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Measures of noncompactness and normal structure in Banach spaces

by

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Abstract. Sufficient conditions for normal structure of a Banach space are given. One of them implies reflexivity for Banach spaces with an unconditional basis, and also for Banach lattices.

1. Introduction. Let $(X, \|\cdot\|)$ be a Banach space and let $B(x,r)$ and $S[x,r]$ denote the open ball and the sphere centered at x and of radius r. For brevity we will write S_X instead of $S[0,1]$, and B_X will be the closed unit ball of X. A convex set $C \subset X$ is said to have *normal structure* (n.s.) if for each closed convex bounded subset $K \subset C$ which is not a singleton, there exists at least one point $x \in C$ such that

$$r(x,C) := \sup\{\|x-y\| : y \in K\} < \operatorname{diam}(K).$$

(Such a point is called *nondiametral* in K.) Similarly, the convex set C has *weakly normal structure* (w.n.s.) provided that each weakly compact convex nontrivial subset K of C has a nondiametral point p in K.

Normal structure was introduced in 1948 by M. S. Brodskiĭ and D. P. Milman. Since then this concept, together with many natural variations, has been widely studied. In particular, n.s. has been significant in the development of fixed point theory. In 1965 W. A. Kirk [K] proved that weakly normal structure is a sufficient condition for the fixed point property of nonexpansive mappings.

To test whether a given convex bounded subset of a Banach space has normal structure is not an easy task. Considerable research has been directed towards finding geometrical conditions which imply normal structure.

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Uniform convexity and related properties are, among many others, such sufficient conditions.

In infinite-dimensional Banach spaces normal structure is a consequence of the fact that some subsets of its unit sphere are "nearly compact" (see [G-K]). For example, $(X, \|\cdot\|)$ has the *weak uniform Kadec–Klee property* (WUKK) if there exist $\varepsilon \in (0,1)$ and $\delta \in (0,1)$ such that $\operatorname{dist}(0, C) \le 1-\delta$ for every weakly compact convex subset C of B_X with $\alpha(C) > \varepsilon > 0$. Here $\alpha(C)$ is the *Kuratowski measure of noncompactness* of the set $C \subset X$, i.e.

$$\alpha(C) := \inf\{r > 0 : C \text{ has a finite } r\text{-cover}\}.$$

If the above condition holds for every $\varepsilon \in (0,1)$, then X is said to have the (UKK) property (see [H]). In [D-S] the authors prove (WUKK)⇒(WNS). For more information about normal structure, see [N-S-W], [G-K] or [L].

In this paper we give sufficient conditions for normal structure in terms of some "measures" of the *slices* of the unit ball B_X. Given $\delta \in (0,1)$ and $f \in S_{X^*}$ we denote by $S(f,\delta)$ the slice $\{x \in B_X : f(x) > 1-\delta\}$.

2. Sufficient conditions for weakly normal structure

DEFINITIONS. Let X be a Banach space, and let $\mathcal{B}$ be the collection of all nonempty bounded subsets of X. A real function μ defined on $\mathcal{B}$ is called *admissible* provided that μ satisfies the following conditions:

(a) For $A, B \in \mathcal{B}$ with $A \subset B$, $\mu(A) \le \mu(B)$.
(b) For every $x \in X$ and $A \in \mathcal{B}$, $\mu(x + A) = \mu(A)$.
(c) For all sequences (x_n) weakly convergent to 0, and *diametral*, i.e.

$$d(x_{n+1}, \operatorname{conv}\{x_1, \ldots, x_n\}) \to \operatorname{diam}(\{x_n : n \ge 1\}) =: d(\{x_n\}),$$

we have

$$\mu(\{x_n : n = 1, 2, \ldots\}) = d(\{x_n\}).$$

Many nonnegative functions satisfying (a)–(c) may be defined on $\mathcal{B}$. The Kuratowski measure of noncompactness α, as well as the diameter $d : \mathcal{B} \to [0,\infty)$, $d(A) := \operatorname{diam}(A)$, are immediate examples.

Additional examples of admissible functions are the measure of noncompactness γ of a set $S \subset X$, defined by (see [S])

$$\gamma(S) = \sup\{\inf\{\|x_m - x_n\| : m \ne n\} : (x_n) \text{ a sequence in } S\},$$

and also every convex combination of admissible functions.

The above functions are admissible in every Banach space. Nevertheless, there are functions which are admissible in some specific Banach spaces. It is easy to see that if a Banach space X satisfies the nonstrict Opial condition, then also for the *Hausdorff measure of noncompactness*

$$\chi(S) = \inf\{r > 0 : S \text{ has a finite cover by balls of radii smaller than } r\}$$

conditions (a)–(c) hold. (Recall that a Banach space X satisfies the *nonstrict Opial condition* provided that if a sequence (x_n) is weakly convergent to $x \in X$, then $\liminf \|x_n - x\| \le \liminf \|x_n - y\|$ for every $y \in X$.)

Let $(X, \|\cdot\|)$ be a Banach space. For a given admissible function μ on X, we say that X has the *property* (μ') if there exists δ with $0 < \delta < 1$ such that $\mu(S(f,\delta)) < 1$ for every $f \in S_{X^*}$.

For example, the finite-dimensional space l_∞^n has the property (α'), but does not have (d'). Obviously, $(d')\Rightarrow(\alpha')$.

THEOREM 1. *Every Banach space with a property (μ') has weakly normal structure.*

Proof. Suppose, for a contradiction, that X does not have w.n.s. Then there exists a diametral sequence (x_n) in X. Moreover, it can be supposed that (see [G-LD]):

(i) $x_n \rightharpoonup 0$,
(ii) $\|x_n\| \le 1$ $(n = 1, 2, \ldots)$, and $\|x_n\| \to 1$,
(iii) $d(\{x_n\}) = 1$.

Since X has the property (μ') there exists $\delta > 0$ such that $\mu(S(f,\delta)) < 1$ for every $f \in S_{X^*}$.

From (ii) we can obtain x_{n_0} such that $\|x_{n_0}\| > 1-\delta$. Let f_0 be a functional with $\|f_0\| = 1$ and $f_0(x_{n_0}) = \|x_{n_0}\|$. Then

$$-x_{n_0} + x_n \rightharpoonup -x_{n_0}$$

and

$$-f_0(-x_{n_0} + x_n) \to -f_0(-x_{n_0}) = \|x_{n_0}\| > 1-\delta.$$

On the other hand, for $n = 1, 2, \ldots$,

$$\|-x_{n_0} + x_n\| \le \operatorname{diam}(\{x_n\}) = 1,$$

and we obtain $n_1 \in \mathbb{N}$ such that, for every $n \ge n_1$, $-x_{n_0} + x_n \in S(-f_0, \delta)$, and therefore

$$\{x_n : n \ge n_1\} \subset x_{n_0} + S(-f_0, \delta).$$

Now we obtain the following contradiction which completes the proof:

$$1 = \mu(\{x_n : n \ge n_1\}) \le \mu(x_{n_0} + S(-f_0,\delta)) = \mu(S(-f_0,\delta)) < 1.$$

COROLLARY 1. *Every Banach space satisfying one of the properties (α') $(\mu = \alpha)$, (d') $(\mu = d)$, or (γ') $(\mu = \gamma)$ has weakly normal structure. Every Banach space with the nonstrict Opial condition satisfying the property (χ') has weakly normal structure.*

DEFINITIONS. If $(X, \|\cdot\|)$ is a Banach space, and there exists an admissible function μ on X such that $\mu(B_X) < 1$, then X has normal structure.

Hence, if $\mu(B_X) \geq 1$, we define the *modulus of convexity of X with respect to* μ as the function $\Delta_\mu : [0, \mu(B_X)] \to [0,1]$ given by

$$\Delta_\mu(\varepsilon) := \inf\{1 - \operatorname{dist}(0, A) : A \subset B_X, A \text{ convex, and } \mu(A) \geq \varepsilon\}.$$

The *coefficient of convexity* of X with respect to μ is defined by

$$\varepsilon_1(\mu) := \sup\{\varepsilon > 0 : \Delta_\mu(\varepsilon) = 0\}.$$

In [G-S] K. Goebel and T. Sękowski introduce the coefficient $\varepsilon_1(\alpha)$, and they show that the Banach space $(X, \|\cdot\|)$ has normal structure if $\varepsilon_1(\alpha) < 1$. On the other hand, the coefficient $\varepsilon_1(\chi)$ was introduced by J. Banaś [B], who proved that a Banach space X has normal structure if $\varepsilon_1(\chi) < 1/2$. The following theorems improve slightly the above results.

THEOREM 2. *Let $(X, \|\cdot\|)$ be a Banach space, and μ an admissible function on X. Then X has the property (μ') if and only if $\Delta_\mu(1) \neq 0$.*

Proof. Suppose that X does not have the property (μ'). Then for $n = 1, 2, \ldots$ we obtain a functional $f_n \in S_{X^*}$ such that $\mu(S(f_n, 1/n)) \geq 1$. For every $x \in S(f_n, 1/n)$ we have $1 - 1/n < f_n(x) \leq \|x\|$, and thus

$$1 - 1/n \leq \inf\{f_n(x) : x \in S(f_n, 1/n)\} \leq \inf\{\|x\| : x \in S(f_n, 1/n)\}.$$

We get

$$\Delta_\mu(1) \leq 1 - \inf\{\|x\| : x \in S(f_n, 1/n)\} \leq 1 - (1 - 1/n) = 1/n,$$

and hence $\Delta_\mu(1) = 0$.

Next, suppose that X has the property (μ'). Then there exists $\delta > 0$ such that $\mu(S(f, \delta)) < 1$ for every $f \in S_{X^*}$. Thus, if we have a convex subset C of B_X such that $\mu(C) \geq 1$, then this set is not contained in any slice $S(f, \delta)$ $(f \in S_{X^*})$.

Let us see that there exist $x \in C$ such that $\|x\| < 1 - \delta$: If $\|x\| \geq 1 - \delta$ for all $x \in C$ then the open set $B(0, 1 - \delta)$ does not intersect the convex set C, and we can find a functional $f_1 \in S_{X^*}$ such that $f_1(y) \leq f_1(x)$ for all $y \in B(0, 1 - \delta)$ and $x \in C$. Then

$$\sup\{f_1(y) : y \in (1 - \delta) B(0,1)\} \leq \inf\{f_1(x) : x \in C\},$$

and therefore, $1 - \delta \leq f_1(x)$ for all $x \in C$. But this means that C is contained in $S(f_1, \delta)$, a contradiction.

Thus $\operatorname{dist}(0, C) < 1 - \delta$, and consequently $\Delta_\mu(1) \neq 0$, and Theorem 2 is proved.

In [G-J-L] one can find the above results for $\mu = \alpha$.

If $\varepsilon_1(\alpha) < 1$ then the space X has the property (α'), and hence (WUKK)$\Rightarrow$ (α') for reflexive Banach spaces, by the equivalence (WUKK)$\Leftrightarrow$$(\varepsilon_1(\alpha) < 1)$. We do not know whether the function $\Delta_\alpha(\varepsilon)$ is continuous at 1, in spite of the fact that it is continuous in [0,1) (see [B]).

COROLLARY 2. *Let $(X, \|\cdot\|)$ be a Banach space with the nonstrict Opial condition. Then X has the property (χ') if and only if $\Delta_\chi(1) \neq 0$.*

It is a seemingly open question whether the nonstrict Opial condition implies the fixed point property for nonexpansive mappings (F.P.P. for short) (see [G-K] for the definitions). On the other hand, it is also unknown whether $1/2 \leq \varepsilon_1(\chi) < 1$ is a sufficient condition for F.P.P. Corollary 2 yields that both conditions together are sufficient for normal structure, and hence for F.P.P.

COROLLARY 3. *Banach spaces with nonstrict Opial condition and $\varepsilon_1(\chi) < 1$ have weakly normal structure.*

It is easy to see that, for every $\varepsilon \in [0, 2)$, $\delta_X(\varepsilon) \leq \Delta_d(\varepsilon)$, where δ_X is the modulus of *uniform* convexity δ_X of the Banach space $(X, \|\cdot\|)$. However, we obtain the following result.

THEOREM 3. *A Banach space $(X, \|\cdot\|)$ has the property (d') if and only if $\delta_X(1) \neq 0$.*

Proof. Suppose that X has the property (d'), and fix $\varepsilon \in (1, 2]$. For every $x, y \in B_X$ with $\|x - y\| \geq \varepsilon$, let $f \in S_{X^*}$ be a functional for which $f(x + y) = \|x + y\|$. There exists $\delta \in (0, 1)$ such that $\operatorname{diam}(S(f, \delta)) < 1$, and thus $\{x, y\}$ is not contained in $S(f, \delta)$. Then either $f(x) \leq 1 - \delta$ or $f(y) \leq 1 - \delta$.

Now we have

$$\|x + y\| = f(x) + f(y) = \min\{f(x), f(y)\} + \max\{f(x), f(y)\} \leq (1 - \delta) + 1,$$

and hence $1 - \|(x + y)/2\| \geq \delta/2$. From the definition of the modulus of convexity δ_X we conclude that $\delta_X(\varepsilon) \geq \delta/2 > 0$, and thus $\delta_X(1) \neq 0$.

Conversely, if $\delta_X(1) \neq 0$ then $\Delta_d(1) \neq 0$ and thus X has the property (d').

The proof of the following result is straightforward.

THEOREM 4. *A Banach space $(X, \|\cdot\|)$ has the property (γ') if X is reflexive and if it has the property (WUKK).*

3. The property (α') in c_0 and l_1

THEOREM 5. *Let $(X, \|\cdot\|)$ be the Banach space $(c_0, \|\cdot\|_\infty)$ or $(l_1, \|\cdot\|_1)$. Let $|\cdot|$ be a norm on X equivalent to $\|\cdot\|$. Then $(X, |\cdot|)$ does not have the property (α').*

Proof. Case $(X, \|\cdot\|) = (l_1, \|\cdot\|_1)$. Fix $\delta \in (0,1)$ and choose ε such that $1 - \delta < 1/(1 + \varepsilon)$. From James' distortion theorem (see [L-Tz], p. 97) there exists a block basis (y_n) of the standard basis (e_k) of l_1 with $|y_n| = 1$ $(n = 1, 2, \ldots)$ such that, for every sequence (a_n) in l_1,

$$\Big|\sum_{n=1}^{\infty} a_n y_n\Big| \le \|(a_n)\|_1 \le (1+\varepsilon)\Big|\sum_{n=1}^{\infty} a_n y_n\Big|.$$

We define a continuous linear functional $f_\delta : l_1 \to \mathbb{R}$ by the standard extension of

$$f_\delta\Big(\sum_{n=1}^{\infty} a_n y_n\Big) := \sum_{n=1}^{\infty} a_n.$$

Then $|f_\delta| \le 1+\varepsilon$, and $h_\delta := (1/(1+\varepsilon))f_\delta \in B_{(X,|\cdot|)^*}$. Moreover, for all $n = 1, 2, \ldots$, $h_\delta(y_n) = 1/(1+\varepsilon) > 1-\delta$, and thus $y_n \in S(h_\delta, \delta)$.

On the other hand, for positive integers m, n $(m < n)$, we also have

$$|y_n - y_m| \ge \|e_n - e_m\|_1 \ge 1.$$

Hence, $\alpha(S(h_\delta, \delta)) \ge 1$ in $(X, |\cdot|)$. Consequently, for each $\delta \in (0,1)$ we can find $h_\delta \in B_{X^*}$ such that $\alpha(S(h_\delta,\delta)) \ge 1$, and this means that $(X, |\cdot|)$ does not have the property (α').

Case $(X, \|\cdot\|) = (c_0, \|\cdot\|_\infty)$. Fix $\delta \in (0,1)$ and choose ε such that

(i) $(1-\varepsilon)(1+\varepsilon)^{-1} > 1-\delta$,

(ii) $2(1-\varepsilon)(1+\varepsilon)^{-1} \ge 1$.

By using James' theorem again we also obtain a block basis (y_n) of the standard basis (e_n) of c_0 such that $|y_n| = 1$ $(n = 1, 2, \ldots)$ and, for every sequence (a_n) in c_0,

$$(1-\varepsilon)\Big|\sum_{n=1}^{\infty} a_n y_n\Big| \le \|(a_n)\|_\infty \le (1+\varepsilon)\Big|\sum_{n=1}^{\infty} a_n y_n\Big|.$$

Let f_δ be a continuous linear functional defined in c_0 via

$$f_\delta(y_1) = 1, \quad f_\delta(y_n) = 0 \quad (n = 2, 3, \ldots).$$

It is straightforward to check that $|f_\delta| \le (1+\varepsilon)$, and thus the functional $h_\delta := (1+\varepsilon)^{-1} f_\delta$ belongs to $B_{(X,|\cdot|)^*}$.

On the other hand, for every $n \in \mathbb{N}$, there exists a unique $k \in \mathbb{N}$ such that $2^{k-1} \le n \le 2^k - 1$. We write

$$B_n := \{y_1 + \varepsilon_2 y_2 + \ldots + \varepsilon_k y_k : \varepsilon_i = \pm 1 \ (i = 2, \ldots, k)\}.$$

If $v \in B_n$ then

$$|v| \le (1-\varepsilon)^{-1}\|(1, \varepsilon_2, \ldots, \varepsilon_k, 0, \ldots)\|_\infty = (1-\varepsilon)^{-1}.$$

Hence, from (i), $(1-\varepsilon)B_n \subset S(h_\delta, \delta)$, and for every $x, y \in (1-\varepsilon)B_n$, $x \ne y$, from (ii) we have

(iii) $$|x - y| \ge (1+\varepsilon)^{-1}(1-\varepsilon)\|(0, \varepsilon_2 - \varepsilon_2', \ldots, \varepsilon_k - \varepsilon_k', 0, \ldots)\|_\infty = 2(1-\varepsilon)(1+\varepsilon)^{-1} \ge 1.$$

Suppose that $\alpha(S(h_\delta, \delta)) < 1$. Then $S(h_\delta, \delta)$ admits a finite r-cover $\{A_1, \ldots, A_m\}$ with $r < 1$. Take k with $\mathrm{card}(B_k) > m$. Then for some A_i there exist x, y $(x \ne y)$ with $x, y \in B_k$ and $x, y \in A_i$. From (iii) we deduce that $\mathrm{diam}(A_i) \ge 1$, a contradiction.

Hence $\alpha(S(h_\delta, \delta)) \ge 1$. Thus $(X, |\cdot|)$ lacks (α').

COROLLARY 4. *If a Banach space X with an unconditional basis has the property (α'), then X is reflexive. If a Banach lattice X has the property (α'), then X is reflexive.*

It is easy to see that (α') is not invariant under topological isomorphisms although it is inherited by closed subspaces.

COROLLARY 5. *If a Banach space X has the property (α') then X has a subspace without the Dunford–Pettis property.*

Remarks. 1. The property (α') seems to be close to the well known property (α) of Rolewicz [R1]: A Banach space X has the property (α) provided that for every $f \in S_{X^*}$,

$$\lim_{\delta \to 0} \alpha(S(f, \delta)) = 0.$$

In fact, S. Rolewicz defines a uniform version, $(u\alpha)$, of the property (α) (see [R2]). It is easy to see that the property (α') is weaker than $(u\alpha)$.

On the other hand, although (α) and (α') are closely related, neither one implies the other.

The reflexive Banach space $l_2(l_2 \oplus l_3 \oplus \ldots \oplus l_n \oplus \ldots)$ has the property (α), but does not have (α') (see [M]).

On the other hand, l_2 with norm $|x|_\beta := \max\{\|x\|_2, (1+\beta)\|x\|_\infty\}$ has (α') for suitably small β, but not (α) (see [D-S]).

In [R1] it is proved that every Banach space $(X, \|\cdot\|)$ with the property (α) is reflexive. We do not know if the analogue for (α') holds, although Corollary 4 provides a partial affirmative answer. On the other hand, from Theorem 3 we know that $(d') \Leftrightarrow (\varepsilon_0(X) < 1) \Rightarrow (X$ is superreflexive), but there are nonsuperreflexive (NUC) Banach spaces (see [KU]).

2. Every renorming of c_0 with normal structure fails (α') and provides a counterexample to the converse implication of Theorem 1.

3. We do not know whether, for the Hausdorff measure of noncompactness χ, condition (c) holds in an arbitrary Banach space.

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On the maximal function for rotation invariant measures in $\mathbb{R}^n$

by

ANA M. VARGAS (Madrid)

Abstract. Given a positive measure μ in $\mathbb{R}^n$, there is a natural variant of the noncentered Hardy–Littlewood maximal operator

$$\mathcal{M}_\mu f(x) = \sup_{x\in B} \frac{1}{\mu(B)} \int_B |f|\,d\mu,$$

where the supremum is taken over all balls containing the point x. In this paper we restrict our attention to rotation invariant, strictly positive measures μ in $\mathbb{R}^n$. We give some necessary and sufficient conditions for $\mathcal{M}_\mu$ to be bounded from $L^1(d\mu)$ to $L^{1,\infty}(d\mu)$.

Let μ be a non-negative measure in $\mathbb{R}^n$, finite on compact sets. Given a function $f \in L^1_{\mathrm{loc}}(d\mu)$, we can define the analogue of the Hardy–Littlewood maximal function

$$\mathcal{M}_\mu f(x) = \sup_{B\in\mathcal{B}_x} \frac{1}{\mu(B)} \int_B |f|\,d\mu,$$

where

$$\mathcal{B}_x = \{B \text{ open ball}\,;\, x\in B \text{ and } \mu(B) > 0\}.$$

In fact, there are two possible definitions of the Hardy–Littlewood maximal operator. The second one corresponds to a smaller basis; namely,

$$\mathcal{B}^c_x = \{\text{open balls } B \text{ centered at } x \text{ with } \mu(B) > 0\}.$$

The operator associated with the latter basis maps $L^1(d\mu)$ into $L^{1,\infty}(d\mu)$. This can be proved using the Besicovitch covering lemma. An operator that satisfies this boundedness property is said to be of weak type 1-1 with respect to the measure μ.

But things are not so easy when dealing with the former basis, the noncentered case:

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