbb{T}$. For a separable Hilbert space $\mathcal{C}$, take $L^p_{\mathcal{C}}(\sigma)$, $H^p_{\mathcal{C}}(\sigma)$, $L^p_{\mathcal{B}(\mathcal{C})}(\sigma)$ and $H^p_{\mathcal{B}(\mathcal{C})}(\sigma)$ to be the Lebesgue and Hardy spaces of $\mathcal{C}$- and $\mathcal{B}(\mathcal{C})$-valued functions on $\mathbb{T}$.

A principal tool of this investigation is the method of [5, 7] for factoring a nonnegative Toeplitz operator. This one-variable result will be utilized in the particular form stated in the lemma below, which relates the factorization to a regularity condition.

Let S be the shift operator $S : f(e^{is}) \to e^{is} f(e^{is})$ on $H^2_{\mathcal{C}}(\sigma)$. We say that an operator A on $H^2_{\mathcal{C}}(\sigma)$ is

- *S-analytic* if A is of the form $Af = Wf$ for some W in $H^\infty_{\mathcal{B}(\mathcal{C})}(\sigma)$;
- *S-outer* if A is S-analytic and $A\mathcal{H}^2_{\mathcal{C}}(\sigma)^- = H^2_M(\sigma)$ for some subspace M of $\mathcal{C}$;
- *S-Toeplitz* if A is of the form $Af = P(Wf)$ for some W in $L^\infty_{\mathcal{B}(\mathcal{C})}(\sigma)$, where P is the projection of $L^2_{\mathcal{C}}(\sigma)$ onto $H^2_{\mathcal{C}}(\sigma)$.

When $\mathcal{C}$ is one-dimensional, these definitions reduce to the familiar notions.

Suppose that T is a nonnegative S-Toeplitz operator on $H^2_{\mathcal{C}}(\sigma)$. We define the Hilbert space $\mathcal{T}$ to be the completion of $(\ker T)^\perp$ under the inner product $\langle \cdot, \cdot \rangle_{\mathcal{T}} = \langle T\cdot, \cdot\rangle_2$. Note that for f in $(\ker T)^\perp$,

$$(3) \qquad \|Sf\|^2_{\mathcal{T}} = \langle TSf, Sf\rangle_2$$
$$= \langle S^*TSf, f\rangle_2 = \langle Tf, f\rangle_2$$
$$(4) \qquad = \|f\|^2_{\mathcal{T}}.$$

Thus S extends to an isometry $S_{\mathcal{T}}$ on $\mathcal{T}$.

LEMMA 2.1. *A nonnegative S-Toeplitz operator T has a factorization $T = A^*A$, where A is S-analytic, if and only if the space $\mathcal{T}$ satisfies the condition*

$$(5) \qquad \bigcap_{n=0}^{\infty} S^n_{\mathcal{T}} \mathcal{T} = (0).$$

In this case, A may be chosen to be S-outer.

The assertion can be deduced in a straightforward way from [8, Theorem 3.4].

The Lebesgue spaces $L^p_{\mathcal{C}}(\sigma^2)$ and $L^p_{\mathcal{B}(\mathcal{C})}(\sigma^2)$ of functions on the torus $\mathbb{T}^2$ are defined in the obvious way. Note that a function $f(e^{is}, e^{it}) \in L^2_{\mathcal{C}}(\sigma^2)$ can be viewed as a member of $L^2_{L^2_{\mathcal{C}}(\sigma(e^{it}))}(\sigma(e^{is}))$, that is, as a function of e^{is}, with values in $L^2_{\mathcal{C}}(\sigma(e^{it}))$. This view will be taken as needed in order to exploit the one-variable theory. Likewise, we can identify $L^\infty_{\mathcal{B}(\mathcal{C})}(\sigma^2)$ with $L^\infty_{L^\infty_{\mathcal{B}(\mathcal{C})}(\sigma)}(\sigma)$.

For any subset Ω of $\mathbb{Z}^2$ we define $\mathcal{M}_{\mathcal{C}}(\Omega)$ to be that subspace of $L^2_{\mathcal{C}}(\sigma^2)$ which is spanned by $\{ce^{ims+int} : c \in \mathcal{C},\ (m,n) \in \Omega\}$. Of particular interest are the subspaces $\mathcal{M}_{\mathcal{C}}(\Lambda)$, where Λ is the halfplane given in (2), and $\mathcal{M}_{\mathcal{C}}(\Pi)$, where

$$(6) \qquad \Pi = \{(m,n) \in \mathbb{Z}^2 : m \geq 0\}.$$

Note that $\mathcal{M}_{\mathcal{C}}(\Pi)$ is isomorphic to $H^2_{L^2_{\mathcal{C}}(\sigma(e^{it}))}(\sigma(e^{is}))$. We omit the subscript, as in $\mathcal{M}(\Omega)$, when the coefficient space is the complex field $\mathbb{C}$.

For any function $f(e^{is}, e^{it})$ on $\mathbb{T}^2$, we define the operations S_1 and S_2 by

$$(7)\qquad (S_1 f)(e^{is}, e^{it}) = e^{is} f(e^{is}, e^{it}), \qquad (S_2 f)(e^{is}, e^{it}) = e^{it} f(e^{is}, e^{it}).$$

3. Weighted spaces and Λ-regularity. Suppose that $\mathcal{C}$ is a separable Hilbert space, and $W(e^{is}, e^{it})$ is a $\mathcal{B}(\mathcal{C})$-valued weight function on $\mathbb{T}^2$. Thus W belongs to $L^1_{\mathcal{B}(\mathcal{C})}(\sigma^2)$, and its values are nonnegative operators on $\mathcal{C}$. Such a W determines a Hilbert space $L^2(W)$ of $\mathcal{C}$-valued functions: Take the completion of the vector space generated by the trigonometric functions $\{ce^{ims+int} : c \in \mathcal{C},\ (m,n) \in \mathbb{Z}^2\}$, under the inner product

$$(8)\qquad \langle f, g\rangle_W = \iint \langle W(e^{is}, e^{it}) f(e^{is}, e^{it}), g(e^{is}, e^{it})\rangle_{\mathcal{C}}\, d\sigma(e^{is})\, d\sigma(e^{it})$$

(as usual identifying f with 0 if $\langle f, f\rangle_W = 0$). Observe that S_1 and S_2 (from (7)) are isometries on $L^2(W)$. Again, the subsets of $\mathbb{Z}^2$ generate natural subspaces of $L^2(W)$. For any $\Omega \subset \mathbb{Z}^2$, we take $\mathcal{M}_W(\Omega)$ to be the subspace spanned by $\{ce^{ims+int} : c \in \mathcal{C},\ (m,n) \in \Omega\}$. (The notation is efficient, if somewhat inconsistent with the previously defined $\mathcal{M}_{\mathcal{C}}(\Omega)$.) The frequency sets Λ (from (2)) and Π (from (6)) are also of particular importance here. Check that for any weight function W,

$$(9)\qquad \mathcal{M}_W(\Pi) = \bigvee_{n=0}^{\infty} S_2^{-n} \mathcal{M}_W(\Lambda).$$

The space $L^2(W)$ is important in prediction theory, where it represents the spectral domain of some $\mathcal{C}$-valued stationary process, and the concept of regularity is analogous to "complete nondeterminism" of the process. Motivated by these considerations, we present the following idea.

DEFINITION 3.1. Let W be a $\mathcal{B}(\mathcal{C})$-valued weight function on $\mathbb{T}^2$. We say that the Hilbert space $L^2(W)$ is *Λ-regular* if

$$(10)\qquad \bigcap_{m=0}^{\infty} S_1^m \mathcal{M}_W(\Pi) = (0),$$

and

$$(11)\qquad \bigcap_{n=0}^{\infty} S_2^n \mathcal{M}_W(\Lambda) = S_1 \mathcal{M}_W(\Pi).$$

Indeed, when $\mathcal{C}$ is finite-dimensional, Definition 3.1 reduces to its classical counterparts in [2, 4].

Remark. Here and elsewhere we speak of "functions" belonging to $L^2(W)$ (or some other Hilbert space), when of course we mean equivalence

classes of Cauchy sequences. If the underlying coefficient space $\mathcal{C}$ is finite-dimensional, then the distinction is minor, for then the elements of $L^2(W)$ admit (possibly non-unique) functional representatives [9, Lemma 7.1]. But when $\mathcal{C}$ is infinite-dimensional this circumstance may fail, even in otherwise simple examples. For instance, let $\{v_n\}_{n=0}^{\infty}$ be an orthonormal basis for $\mathcal{C}$, and suppose that W is the constant weight function such that $\langle Wv_m, v_n\rangle_{\mathcal{C}} = 2^{-m}\delta_{mn}$, where δ is the Kronecker symbol. Then the sequence of constant functions $\{f_n\}_{n=0}^{\infty}$, where $f_n = v_0 + v_1 + \ldots + v_n$, is Cauchy in $L^2(W)$, but evidently fails to converge to a $\mathcal{C}$-valued function. In what follows, it is therefore necessary to interpret these spaces with caution.

4. The scalar mollifier and Λ-outer functions. We now seek the definition of "outer function" which is appropriate for the present factorization problem. The first step is to introduce the scalar mollifier. Let $\nu(e^{is}, e^{it})$ be a measurable, nonnegative real valued function such that $\log \nu$ is integrable. Define, for z in the unit disc,

$$(12)\qquad a(z, e^{it}) = \exp \int \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \nu(e^{i\theta}, e^{it})\, d\sigma(e^{i\theta}),$$

$$g(z) = \exp \int \frac{e^{i\theta} + z}{e^{i\theta} - z} \log a(0, e^{i\theta})\, d\sigma(e^{i\theta}).$$

Now extract radial limits in z, and put

$$(13)\qquad \phi(e^{is}, e^{it}) = g(e^{it}) a(e^{is}, e^{it}) / a(0, e^{it}).$$

We say that a scalar valued function $\phi(e^{is}, e^{it})$ is *Λ-outer* if it can be constructed in the above manner. (Once again, the terminology sacrifices typographical consistency for efficiency.)

This notion of outer function has its roots in [2], in which a square integrable ϕ is constructed using a geometric argument. The scheme leading to (13) is due to [3, Theorem 1.2.3]. The factorization of operator weight functions will be patterned after this scheme.

Observe that for e^{it} fixed, each $\phi(\cdot, e^{it})$ is outer (in the standard sense); $g(\cdot)$ is outer; and $|\phi| = \nu$. Replacing ν with $1/\nu$, we see that $1/\phi$ is also Λ-outer. If ϕ is both Λ-outer and square integrable, then ϕ belongs to $\mathcal{M}(\Lambda)$. Furthermore, [3] then provides that

$$(14)\qquad \bigvee\{\phi(e^{is}, e^{it}) e^{ims+int} : (m,n) \in \Lambda\} = \mathcal{M}(\Lambda).$$

This, together with (9) (with constant weight 1), gives

$$(15)\qquad \bigvee\{\phi(e^{is}, e^{it}) e^{ims+int} : (m,n) \in \Pi\} = \mathcal{M}(\Pi).$$

That is, ϕ can be interpreted as an S_1-outer operator on the space $H^2_{L^2(\sigma)}(\sigma)$ (which is isomorphic to $\mathcal{M}(\Pi)$).

All of this suggests the following definition of Λ-outer for bounded operator valued functions. Let $\mathcal{C}$ be a separable Hilbert space. Put $\mathcal{H} = \mathcal{M}_{\mathcal{C}}(\Pi)$ (isomorphic to $H^2_{L^2_{\mathcal{C}}(\sigma)}(\sigma)$), so that S_1 is a shift operator on $\mathcal{H}$. Define $\mathcal{K} = \ker S_1^*$, a copy of $L^2_{\mathcal{C}}(\sigma(e^{it}))$, and let P_0 be the projection of $\mathcal{H}$ onto $\mathcal{K}$. Write Φ_0 for $P_0 \Phi P_0$.

DEFINITION 4.1. Let $\Phi \in L^\infty_{\mathcal{B}(\mathcal{C})}(\sigma^2)$. We say that Φ is *Λ-outer* if

(i) Φ determines an S_1-outer operator on $H^2_{L^2_{\mathcal{C}}(\sigma(e^{it}))}(\sigma(e^{is}))$; and

(ii) Φ_0 determines an S_2-outer operator on $H^2_{\mathcal{C}}(\sigma(e^{it}))$.

Indeed, in the simple case $\mathcal{C} = \mathbb{C}$, the conditions of Definition 4.1 guarantee that $\log|\Phi|$ is integrable, and hence Φ can then be recovered from (12) and (13) by taking $\nu = |\Phi|$.

This definition is structurally consistent with that from the standard one-variable theory.

PROPOSITION 4.2. *Let $\Phi \in L^\infty_{\mathcal{B}(\mathcal{C})}(\sigma^2)$. Then Φ is Λ-outer if and only if*

$$(\Phi \mathcal{M}_{\mathcal{C}}(\Lambda))^- = \mathcal{M}_N(\Lambda) \tag{16}$$

for some subspace N of $\mathcal{C}$.

Proof. Assume first that Φ is Λ-outer. Then Φ is S_1-outer, so that $\overline{\Phi\mathcal{H}} = H^2_M(\sigma)$, for some subspace M of $L^2_{\mathcal{C}}(\sigma)$. Consider Φ_0, viewed as a function of e^{it}. By hypothesis Φ_0 is S_2-outer, hence $(\Phi_0 H^2_{\mathcal{C}}(\sigma))^- = H^2_N(\sigma)$ for some subspace N of $\mathcal{C}$. It follows that $(\Phi_0 L^2_{\mathcal{C}}(\sigma))^- = L^2_N(\sigma)$. But from the S_1-outer property of Φ, [8, Theorem B, p. 98] provides that $(\Phi_0 L^2_{\mathcal{C}}(\sigma))^- = M$. Thus $M = L^2_N(\sigma)$, and consequently $\overline{\Phi\mathcal{H}} = H^2_{L^2_N(\sigma)}(\sigma)$. This can be expressed as $(\Phi\mathcal{M}_{\mathcal{C}}(\Pi))^- = \mathcal{M}_N(\Pi)$. Finally, let $\mathcal{L}$ be the left side of (16). Pick a trigonometric polynomial $f \in \mathcal{M}_{\mathcal{C}}(\Lambda)$, and write $f(e^{is}, e^{it}) = \sum f_j(e^{it}) e^{ijs}$. Note that $\Phi_0 f_0$ lies in $H^2_N(\sigma(e^{it}))$, that $\Phi f - \Phi_0 f_0$ lies in $S_1 \mathcal{M}_N(\Pi)$, and that $\Phi_0 f_0$ is orthogonal to $S_1 \mathcal{M}_N(\Pi)$. Since $S_1 \mathcal{M}_N(\Pi)$ is already a subspace of $\mathcal{L}$, and such functions Φf are dense in $\mathcal{L}$, it must be that $\mathcal{L}$ is the orthogonal sum of $S_1 \mathcal{M}_N(\Pi)$ and the span of all the $\Phi_0 f_0$ (that is, a copy of $H^2_N(\sigma)$, viewed as a subspace of $\mathcal{K}$). But this is to say that $\mathcal{L} = \mathcal{M}_N(\Lambda)$, which is (16).

Conversely, suppose that (16) holds. Then by (9), we have $(\Phi\mathcal{M}_{\mathcal{C}}(\Pi))^- = \mathcal{M}_N(\Pi)$, showing that Φ is S_1-outer. Define $\mathcal{K}$, P_0, Φ_0 and M as before. From (16) we see that $M = \overline{\Phi_0 \mathcal{K}} = (\Phi_0 L^2_{\mathcal{C}}(\sigma))^- = L^2_N(\sigma)$. Again, represent an arbitrary trigonometric polynomial $f \in \mathcal{M}_{\mathcal{C}}(\Lambda)$ as $f(e^{is}, e^{it}) = \sum f_j(e^{it}) e^{ijs}$, and consider $\Phi f = \Phi_0 f_0 + (\Phi f - \Phi_0 f_0)$. The second term lies in $S_1 \mathcal{M}_N(\Pi)$, which is already a subspace of $\mathcal{M}_N(\Lambda)$. In order for (16) to hold, it must be that the collection of all $\Phi_0 f_0$ generates $H^2_N(\sigma(e^{it}))$; in other words, Φ_0 is S_2-outer. Thus Φ is Λ-outer. ∎

The next observation allows us to extend the definition of Λ-outer to operator valued functions which are not necessarily bounded. Its proof is a simple modification of [8, Theorem A, p. 97].

PROPOSITION 4.3. *Let Φ be a weakly measurable $\mathcal{B}(\mathcal{C})$-valued function on $\mathbb{T}^2$, and let ϕ and ψ be bounded, scalar valued, Λ-outer functions such that $\phi\Phi$ and $\psi\Phi$ belong to $L^\infty_{\mathcal{B}(\mathcal{C})}(\sigma^2)$. Then*

$$(\phi\Phi\mathcal{M}_{\mathcal{C}}(\Lambda))^- = (\psi\Phi\mathcal{M}_{\mathcal{C}}(\Lambda))^-. \tag{17}$$

DEFINITION 4.4. Let Φ be a weakly measurable $\mathcal{B}(\mathcal{C})$-valued function on $\mathbb{T}^2$. We say that Φ is *Λ-outer* if, for some bounded, scalar valued, Λ-outer function ϕ, the $\mathcal{B}(\mathcal{C})$-valued function $\phi\Phi$ is bounded and Λ-outer in the sense of Definition 4.1. In this situation, we refer to ϕ as a *scalar mollifier* for Φ, and we write $N(\Phi) = N$ for the subspace N associated with $\phi\Phi$ in Proposition 4.2.

The definition is sound. For if Φ is Λ-outer under two scalar mollifiers ϕ and ψ, then Propositions 4.2 and 4.3 give $(\phi\Phi\mathcal{M}_{\mathcal{C}}(\Lambda))^- = (\psi\Phi\mathcal{M}_{\mathcal{C}}(\Lambda))^- = \mathcal{M}_N(\Lambda)$, for some subspace N of $\mathcal{C}$. Evidently, N does not depend on the choice of scalar mollifier.

DEFINITION 4.5. Let W be a $\mathcal{B}(\mathcal{C})$-valued weight function on $\mathbb{T}^2$. We say that W has a *Λ-outer factorization* if

$$W(e^{is}, e^{it}) = \Phi(e^{is}, e^{it})^* \Phi(e^{is}, e^{it}) \tag{18}$$

for some Λ-outer function Φ. In this case Φ is said to be a *Λ-outer factor* of W.

Finally, we use the scalar mollifier to reduce the main factorization problem to the case of bounded weight functions.

LEMMA 4.6. *Suppose that W is a $\mathcal{B}(\mathcal{C})$-valued weight function on $\mathbb{T}^2$, and let $u(e^{is}, e^{it}) = (\max\{1, \|W(e^{is}, e^{it})\|_{\mathcal{B}(\mathcal{C})}\})^{-1}$. Then*

(i) W has a Λ-outer factorization if and only if uW has a Λ-outer factorization;

(ii) $L^2(W)$ is Λ-regular if and only if $L^2(uW)$ is Λ-regular.

Proof. In any case, the scalar valued function u is nonnegative and bounded, and $\log u$ is integrable. Let ϕ be the bounded, scalar valued, Λ-outer function defined through (13) with $\nu = u^{1/2}$ in (12).

If W has a Λ-outer factor Φ, then $\phi\Phi$ is a Λ-outer factor for uW. Conversely, if uW has a Λ-outer factor Ψ, then Ψ/ϕ is a Λ-outer factor for W. Indeed, Ψ/ϕ is Λ-outer with scalar mollifier ϕ. This proves (i).

For (ii), let $c \in \mathcal{C}$ be a unit vector, let f and g be polynomials in $\mathcal{M}(\Lambda)$, and consider the following estimates.

$$\begin{aligned}(19)\quad \iint \langle Wc(\phi f+g), c(\phi f+g)\rangle_{\mathcal{C}}\,d\sigma^2 &\le \iint \|W\|_{\mathcal{B}(\mathcal{C})}\|c\|_{\mathcal{C}}^2|\phi f+g|^2\,d\sigma^2\\ &\le \iint u^{-1}\cdot 1\cdot|\phi f+g|^2\,d\sigma^2\\ (20)\quad &= \iint |f+\phi^{-1}g|^2\,d\sigma^2.\end{aligned}$$

Let $f=1$. Since $1/\phi$ is square integrable and Λ-outer, (14) implies that the last quantity can be made arbitrarily close to zero by choosing g appropriately. It follows that $c\phi$ belongs to $\mathcal{M}_W(\Lambda)$ for all $c \in \mathcal{C}$, and consequently ϕ determines an isometry from $L^2(uW)$ into $L^2(W)$ such that $(\phi\mathcal{M}_{uW}(\Lambda))^- \subset \mathcal{M}_W(\Lambda)$. On the other hand, fix $g=1$. Choosing f close to $1/\phi$ in $\mathcal{M}(\Lambda)$, we see that c lies in the closure of $\phi\mathcal{M}_{uW}(\Lambda)$. Thus $(\phi\mathcal{M}_{uW}(\Lambda))^- = \mathcal{M}_W(\Lambda)$. From (9) we also get $(\phi\mathcal{M}_{uW}(\Pi))^- = \mathcal{M}_W(\Pi)$. The assertion (ii) follows from these observations. ■

5. A preliminary factorization. We now establish a factorization of an operator weight function W into A^*A, where the factor A is analogous to the function $a(e^{is},e^{it})$ in (12) in the scalar valued case. With that done, the desired Λ-outer factor will be constructed in the next section by making an adjustment to A corresponding to (13).

LEMMA 5.1. *Let $W(e^{is},e^{it})$ be a bounded $\mathcal{B}(\mathcal{C})$-valued weight function on $\mathbb{T}^2$. If $L^2(W)$ satisfies the regularity condition* (10), *then there exists a $\mathcal{B}(\mathcal{C})$-valued function $A(e^{is},e^{it})$ on $\mathbb{T}^2$ such that A determines an S_1-outer operator on $H^2_{L^2_{\mathcal{C}}(\sigma(e^{it}))}(\sigma(e^{is}))$, and $W(e^{is},e^{it}) = A(e^{is},e^{it})^*A(e^{is},e^{it})$.*

Proof. Let $\mathcal{H} = H^2_{L^2_{\mathcal{C}}(\sigma)}(\sigma)$ $(\cong \mathcal{M}_{\mathcal{C}}(\Pi))$ and let P be the projection of $L^2_{L^2_{\mathcal{C}}(\sigma)}(\sigma)$ $(\cong L^2_{\mathcal{C}}(\sigma^2))$ onto $\mathcal{H}$. The restriction of S_1 to $\mathcal{H}$ is a shift operator on $\mathcal{H}$; let $\mathcal{K} = \ker S_1^*$ $(\cong L^2_{\mathcal{C}}(\sigma))$, and let P_0 be the projection of $\mathcal{H}$ onto $\mathcal{K}$.

The mapping $T: f \to P(Wf)$ is an S_1-Toeplitz operator on $\mathcal{H}$. By the assumption (10), the condition (5) of Lemma 2.1 is met, where $\mathcal{M}_W(\Pi)$ plays the role of $\mathcal{T}$. The conclusion is that $T = A^*A$, for some S_1-outer operator A. As yet, A is a function of e^{is}, with values in $\mathcal{B}(L^2_{\mathcal{C}}(\sigma(e^{it})))$. It remains to show that A can be interpreted as a $\mathcal{B}(\mathcal{C})$-valued function of the two variables (e^{is},e^{it}). To do this, it suffices to show that A commutes with S_2.

Thus we turn to the construction of A in [8, Section 3.4]. With $\mathcal{H}$, S_1, $\mathcal{K}$ and T given as above, define $\mathcal{H}_T = \overline{T^{1/2}\mathcal{H}}$, viewed as a Hilbert space with the inner product of $\mathcal{H}$. Let S_T be Lowdenslager's isometry, the unique isometry on $\mathcal{H}_T$ such that

$$(21)\qquad S_T(T^{1/2}f) = T^{1/2}(S_1 f)$$

for all $f \in \mathcal{H}$. By the hypotheses, S_T is a shift operator. Write $\mathcal{K}_T = \ker S_T^*$, and define $J = T^{1/2}|\mathcal{K}_T \in \mathcal{B}(\mathcal{K}_T,\mathcal{K})$. Now J^* has a polar decomposition V^*R, where $R = (JJ^*)^{1/2} \in \mathcal{B}(\mathcal{K})$, and $V \in \mathcal{B}(\mathcal{K}_T,\mathcal{K})$ is a partial isometry. In fact, [8, Section 3.4] shows that V is an isometry. With that, the S_1-outer operator A has the operator-strongly convergent series representation

$$(22)\qquad A = \sum_{j=0}^{\infty} S_1^j V(I - S_T S_T^*) S_T^{*j} T^{1/2}.$$

Evidently, it is now enough to show that S_2 (or rather, suitable restrictions of that operator) commutes with V, S_T, S_T^* and $T^{1/2}$.

Obviously, S_2 commutes with the nonnegative operator T, and hence with $T^{1/2}$. Next, observe that $\mathcal{H}_T$ is reducing for S_2, and consider S_2 restricted to $\mathcal{H}_T$. For any $f \in \mathcal{H}$, and for $k = \pm 1$, we have

$$\begin{aligned}(23)\qquad S_T S_2^k T^{1/2} f &= S_T T^{1/2} S_2^k f = T^{1/2} S_1 S_2^k f\\ &= S_2^k T^{1/2} S_1 f = S_2^k S_T T^{1/2} f.\end{aligned}$$

Therefore S_T commutes with S_2^k, and consequently so does S_T^*. We now turn to J, which can be written

$$(24)\qquad J = (I - S_1S_1^*)T^{1/2}(I - S_TS_T^*).$$

Since $\mathcal{K}$ and $\mathcal{K}_T$ are reducing for S_2, and S_2 doubly commutes with S_1, S_T and $T^{1/2}$, it follows that $JS_2 = S_2J$. A similar argument yields $J^*S_2 = S_2J^*$. These in turn show that S_2 commutes with R. Lastly, the defining condition $J^* = V^*R$ allows

$$(25)\qquad (S_2|\mathcal{K})RV = (S_2|\mathcal{K})J = J(S_2|\mathcal{K}_T) = RV(S_2|\mathcal{K}_T);$$

at the same time,

$$(26)\qquad (S_2|\mathcal{K})RV = R(S_2|\mathcal{K})V.$$

Consequently, $R[(S_2|\mathcal{K})V - V(S_2|\mathcal{K}_T)] = 0$. But the final space of V is $\overline{R\mathcal{K}}$, which is reducing for S_2. We conclude that $(S_2|\mathcal{K})V = V(S_2|\mathcal{K}_T)$. This verifies all the needed commutations. ■

With the notation of Lemma 5.1 and its proof, let $A_0 = P_0AP_0$. The proof in [8, Section 3.4] shows that A_0 coincides with the nonnegative operator R; by the above we see that it can be represented as a function of e^{it}. Henceforth, let us write $A_0(e^{it})$ for the $\mathcal{B}(\mathcal{C})$-valued function that arises in this way.

There is a characterization of S_1-outer functions in terms of only one-variable functions.

PROPOSITION 5.2. *Suppose that $\Phi \in L^\infty_{\mathcal{B}(\mathcal{C})}(\sigma^2)$ determines an S_1-analytic operator on $\mathcal{H} = H^2_{L^2_{\mathcal{C}}(\sigma(e^{it}))}(\sigma(e^{is}))$. Then Φ is S_1-outer on $\mathcal{H}$ if and only if $\Phi(\cdot, e^{it})$ is S_1-outer on $H^2_{\mathcal{C}}(\sigma(e^{is}))$ for σ-almost every fixed e^{it}.*

Proof. With Φ given, let $W = \Phi^*\Phi$ and adopt the notation from Lemma 5.1 and its proof. The S_1-Toeplitz operator T commutes with S_2, and can be seen as a $\mathcal{B}(H^2_{\mathcal{C}}(\sigma(e^{is})))$-valued function of e^{it}. In fact, for e^{it} fixed, the mapping $T(e^{it}) : f(e^{is}) \to P(W(e^{is}, e^{it})f(e^{is}))$ is a nonnegative S_1-Toeplitz operator on $H^2_{\mathcal{C}}(\sigma(e^{is}))$. Let $S_T(e^{it})$ be the Lowdenslager isometry corresponding to $T(e^{it})$ for e^{it} fixed. Then $S_T(e^{it})$, as an operator on $\mathcal{H}$, is an isometry satisfying

$$S_T(e^{it})(T(e^{it})^{1/2} f(e^{is}, e^{it})) = T(e^{it})^{1/2}(S_1 f(e^{is}, e^{it})).$$

Thus $S_T(e^{it})$ agrees with S_T as an operator on $\mathcal{H}$. In an analogous way, for each e^{it} we can define $J(e^{it})$, $R(e^{it})$ and $V(e^{it})$ as operators on $\mathcal{H}$; these are likewise seen to coincide with the operators J, R and V, respectively. Now let A be constructed as in Lemma 5.1. Then by the above reasoning, $A(\cdot, e^{it})$ is S_1-outer on $H^2_{\mathcal{C}}(\sigma(e^{is}))$ for every e^{it}. In any case, $\Phi = BA$, where B is S_1-inner. It is straightforward to check that B also commutes with S_2, and hence we may write $B = B(e^{is}, e^{it})$. So Φ is S_1-outer exactly when B is S_1-constant, which is to say that $\Phi(e^{is}, e^{it}) = B(e^{it})A(e^{is}, e^{it})$. This, in turn, occurs if and only if $\Phi(\cdot, e^{it})$ is S_1-outer on $H^2_{\mathcal{C}}(\sigma)$ for σ-almost every e^{it}. ∎

In light of Proposition 5.2, there is also another description of Λ-outer functions.

COROLLARY 5.3. *Let $\Phi \in L^\infty_{\mathcal{B}(\mathcal{C})}(\sigma^2)$ determine an S_1-analytic operator. Then Φ is Λ-outer if and only if*

(i) *$\Phi(\cdot, e^{it})$ is S_1-outer on $H^2_{\mathcal{C}}(\sigma)$ for σ-almost every e^{it}; and*
(ii) *$\Phi_0(\cdot)$ is S_2-outer on $H^2_{\mathcal{C}}(\sigma)$.*

Furthermore, we can use Proposition 5.2 to describe the space M associated with a bounded S_1-outer function $A(e^{is}, e^{it})$. Specifically, M is defined to be the subspace of $L^2_{\mathcal{C}}(\sigma)$ such that $(AH^2_{L^2_{\mathcal{C}}}(\sigma))^- = H^2_M(\sigma)$. It also has the property that $(A_0 L^2_{\mathcal{C}}(\sigma))^- = M$. But Proposition 5.2 asserts that for each e^{it} fixed, there is a subspace $M(e^{it})$ of $\mathcal{C}$ such that $(A(\cdot, e^{it})H^2_{\mathcal{C}}(\sigma))^- = H^2_{M(e^{it})}(\sigma)$, and $(A_0(e^{it})\mathcal{C})^- = M(e^{it})$. These spaces must be related in the following way.

COROLLARY 5.4. *Let M and $M(e^{it})$ be the spaces associated with the S_1-outer function $A(e^{is}, e^{it})$. Then*

$$M = \int \oplus\, M(e^{it})\, d\sigma(e^{it}). \tag{27}$$

Note that in Proposition 5.2 and Corollary 5.3 it was assumed at the start that the S_1-outer operator Φ commutes with S_2, that is, Φ admits the functional form $\Phi(e^{is}, e^{it})$. Of course, not all S_1-analytic operators admit this form, and these results do not apply in such generality.

6. The Λ-outer factorization. Assume the notation and conditions of Lemma 5.1. Thus $W(e^{is}, e^{it})$ is a bounded weight function on $\mathbb{T}^2$, and W factors into A^*A, where $A \in L^\infty_{\mathcal{B}(\mathcal{C})}(\sigma^2)$ is S_1-outer, and A_0 is a nonnegative $\mathcal{B}(\mathcal{C})$-valued function. The factor A generally fails to be Λ-outer, since A_0 is not guaranteed to be S_2-outer: This is not surprising, as we have yet to impose the regularity condition (11). When this is done we will produce the adjusting factor corresponding to $g(e^{it})/a(0, e^{it})$ in the scalar case (13). Then the construction of the Λ-outer factor is completed in an analogous way.

We first check that A determines an isometry from $\mathcal{M}_W(\Pi)$ onto a dense set in $H^2_M(\sigma)$ (with M defined as for Corollary 5.4). For if f and g belong to $\mathcal{M}_W(\Pi)$, then (writing $\mathcal{H} = H^2_{L^2_{\mathcal{C}}(\sigma)}(\sigma) \cong \mathcal{M}_{\mathcal{C}}(\Pi)$)

$$\langle f, g\rangle_W = \langle Tf, g\rangle_{\mathcal{H}} = \langle A^*Af, g\rangle_{\mathcal{H}} = \langle Af, Ag\rangle_{\mathcal{H}}. \tag{28}$$

If f lies in $\mathcal{M}_W(\Lambda) \ominus S_1\mathcal{M}_W(\Pi)$, then for all $g \in \mathcal{M}_W(\Pi)$,

$$\begin{aligned} 0 = \langle f, S_1 g\rangle_W &= \langle Tf, S_1 g\rangle_{\mathcal{H}} \\ &= \langle A^*Af, S_1 g\rangle_{\mathcal{H}} = \langle Af, AS_1 g\rangle_{\mathcal{H}} = \langle Af, S_1 Ag\rangle_{\mathcal{H}}. \end{aligned} \tag{29}$$

That is, Af belongs to $H^2_M \ominus S_1 H^2_M$ (a copy of M). But more can be said. For any polynomial $f(e^{is}, e^{it}) = \sum f_j(e^{it})e^{ijs} \in \mathcal{M}_W(\Lambda)$, consider $Af = A_0 f_0 + (Af - A_0 f_0)$. The first term is in the span $\mathcal{F}$ of $\{A_0 c e^{int} : c \in \mathcal{C}, n \ge 0\}$ in $L^2_{\mathcal{C}}(\sigma)$, which is a subspace of $H^2_M \ominus S_1 H^2_M$, while the rest sits in $S_1 H^2_M(\sigma)$. Therefore the image of $\mathcal{M}_W(\Lambda) \ominus S_1\mathcal{M}_W(\Pi)$ under A is dense in $\mathcal{F}$. Assume that the regularity condition (11) holds, so that by the above correspondence

$$\bigcap_{n=0}^{\infty} S_2^n \mathcal{F} = (0). \tag{30}$$

Accordingly, the one-variable weighted space $\mathcal{T} = L^2(A_0^2)$ satisfies condition (5) of Lemma 2.1. The conclusion is that $A_0(e^{it})^2 = G(e^{it})^* G(e^{it})$ for some S_2-outer $\mathcal{B}(\mathcal{C})$-valued function G. We may choose G_0 to be a nonnegative operator. There is a subspace N of $\mathcal{C}$ such that $(GH^2_{\mathcal{C}}(\sigma))^- = H^2_N(\sigma)$ and $\overline{G_0\mathcal{C}} = N$. Finally, for e^{it} fixed, let $\Theta(e^{it})$ be the partial isometry with initial space $(A_0(e^{it}))^- = M(e^{it})$, and final space N, such that

$$G(e^{it}) = \Theta(e^{it}) A_0(e^{it}) \tag{31}$$

(thus $\Theta(e^{it})$ corresponds to $g(e^{it})/a(0,e^{it})$ in (13)). At last we define

$$\Phi(e^{is},e^{it}) = \Theta(e^{it})A(e^{is},e^{it}). \tag{32}$$

Check that Φ is S_1-outer and $\Phi_0 = \Theta A_0 = G$ is S_2-outer: In other words, Φ is Λ-outer. Furthermore, $\Phi^*\Phi = A^*\Theta^*\Theta A = A^*A = W$, which means that Φ is a Λ-outer factor of W.

If $\Phi(e^{is},e^{it})$ is a $\mathcal{B}(\mathcal{C})$-valued function, we define

$$\Phi_{0,0} = \iint \Phi\, d(\sigma\times\sigma)$$

when the integral exists in the weak sense. If Φ is constructed as above, then $\Phi_{0,0}$ coincides with the nonnegative operator G_0. More generally, if Φ is Λ-outer, $\Phi^*\Phi$ is integrable, and ϕ is a scalar mollifier for Φ, then $(\phi\Phi)_{0,0} = \phi_{0,0}\Phi_{0,0}$.

We have established part of the main result.

THEOREM 6.1. *Let $\mathcal{C}$ be a separable Hilbert space, and suppose that $W(e^{is},e^{it})$ is a $\mathcal{B}(\mathcal{C})$-valued weight function on $\mathbb{T}^2$. Then W has a Λ-outer factorization $W = \Phi^*\Phi$ if and only if the weighted Hilbert space $L^2(W)$ is Λ-regular. In this case*

(i) *if $W = \Psi^*\Psi$ is another Λ-outer factorization of W, then $\Psi = B\Phi$, where $B \in \mathcal{B}(\mathcal{C})$ is a partial isometry with inital space $N(\Phi)$ and final space $N(\Psi)$;*

(ii) *there exists a unique Λ-outer factor Φ of W such that $\Phi_{0,0}$ is a nonnegative operator.*

Proof. To begin, we may assume that W is bounded, by virtue of Lemma 4.6. If $L^2(W)$ is Λ-regular, then the above argument provides a Λ-outer factorization of W, with the constructed Φ in (32) being a Λ-outer factor.

Conversely, let W have a Λ-outer factor Φ. Then Φ is S_1-outer, so now Lemma 2.1 shows that the space $L^2(W)$ satisfies the regularity condition (10). Under the isometry $f \to \Phi f$, which maps $\mathcal{M}_W(\Pi)$ into $\mathcal{M}_{\mathcal{C}}(\Pi)$, the image of the subspace $\mathcal{M}_W(\Lambda) \ominus S_1\mathcal{M}_W(\Pi)$ is dense in the span $\mathcal{F}$ of $\{\Phi_0(e^{it})ce^{int} : c \in \mathcal{C},\ n \geq 0\}$. Since Φ_0 is S_2-outer, we have

$$\bigcap_{n=0}^{\infty} S_2^n\mathcal{F} = (0). \tag{33}$$

It follows that $L^2(W)$ satisfies (11) as well, and is therefore Λ-regular. This proves the principal claim.

Let ϕ be the scalar valued Λ-outer function defined as in the proof of Lemma 4.6. This ϕ serves as a scalar mollifier for both Φ and Ψ as needed. For the uniqueness conditions (i) and (ii), it suffices to proceed with W bounded; the general case is established by considering Φ and Ψ replaced with $\phi\Phi$ and $\phi\Psi$, respectively.

Assume that $W = \Phi^*\Phi = \Psi^*\Psi$, where Φ and Ψ are Λ-outer. Both Φ and Ψ are then S_1-outer, so [8, Section 3.5, Corollary] asserts that $\Psi = B\Phi$, where B determines an S_1-constant partial isometry with initial space $\overline{\Phi\mathcal{H}}$ and final space $\overline{\Psi\mathcal{H}}$ (as usual, $\mathcal{H} = H^2_{L^2_{\mathcal{C}}(\sigma)}(\sigma)$). It is easy to see that B commutes with S_2. From $\Psi_0(e^{it}) = B(e^{it})\Phi_0(e^{it})$, we see that $B(e^{it})$ is an S_2-constant partial isometry with initial space $N(\Phi) = (\Phi_0 H^2_{\mathcal{C}}(\sigma))^-$ and final space $N(\Psi) = (\Psi_0 H^2_{\mathcal{C}}(\sigma))^-$. Hence the claim (i) holds with this operator B.

Lastly, observe that from the construction of Φ as in (32), we can choose $\Phi_{0,0}$ to be a nonnegative operator. Let Φ and Ψ be Λ-outer factors of W such that both $\Phi_{0,0}$ and $\Psi_{0,0}$ are nonnegative. Then with B as in the statement (i), we have $\Psi_{0,0} = B\Phi_{0,0}$, which forces B to be the projection on $N(\Phi)$. Thus $\Phi = \Psi$. ■

We summarize the structural features of a Λ-outer function as constructed for Theorem 6.1.

COROLLARY 6.2. *Let Φ be a weakly measurable $\mathcal{B}(\mathcal{C})$-valued function on $\mathbb{T}^2$. Then Φ is Λ-outer if and only if it has a representation*

$$\Phi(e^{is},e^{it}) = \Theta(e^{it})A(e^{is},e^{it})/\phi(e^{is},e^{it}), \tag{34}$$

where $\phi(e^{is},e^{it})$ is a bounded, scalar valued, Λ-outer function such that $\phi\Phi$ is bounded; $A(e^{is},e^{it})$ is a bounded, $\mathcal{B}(\mathcal{C})$-valued, S_1-outer function; and $\Theta(e^{it})$ is a partially isometric, $\mathcal{B}(\mathcal{C})$-valued function such that $\Theta(e^{it})A_0(e^{it})$ is S_2-outer.

The notion of Λ-outer is associated with a version of Szegö's infimum. The problem is to estimate a constant function by polynomials in the subspace $S_2\mathcal{M}_W(\Lambda)$ of the weighted space $L^2(W)$.

COROLLARY 6.3. *Suppose that the $\mathcal{B}(\mathcal{C})$-valued weight function W on $\mathbb{T}^2$ has a Λ-outer factor Φ. Then for all $c \in \mathcal{C}$,*

$$\inf \|c+p\|_W = \|\Phi_{0,0}c\|_{\mathcal{C}}, \tag{35}$$

where the infimum is taken over polynomials $p \in S_2\mathcal{M}_W(\Lambda)$.

Proof. The assertion is established by the calculation

$$\begin{aligned} \inf\{\|c+p\|_W : p \in S_2\mathcal{M}_W(\Lambda)\} &= \inf\{\|\Phi(c+p)\|_{\mathcal{H}} : p \in S_2\mathcal{M}_W(\Lambda)\} \\ &= \inf\{\|\Phi c+q\|_{\mathcal{H}} : q \in S_2\mathcal{M}_{N(\Phi)}(\Lambda)\} \end{aligned} \tag{36}$$

$$= \inf\{\|\Phi c + q\|_{\mathcal{H}} : q \in S_2\mathcal{M}_C(\Lambda)\}$$
$$= \inf\{\|\Phi_0 c + r\|_{\mathcal{H}} : r \in S_2 H^2_C(\sigma)\} = \|\Phi_{0,0}c\|_C. \ \blacksquare$$

The value of this approach as a practical formula is limited, however, in view of the situation described in the concluding remarks of Section 3: The extremal p need not be a function.

A logarithmic integrability condition, familiar from the classical setting [2], is here sufficient for the factorization.

THEOREM 6.4. *Let $W(e^{is}, e^{it})$ be a $\mathcal{B}(C)$-valued weight function on $\mathbb{T}^2$ with pointwise invertible values such that*

$$\iint \log^+ \|W(e^{is}, e^{it})^{-1}\|_{\mathcal{B}(C)}\, d\sigma(e^{is})\, d\sigma(e^{it}) < \infty. \tag{37}$$

Then W has a Λ-outer factorization.

Proof. If W is replaced by uW, where $u(e^{is}, e^{it})$ is the scalar valued function $(\max\{1, \|W(e^{is}, e^{it})\|_{\mathcal{B}(C)}\})^{-1}$, then the above hypotheses are preserved. So by Lemma 4.6 we may assume that W is bounded.

From (37) and [8, Theorem 6.14] we deduce that for e^{it} fixed, $W(\cdot, e^{it})$ has a factorization $A(\cdot, e^{it})^* A(\cdot, e^{it})$, where $A(\cdot, e^{it})$ is S_1-outer on $H^2_C(\sigma)$. We may implement these factorizations through (22), so that in fact $A(e^{is}, e^{it})$ is S_1-outer on $\mathcal{H} = H^2_{L^2_C(\sigma)}(\sigma)$. Note that A has invertible values, and $\overline{A\mathcal{H}} = \mathcal{H}$.

Put $\nu(e^{is}, e^{it}) = (\max\{1, \|A(e^{is}, e^{it})^{-1}\|_{\mathcal{B}(C)}\})^{-1}$, and define the scalar mollifier $\phi(e^{is}, e^{it})$ using (12), (13) and this ν. Then the bounded function ϕA^{-1} is S_1-outer. To see this, observe that $(\phi A^{-1}\mathcal{H})^- = (\phi A^{-1}[A\mathcal{H}]^-)^- = \overline{\phi\mathcal{H}} = \mathcal{H}$, using $|\phi| \le \|A^{-1}\|^{-1}_{\mathcal{B}(C)} \le \|A\|_{\mathcal{B}(C)}$ to get ϕ bounded. Thus A^{-1} is S_1-outer, and each $A(\cdot, e^{it})^{-1}$ for e^{it} fixed is S_1-outer. Observe also that $(A^{-1})_0 = (A_0)^{-1}$. It follows from (37) and the subharmonicity of $\log^+ \|A(\cdot, e^{it})^{-1}\|_{\mathcal{B}(C)}$ that

$$\int \log^+ \|A_0^{-1}\|_{\mathcal{B}(C)}\, d\sigma(e^{it}) \le \iint \log^+ \|A(e^{is}, e^{it})^{-1}\|_{\mathcal{B}(C)}\, d\sigma(e^{is})\, d\sigma(e^{it}) < \infty. \tag{38}$$

Another application of [8, Theorem 6.14] gives $A_0(e^{it})^2 = G(e^{it})^* G(e^{it})$ for some S_2-outer function G. Now a Λ-outer factor of W can be constructed from these ϕ, A and G as before. $\blacksquare$

7. Final comments. The classical result connected to (1) is due to Szegö [10]. It was extended to the case of matrix valued weight functions by Wiener and Masani [12], Helson and Lowdenslager [2], and Wiener and Akutowicz [11]. Devinatz [1] showed that a logarithmic integrability condition is sufficient for the outer factorization of an operator valued weight function. An exact criterion for outer factorization in the operator valued

case was found by Rosenblum [7] and Lowdenslager [5]. Their approach was to apply the analysis of Toeplitz operators to objects derived from the given weight function, yielding a series formula for the outer factor. An entirely different method, based on the Cholesky decomposition, was developed by Power [6].

Further generalizations are concerned with factorizations of weight functions in two (or more) variables. Helson and Lowdenslager [2] treated scalar and full-rank matrix valued weight functions on the torus; analyticity was defined by replacing the nonnegative frequencies in the one-variable picture with frequencies lying in a "halfplane" of the integer lattice. The corresponding non-full-rank problem was addressed by Loubaton [4], who explored the related invariant subspace theory, and used the notion of analytic range function from [2] to overcome the rank deficiency. The present work extends many of Loubaton's results, but it does not displace them. Indeed, Loubaton obtained a general logarithmic integrability criterion for regularity, which cannot fully extend to the infinite-rank case. The approach taken here has the advantage of providing an explicit construction of the outer factor, in terms of operators associated with the weight function.

The results of the previous sections extend in a direct way to the situation with the halfplane Λ replaced by a halfplane at rational slope; this amounts to a change of variables. Halfplanes at irrational slope appear to pose some fundamentally different issues.

References

[1] D. A. Devinatz, *The factorization of operator valued functions*, Ann. of Math. (2) 73 (1961), 458–495.

[2] H. Helson and D. Lowdenslager, *Prediction theory and Fourier series in several variables I, II*, Acta Math. 99 (1958), 165–202; ibid. 106 (1961), 175–213.

[3] H. Korezlioglu and Ph. Loubaton, *Spectral factorization of wide sense stationary processes on $\mathbb{Z}^2$*, J. Multivariate Anal. 19 (1986), 24–47.

[4] Ph. Loubaton, *A regularity criterion for lexicographical prediction of multivariate wide-sense stationary processes on $\mathbb{Z}^2$ with non-full-rank spectral densities*, J. Funct. Anal. 104 (1992), 198–228.

[5] D. Lowdenslager, *On factoring matrix valued functions*, Ann. of Math. (2) 78 (1963), 450–454.

[6] S. C. Power, *Spectral characterization of the Wold–Zasuhin decomposition and prediction-error operator*, Math. Proc. Cambridge Philos. Soc. 110 (1991), 559–567.

[7] M. Rosenblum, *Vectorial Toeplitz operators and the Fejér–Riesz theorem*, J. Math. Anal. Appl. 23 (1968), 139–147.

[8] M. Rosenblum and J. Rovnyak, *Hardy Classes and Operator Theory*, Oxford Univ. Press, New York, 1985.

[9] Yu. A. Rozanov, *Stationary Random Processes*, Holden-Day, San Francisco, 1967.

[10] G. Szegö, *Über die Randwerte analytischer Funktionen*, Math. Ann. 84 (1921), 232–244.
[11] N. Wiener and E. J. Akutowicz, *A factorization of positive Hermitian matrices*, J. Math. Mech. 8 (1959), 111–120.
[12] N. Wiener and P. Masani, *The prediction theory of multivariate stochastic processes I, II*, Acta Math. 98 (1957), 111–150; ibid. 99 (1958), 93–137.

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Weighted L_Φ integral inequalities for operators of Hardy type

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Abstract. Necessary and sufficient conditions are given on the weights t, u, v, and w, in order for

$$\Phi_2^{-1}\big(\int \Phi_2(w(x)|Tf(x)|)t(x)\,dx\big) \le \Phi_1^{-1}\big(\int \Phi_1(Cu(x)|f(x)|)v(x)\,dx\big)$$

to hold when Φ_1 and Φ_2 are N-functions with $\Phi_2\circ\Phi_1^{-1}$ convex, and T is the Hardy operator or a generalized Hardy operator.

Weak-type characterizations are given for monotone operators and the connection between weak-type and strong-type inequalities is explored.

1. Introduction. In this paper, we will extend some weighted norm inequalities from the Lebesgue setting to the Orlicz space setting. Given a σ-finite measure space $(X, d\mu)$ and an N-function Φ, the Orlicz space $L_\Phi(X, d\mu)$ is the Banach space normed by

$$\|f\|_{L_\Phi} = \|f\|_{L_\Phi(X,d\mu)} = \inf_{\lambda>0}\Big\{ \int_X \Phi\Big(\frac{|f(x)|}{\lambda}\Big)\,d\mu(x) \le 1\Big\}.$$

In this paper, with the exception of Section 2, X will be either $\mathbb{R}^+ = (0,\infty)$ or $\mathbb{R}^n$, and μ will be defined on the Lebesgue measurable sets.

A weight is a measurable function on X that is positive almost everywhere. For the Lebesgue space, $L^r(X)$, $1 < r < \infty$, which corresponds to the N-function $\Phi(x) = x^r/r$, a weighted norm inequality for an operator T has the form

$$\|Tf\|_{L^q(X,w(x)\,dx)} \le C\|f\|_{L^p(X,v(x)\,dx)}.$$

This has a number of useful equivalent formulations, such as

$$\Big(\int |Tf(x)|^q w(x)\,dx\Big)^{1/q} \le C\Big(\int |f(x)|^p v(x)\,dx\Big)^{1/p}$$

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