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On the maximal function for rotation invariant measures in $\mathbb{R}^n$

by

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Abstract. Given a positive measure μ in $\mathbb{R}^n$, there is a natural variant of the non-centered Hardy–Littlewood maximal operator

$$\mathcal{M}_\mu f(x) = \sup_{x\in B} \frac{1}{\mu(B)} \int_B |f|\,d\mu,$$

where the supremum is taken over all balls containing the point x. In this paper we restrict our attention to rotation invariant, strictly positive measures μ in $\mathbb{R}^n$. We give some necessary and sufficient conditions for $\mathcal{M}_\mu$ to be bounded from $L^1(d\mu)$ to $L^{1,\infty}(d\mu)$.

Let μ be a non-negative measure in $\mathbb{R}^n$, finite on compact sets. Given a function $f \in L^1_{\text{loc}}(d\mu)$, we can define the analogue of the Hardy–Littlewood maximal function

$$\mathcal{M}_\mu f(x) = \sup_{B\in\mathcal{B}_x} \frac{1}{\mu(B)} \int_B |f|\,d\mu,$$

where

$$\mathcal{B}_x = \{B \text{ open ball}\,;\, x \in B \text{ and } \mu(B) > 0\}.$$

In fact, there are two possible definitions of the Hardy–Littlewood maximal operator. The second one corresponds to a smaller basis; namely,

$$\mathcal{B}_x^{c} = \{\text{open balls } B \text{ centered at } x \text{ with } \mu(B) > 0\}.$$

The operator associated with the latter basis maps $L^1(d\mu)$ into $L^{1,\infty}(d\mu)$. This can be proved using the Besicovitch covering lemma. An operator that satisfies this boundedness property is said to be of weak type 1-1 with respect to the measure μ.

But things are not so easy when dealing with the former basis, the non-centered case:

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It is known ([M-S], [S]) that, for $n = 1$, $\mathcal{M}_\mu$ always maps $L^1(d\mu)$ into $L^{1,\infty}(d\mu)$, no matter what μ is.

In $\mathbb{R}^n$, and if μ is a doubling measure (which means that there is a $C > 0$ such that $\mu(B_{2r}(x_0)) \le C\mu(B_r(x_0))$ for all $x_0 \in \mathbb{R}^n$, $r > 0$), then $\mathcal{M}_\mu$ is of weak type 1-1. This can be proved by using a Vitali type covering lemma.

For $n = 2$, P. Sjögren [S] showed that the operator associated with the measure $d\mu(x) = e^{-|x|^2/2}\,dx$ does not have the same boundedness property.

So, there are two questions that arise from these observations:

(1) Is there any non-doubling measure μ in $\mathbb{R}^n$, $n > 1$, such that $\mathcal{M}_\mu$ is of weak type 1-1?

(2) How can we know whether or not a measure μ provides an operator of weak type 1-1?

From now onwards, throughout this paper, μ (or $d\mu$) will be a rotation invariant and strictly positive measure on $\mathbb{R}^n$ which is finite on compact sets. "Rotation invariant" means that, for every measurable set U and every rotation ϱ around the origin, $\mu(U) = \mu(\varrho(U))$, and "strictly positive" that, for every open ball B, $\mu(B) > 0$. In the case of μ absolutely continuous with respect to Lebesgue measure dx, i.e., $d\mu(x) = f(x)\,dx$ with $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, the rotation invariance property of μ is equivalent to f being a radial function. For that reason, and with a slight abuse of language, we will often refer to μ as a *radial measure*.

We are going to show some conditions that characterize the measures μ of this type for which $\mathcal{M}_\mu$ is bounded from $L^1(d\mu)$ to $L^{1,\infty}(d\mu)$. This answers the second question (for this particular kind of measures).

THEOREM. *Let μ be a rotation invariant and strictly positive measure on $\mathbb{R}^n$ which is finite on compact sets. The following assertions are equivalent*:

(i) *$\mathcal{M}_\mu : L^1(d\mu) \to L^{1,\infty}(d\mu)$ is bounded.*

(ii) *There is a constant C_2 such that, for all $r \le 10a$,*

$$\mu(\{a < |x| < a + 2r\}) \le C_2\mu(\{a + r/2 < |x| < a + 3r/2\}).$$

(iii) *μ is a doubling measure away from the origin, that is,*

$$\mu(B_{2s}(x_0)) \le C_3\mu(B_s(x_0)) \quad \textit{for all } s \le |x_0|/4,$$

with C_3 independent of s and x_0.

Remark. If $d\mu(x) = g(|x|)dx$, we can write (ii) in the equivalent form

$$\text{(ii}'\text{)} \quad \int_a^{a+2r} g(s)\,ds \le C_2' \int_{a+r/2}^{a+3r/2} g(s)\,ds \quad \text{for all } r \le 10a.$$

EXAMPLES. 1) $d\mu(x) = (1 + |x|^\alpha)^{-1}\,dx$ is doubling away from the origin, so $\mathcal{M}_\mu$ is of weak type 1-1. But, if $\alpha \ge n$, $d\mu$ is not a doubling measure, as one can easily check.

This example provides an affirmative answer to the first question.

2) It is not hard to see that condition (ii), together with the regularity of μ, implies that

$$\mu(\{|x| = a\}) = 0 \quad \text{for all } a > 0,$$

and, in particular, if $d\sigma$ denotes the usual measure over the unit sphere, the measure $d\mu = d\sigma + dx$ does not give us a bounded operator.

On the other hand, $d\sigma$ gives a bounded operator, as can be easily seen with the same argument used for the Hardy–Littlewood maximal operator in $\mathbb{R}^{n-1}$. Observe that $d\sigma$ is not strictly positive, and our theorem does not apply.

Remark. Condition (iii) tells us that, for radial and strictly positive measures on $\mathbb{R}^n$, μ being a doubling measure is an almost necessary condition for $\mathcal{M}_\mu$ to be bounded. This is a remarkable difference with the one-dimensional case in which, as we mentioned before, any positive measure gives rise to an associated bounded operator.

For the case of a measure $d\mu(x) = g(|x|)\,dx$ with g monotonic, (ii) has an even simpler statement:

COROLLARY. *Let $d\mu(x) = g(|x|)\,dx$ be a measure in $\mathbb{R}^n$ with g monotonic and strictly positive on $(0, \infty)$. Then $\mathcal{M}_\mu$ is of weak type 1-1 if and only if there are some constants $c_k > 0$, $k \in \mathbb{Z}$ and $C > 0$ such that*

$$c_k \le g(r) \le Cc_k \quad \textit{for } 2^{k-1} \le r \le 2^{k+1}.$$

Proof of the Theorem. Before writing down the proof, we need to introduce a new geometric object which appears in a natural way when considering this problem.

DEFINITION. Given a ball $B = B_r(x_0)$, we define its *associated sector* $S = S_B$ as

$$\text{(1)} \quad S = \{x \in \mathbb{R}^n : |x_0| - r < |x| < |x_0| + r \text{ and } \arg(x, x_0) < \arcsin(r/|x_0|)\} \quad \text{if } |x_0| \ge r,$$

where $\operatorname{ang}(x, x_0)$ denotes the angle determined by the vectors $\overrightarrow{0x}$, $\overrightarrow{0x_0}$, and

$$\text{(2)} \quad S = \{x \in \mathbb{R}^n : |x| < |x_0| + r\} \quad \text{if } |x_0| < r.$$

We set

$$\arg(S) = \begin{cases} \arcsin(r/|x_0|) & \text{if } |x_0| \le r, \\ 2\pi & \text{if } |x_0| > r. \end{cases}$$

Also, it will be useful to state another equivalent condition to (i), involving sectors and balls:

(iv) $\mu(S_B) \le C_4\mu(B)$ for all open balls B in $\mathbb{R}^n$, with C_4 independent of B.

We will prove the following implications: (i)⇒(iv)⇒(ii)⇒(iii)⇒(iv), and (iii) + (iv) ⇒ (i).

(i)⇒(iv). Let $B = B_r(x_0)$ and $\varepsilon > 0$ be given. Consider the set

$$\mathcal{X} = \{x_0' \in \mathbb{R}^n : |x_0| = |x_0'| \text{ and } |x_0 - x_0'| < r/2\}.$$

Then, for each $x_0' \in \mathcal{X}$, we have $B_{r/2}(x_0) \subset B_r(x_0')$ and $\mu(B) = \mu(B_r(x_0'))$. Hence

$$\bigcup_{x_0' \in \mathcal{X}} B_r(x_0') \subset \left\{x : \mathcal{M}_\mu\left(\frac{\chi_{B_{r/2}(x_0)}}{\mu(B_{r/2}(x_0))}\right) > \frac{1}{\mu(B)+\varepsilon}\right\}.$$

Now, we realize that if $|x_0| \ge r$, then

$$\widetilde{S}_B := \{x : |x_0| - r < |x| < |x_0| + r \text{ and } \operatorname{ang}(x, x_0) < r/(2|x_0|)\} \subset \bigcup_{x_0' \in \mathcal{X}} B_r(x_0'),$$

and if $0 < |x_0| < r$, then

$$\widetilde{S}_B := S_B \cap \{\operatorname{ang}(x, x_0) < 1/2 \text{ or } x = (0,0)\} \subset \bigcup_{x_0' \in \mathcal{X}} B_r(x_0').$$

In both cases, we can conclude from our hypothesis (i) and the fact that $\mu(\widetilde{S}_B) \sim \mu(S_B)$ that

$$\mu(S_B) \le C\mu\Big(\bigcup_{x_0' \in \mathcal{X}} B_r(x_0')\Big) \le CC_1(\mu(B)+\varepsilon)\int \frac{\chi_{B_{r/2}(x_0)}}{\mu(B_{r/2}(x_0))}\,d\mu = CC_1(\mu(B)+\varepsilon).$$

Letting $\varepsilon \to 0$, we obtain (iv).

(iv)⇒(ii). Given the annulus $\{a < |x| < a + 2r\}$, with $r \le 10a$, we consider the inscribed ball $B = B_r(a+r, 0, \ldots, 0) =: B_r(x_0)$ and its associated sector, S.

We claim that we can select $0 < \varepsilon < 1/2$, independent of a and r under the sole condition $r \le 10a$, such that if $x \in B$, and $|x| \le a + r\varepsilon$ or $|x| \ge a + 2r - r\varepsilon$, then

$$\operatorname{ang}(x, x_0) < \arg(S)/(2C_4).$$

Actually, the following can be proved:

LEMMA. *Let $B = B_r(x^0)$, with $0 < r < \frac{10}{11}|x^0|$ and $0 < \varepsilon \le 1/2$. Assume that $y \in \partial B \cap \{\partial B_{|x^0|-r+r\varepsilon}(0) \cup \partial B_{|x^0|+r-r\varepsilon}(0)\}$. Then*

$$\operatorname{ang}(y, x^0) \sim \sqrt{\varepsilon}\arg(S).$$

The proof is straightforward and left to the reader.

Then, by (iv) and our selection of ε,

$$\begin{aligned}
\mu(\{a < |x| \le a + \varepsilon r &\text{ or } a + 2r - \varepsilon r \le |x| < a + 2r\} \\
&\cap \{\operatorname{ang}(x, x_0) < \arg(S)\}) \\
&\le \mu(S) \le C_4\mu(B) \\
&\le C_4\mu(\{a < |x| \le a + \varepsilon r,\ \operatorname{ang}(x, x_0) < \arg(S)/(2C_4)\}) \\
&\quad + C_4\mu(\{a + 2r - \varepsilon r \le |x| < a + 2r,\ \operatorname{ang}(x, x_0) < \arg(S)/(2C_4)\}) \\
&\quad + C_4\mu(\{a + \varepsilon r < |x| < a + 2r - \varepsilon r,\ \operatorname{ang}(x, x_0) < \arg(S)\}) \\
&\le \tfrac{1}{2}\mu(\{a < |x| \le a + \varepsilon r,\ \operatorname{ang}(x, x_0) < \arg(S)\}) \\
&\quad + \tfrac{1}{2}\mu(\{a + 2r - \varepsilon r \le |x| < a + 2r,\ \operatorname{ang}(x, x_0) < \arg(S)\}) \\
&\quad + C_4\mu(\{a + \varepsilon r < |x| < a + 2r - \varepsilon r,\ \operatorname{ang}(x, x_0) < \arg(S)\}),
\end{aligned}$$

where we have used in the last inequality the radiality of μ.

Reorganizing the inequality, we get

$$\begin{aligned}
\mu(\{a < |x| \le a + \varepsilon r &\text{ or } a + 2r - \varepsilon r \le |x| < a + 2r\} \\
&\cap \{\operatorname{ang}(x, x_0) < \arg(S)\}) \\
&\le 2C_4\mu(\{a + \varepsilon r < |x| < a + 2r - \varepsilon r,\ \operatorname{ang}(x, x_0) < \arg(S)\})
\end{aligned}$$

and, therefore, by the radiality of μ again, we conclude

$$\begin{aligned}
\mu(\{x : a < |x| \le a + \varepsilon r \text{ or } a + 2r - \varepsilon r \le |x| < a + 2r\} \\
\le 2C_4\mu(\{a + \varepsilon r < |x| < a + 2r - \varepsilon r\}).
\end{aligned}$$

Now, we repeat this argument with the annulus $\{a + \varepsilon r < |x| < a + 2r - \varepsilon r\}$. In a few steps ($\sim 1/\varepsilon$) we obtain (ii).

(ii)⇒(iii). Let us set $B = B_s(x_0)$ and $2B = B_{2s}(x_0)$. Then

$$\begin{aligned}
\mu(2B) &\le \mu(S_{2B}) \\
&= \mu(\{x : |x_0| - 2s < |x| < |x_0| + 2s,\ \operatorname{ang}(x, x_0) < \arcsin(2s/|x_0|)\}) \\
&\le c(2s/|x_0|)^{n-1}\mu(\{x : |x_0| - 2s < |x| < |x_0| + 2s\}),
\end{aligned}$$

because of the radiality.

We can apply (ii) twice, so that the last expression can be seen to be bounded above by

$$cC_2^2(2s/|x_0|)^{n-1}\mu(\{x : |x_0| - s/2 < |x| < |x_0| + s/2\})$$
$$\le CC_2^2\mu(\{x : |x_0| - s/2 < |x| < |x_0| + s/2,\ \operatorname{ang}(x, x_0) < cs/|x_0|\}).$$

Now, notice that the hypothesis $s \le |x_0|/4$ and the Lemma imply that

$$\{x : |x_0| - s/2 < |x| < |x_0| + s/2,\ \operatorname{ang}(x, x_0) < cs/|x_0|\} \subset B,$$

for a suitable c, which is enough to conclude the proof of (ii)⇒(iii).

(iii)⇒(iv). Let $B = B_r(x_0)$. Assume that $x_0 = (x_0^1, 0, \ldots, 0)$. We will distinguish three different cases:

First case: $r \le |x_0|/4$. As in the proof above, the point here is that there is a constant $c > 0$ such that

$$\widehat{S}_B := \{x : |x_0| - r < |x| < |x_0| + r,\ \operatorname{ang}(x, x_0) < cr/|x_0|\} \subset 2B$$

and, therefore,

$$\mu(S_B) \sim \mu(\widehat{S}_B) \le \mu(2B) \le C_3\mu(B).$$

Second case: $|x_0|/4 < r \le |x_0|$. First, define two auxiliary balls:

$$B_1 = B_{(|x_0|-r)/3}\big(\tfrac{4}{3}(|x_0| - r), 0, \ldots, 0\big),$$
$$B_2 = B_{(|x_0|+r)/5}\big(\tfrac{4}{5}(|x_0| + r), 0, \ldots, 0\big).$$

Both are as in the first case considered. Hence,

$$\mu(B) \ge \sum_{i=1}^{2}\mu(B_i) \ge CC_3^{-1}\sum\mu(S_{B_i})$$
$$\ge C'C_3^{-1}\big[\mu\big(\{|x_0| - r < |x| < \tfrac{5}{3}(|x_0| - r)\}\big)$$
$$+ \mu\big(\{\tfrac{3}{5}(|x_0| + r) < |x| < |x_0| + r\}\big)\big],$$

the last inequality being true since $\arg(S_{B_i}) = \arcsin\frac{1}{4}$, $i = 1, 2$, and μ is radial.

Secondly, there is a constant $c > 0$ such that

$$\{\tfrac{5}{3}(|x_0| - r) < |x| < \tfrac{3}{5}(|x_0| + r),\ \operatorname{ang}(x, x_0) < c\} \subset B.$$

Therefore,

$$\mu(B) \ge C\mu\big(\{\tfrac{5}{3}(|x_0| - r) < |x| < \tfrac{3}{5}(|x_0| + r)\}\big).$$

Third case: $r > |x_0|$. Consider $B' = B_{(|x_0|+r)/2}((|x_0| + r)/2, 0, \ldots, 0)$. $B' \subset B$, $(0, \ldots, 0) \in B$ and $S_{B'} = [S_B \setminus \{(0, \ldots, 0)\}] \cap \{\operatorname{ang}(x, x_0) < \pi/2\}$. By the radiality of μ, $\mu(S_B \setminus \{(0, \ldots, 0)\}) \le 2\mu(S_{B'})$. Hence

$$\mu(S_B) \le 2\mu(S_{B'}) + \mu(B).$$

Furthermore, B' is as in the second case. We conclude, from our previous calculations, that

$$\mu(S_B) \le CC_3\mu(B') + \mu(B) \le C'C_3\mu(B).$$

(iii) + (iv) ⇒ (i). Classify balls in $\mathbb{R}^2$ into two families:

$$\mathcal{F}_1 = \{B = B_r(x_0) : r \le |x_0|/4\}, \qquad \mathcal{F}_2 = \{B = B_r(x_0) : r > |x_0|/4\}.$$

We define

$$\mathcal{M}_\mu^1 f(x) = \sup_{x\in B, B\in\mathcal{F}_1}\frac{1}{\mu(B)}\int_B |f(y)|\,d\mu(y).$$

Observe that there is a constant $c > 0$ such that, if $B \in \mathcal{F}_2$, then $\arg(S_B) \ge c$, so

$$\mu(B) \ge C_4^{-1}\mu(S_B) \ge C_4^{-1}C\mu(\{|x_0| - r < |x| < |x_0| + r\}).$$

That is why we define the following maximal operator:

$$\mathcal{M}_\mu^2 f(x) = \sup_{A\in\mathcal{A}_x}\frac{1}{\mu(A)}\int_A |f(y)|\,d\mu(y),$$

where

$$\mathcal{A}_x = \{A \text{ open annulus centered at the origin} : x \in A\}.$$

We have just seen that

$$\mathcal{M}_\mu f(x) \le \mathcal{M}_\mu^1 f(x) + C\mathcal{M}_\mu^2 f(x).$$

We must prove the boundedness of $\mathcal{M}_\mu^1$ and $\mathcal{M}_\mu^2$.

$\mathcal{M}_\mu^1$ is easy to handle: the doubling condition (iv) allows us to write down a proof copying the standard one for the Hardy–Littlewood maximal operator.

To obtain the boundedness of $\mathcal{M}_\mu^2$, observe that this is essentially a one-dimensional object.

Remark. There are two more equivalent conditions that we would like to point out here:

(v) There exists a constant $C_5 > 0$ such that

$$\mu(\{x : \sup_{x\in B}\chi_B(x_0)/\mu(B) > \lambda\}) \le C_5/\lambda \quad \text{for all } \lambda > 0,\ x_0 \in \mathbb{R}^n.$$

(vi) For every ball B we define

$$\mathfrak{B}_B = \{B' \text{ open ball} : \mu(B') \le \mu(B) \text{ and } B \cap B' \ne \emptyset\}.$$

Then

$$\mu\Big(\bigcup_{B'\in\mathfrak{B}_B} B'\Big) \le C_6\mu(B).$$

It is easy to see that (i)⇒(v)⇒(iv). For (i)⇒(v) consider a family of balls converging to x_0: $B_n = B_{1/n}(x_0)$, and apply (i) to $\mathcal{M}_\mu(\chi_{B_n}/\mu(B_n))$. In order to prove (v)⇒(iv) one just has to modify the proof of (i)⇒(iv) noting that

$$\bigcup_{x_0'\in\mathcal{X}} B_r(x_0') \subset \{x : \sup_{x\in B'} \chi_{B'}(x_0)/\mu(B') > 1/(\mu(B)+\varepsilon)\}.$$

The proof of (vi)⇒(i) is also trivial. The only difficulty is then to prove that any of the conditions (i) to (iv) implies (vi). We will prove (iv) + (ii) ⇒ (vi):

(iv) + (ii) ⇒ (vi). First of all, we prove that there is a constant κ such that, if we have balls B_1, B_2, $B_1 \cap B_2 \neq \emptyset$, with sectors S_1, S_2 such that $\arg(S_i) = \theta_i$ and $\theta_1 > \kappa\theta_2$ then $\mu(B_1) \geq 2\mu(B_2)$:

If $B_i = B_{r_i}(x_i)$, then $\theta_1 > \kappa\theta_2$ implies $2\pi > \kappa\theta_2$, so $r_2 \leq |x_2|/1000$ (taking κ large enough), and $r_1 \geq 10r_2$. Therefore,

$$\{x : |x_2| - 10r_2 \leq |x| \leq |x_2| - r_2,\ \operatorname{ang}(x,x_1) < \theta_1\} \subset S_1 \quad \text{if } |x_1| < |x_2|,$$
$$\{x : |x_2| + r \leq |x| \leq |x_2| + 10r_2,\ \operatorname{ang}(x,x_1) < \theta_1\} \subset S_1 \quad \text{if } |x_1| \geq |x_2|.$$

In the first case

$$\mu(B_2) \leq \mu(S_2) = \frac{\theta_2}{\pi}\mu(\{|x_2| - r_2 < |x| < |x_2| + r_2\}).$$

Apply (ii) with $r = 4r_2$ and $a = |x_2| - 7r_2$ to get

$$\leq \frac{\theta_2}{\pi} C_2 \mu(\{|x_2| - 5r_2 < |x| < |x_2| - r_2\})$$
$$= \frac{\theta_2}{\theta_1} C_2 \mu(S_1)$$

and, applying (iv),

$$\leq \kappa^{-1} C_4 C_2 \mu(B_1) \leq \tfrac{1}{2}\mu(B_1),$$

for a suitable κ.

In the second case the proof is similar.

Let $B = B_r(x_0)$. In order to prove (vi), and by the reasoning above, we only have to take care of balls $B' \in \mathfrak{B}_B$ with $\kappa^{-1}\theta \leq \theta' \leq \kappa\theta$.

From them, we select two balls:

- B_l' nearest to the origin;
- B_u' with center x_u and radius r_u such that $|x_u| + r_u$ is maximal.

If $B' \in \mathfrak{B}_B$ with $\kappa^{-1}\theta \leq \theta' \leq \kappa\theta$, then there is a constant $c(\kappa) \sim 3\kappa^2$ such that

$$B' \subset c(\kappa)S_{B_l'} \cup c(\kappa)S_{B_u'} \cup c(\kappa)S_B$$

where $c(\kappa)S_B$ denotes the sector

$$\{x \in \mathbb{R}^2 : |x_0| - r < |x| < |x_0| + r \text{ and } \operatorname{ang}(x,x_0) < c(\kappa)\arg(S_B)\}$$

and $c(\kappa)S_{B_l'}$ and $c(\kappa)S_{B_u'}$ are defined analogously.

Using the radiality of μ and (iv) again, we get

$$\mu\Big(\bigcup_{B'\in\mathfrak{B}_B} B'\Big) \leq \mu(c(\kappa)S_{B_l'} \cup c(\kappa)S_{B_u'} \cup c(\kappa)S_B)$$
$$\leq c(\kappa)C_4(\mu(B_l') + \mu(B_u') + \mu(B)) \leq 3c(\kappa)C_4\mu(B).$$

A final comment. Notice that (ii) or (iii) trivially imply that the measure μ is either strictly positive or a constant times the Dirac delta at the origin. Hence, the Theorem cannot be generalized.

We know the behaviour of $\mathcal{M}_\nu$ for some measures ν not strictly positive. For instance, if ν is supported on the annulus $A = \{1 \leq |x| \leq 2\}$ and is strictly positive on A (that is, if B is an open ball satisfying $A \cap B \neq \emptyset$, then $\nu(B) > 0$), then $\mathcal{M}_\nu$ is never of weak type 1-1. Notice the difference between this behaviour and that of $\mathcal{M}_\sigma$, in spite of $d\sigma$ being a limit of measures of this kind.

Many particular cases can be done with easy arguments, but we think much more strong ones will be needed in order to get a global result.

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