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Calderón–Zygmund operators and unconditional bases of weighted Hardy spaces

by

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Abstract. We study sufficient conditions on the weight w, in terms of membership in the A_p classes, for the spline wavelet systems to be unconditional bases of the weighted space $H^p(w)$. The main tool to obtain these results is a very simple theory of regular Calderón–Zygmund operators.

0. Introduction. The purpose of this article is twofold. First of all, we present (in Section 2) a very simple theory of regular Calderón–Zygmund operators, based upon the notion of weighted atom and a general extrapolation principle. The whole theory develops almost immediately from the basic estimate in Theorem 2.3 below. This estimate contains almost all the information about the boundedness properties of the operator.

Secondly, as an illustration and an extension of the theory, we find (in Section 3) sufficient conditions on the weight w, in terms of membership in the A_p classes, for the systems of m-splines to be unconditional bases of $H^p(w)$. Only the unweighted case has been treated so far in the literature. For this problem, the operators to be studied are different from the Calderón–Zygmund operators of Section 2, but the basic estimates they satisfy turn out to be the same. This unity makes the theory transparent. The first estimates for the basic m-splines appear in the work of Z. Ciesielski. We improve the estimates which were obtained in [St] to deal with the unweighted case. Moreover, we show that the high-dimensional case, which is treated in [St] in a way far from satisfactory, is not essentially different from the one-dimensional case.

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We make no effort to get the weakest sufficient conditions on w outside the A_p classes. With our same proof, it is obvious that one can formulate weaker conditions of the type of those appearing in [S-T]. We have not included these refinements in the present paper, since we mainly want to illustrate the general philosophy.

We shall address the problem of necessity elsewhere and we shall also consider the case $p > 1$ on another occasion. However, we have felt the need to establish a comparison with the dyadic case, that is, the case associated with the Haar system and dyadic martingales. We treat it in Section 4.

1. Basic facts. Let E be a separable F-space (i.e. a complete invariant metric topological vector space). A sequence of vectors $\{e_k\}_{k=1}^{\infty}$ is called a *(Schauder) basis* of E if for every $e \in E$ there exists a unique sequence of numbers $\{\xi_k\}_{k=1}^{\infty}$ such that the sequence of partial sums

$$S_N(e) = \sum_{k=1}^{N} \xi_k e_k$$

converges to e in the metric of E.

The following theorem was proved by S. Banach (see [B]) for Banach spaces, but it is also true for general F-spaces because the only tool used in the proof is the open mapping theorem which holds for F-spaces as well.

THEOREM (Banach). *Let E be a separable F-space and let $\{e_k\}_{k=1}^{\infty}$ be a sequence of vectors of E. Then $\{e_k\}_{k=1}^{\infty}$ is a basis of E if and only if the following conditions hold*:

(1) *The system $\{e_k\}_{k=1}^{\infty}$ is complete in E, i.e. the linear span of the vectors $\{e_k\}_{k=1}^{\infty}$ is dense in E.*

(2) *There exists a system of functionals $\{e_k^*\}_{k=1}^{\infty}$ belonging to the dual space E^* such that $e_m^*(e_k) = \delta_{mk}$, the Kronecker δ ($\{e_k^*\}_{k=1}^{\infty}$ is called the biorthogonal or conjugate system).*

(3) *The partial sum operators*

$$S_N(e) = \sum_{k=1}^{N} e_k^*(e) e_k$$

are uniformly bounded in E.

A basis $\{e_k\}_{k=1}^{\infty}$ of E is called *unconditional* if it remains a basis after every rearrangement of its elements.

The following theorem is well known for Banach spaces (for that case it is due to Banach and Orlicz, see [W1]) and it can be proved analogously for F-spaces.

THEOREM. *Let E be a separable F-space and let $\{e_k\}_{k=1}^{\infty}$ be a sequence of vectors of E. Then $\{e_k\}_{k=1}^{\infty}$ is an unconditional basis of E if and only if the conditions of the Banach theorem hold and also the operators $S_{N,\varepsilon}$ or $S_{N,\eta}$ given by*

$$S_{N,\varepsilon}(e) = \sum_{k=1}^{N} \varepsilon_k e_k^*(e) e_k \quad \text{and} \quad S_{N,\eta} = \sum_{k=1}^{N} \eta_k e_k^*(e) e_k$$

are uniformly bounded in E, where $\varepsilon = \{\varepsilon_k\}_{k=1}^{\infty}$ is any sequence of ± 1's and $\eta = \{\eta_k\}_{k=1}^{\infty}$ is any sequence of 0's and 1's.

We shall mainly be concerned with the spaces $L^p(w)$, $1 \le p < \infty$, and $H^p(w)$, $0 < p \le 1$. We always work on $\mathbb{R}^n$, and we only make a few comments on how the theory works in other contexts, like $\mathbb{I} = [0,1]$ or the torus $\mathbb{T}$.

Our weights w will mostly belong to the class A_∞, which is the union of all the classes A_p, $1 \le p < \infty$. We give the definition of A_p in $\mathbb{R}^n$. A weight $w \ge 0$ is said to belong to A_p for $1 < p < \infty$ if

$$(A_p) \qquad \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}\,dx\right)^{p-1} \le C$$

with C finite independent of the cube Q.

The class A_1 is defined by letting $p \to 1$, namely

$$(A_1) \qquad \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\|w^{-1}\|_{L^\infty(Q)} \le C$$

with C finite independent of Q.

These classes were introduced by Muckenhoupt in [Mu], and their theory was further developed in [C-F]. See also [G-R].

EXAMPLE. On the torus $\mathbb{T}$, the trigonometric system $\{e^{inx}\}_{n\in\mathbb{Z}}$ is a basis of $L^p(\mathbb{T}, w)$, $1 < p < \infty$, if and only if w satisfies the (A_p) condition (the analogue for $\mathbb{T}$). This is just a restatement of the well known theorem of R. Hunt, B. Muckenhoupt and R. Wheeden [H-M-W] about the boundedness of the conjugate function on weighted L^p spaces.

However, it follows from a theorem of Gaposhkin (see [K]) that the trigonometric system can be an unconditional basis of $L^p(w)$ only for $p = 2$.

DEFINITION 1.1. Given a weight $w \ge 0$ on $\mathbb{R}^n$, we shall denote by $H^p(w)$ the space of those $f \in \mathcal{S}'(\mathbb{R}^n)$ (tempered distributions) for which the maximal function

$$\phi^*(f)(x) = \sup_{t>0} |\phi_t * f(x)| \in L^p(w)$$

where $\phi \in \mathcal{S}(\mathbb{R}^n)$ is a fixed Schwartz function with $\int_{\mathbb{R}^n} \phi \neq 0$.

Under mild conditions on w (in particular for $w \in A_\infty$), the definition does not depend on the choice of ϕ (see [S-T]) and we can write

$$\|f\|_{H^p(w)} = \|\phi^*(f)\|_{L^p(w)}$$

obtaining a norm if $p \geq 1$ and a p-norm otherwise.

We shall always assume $w \in A_\infty$ and define

$$q_w = \inf\{q > 1 : w \in A_q\},$$

the *critical index* of w.

Also for $0 < p \leq 1$ we write

$$N_p(w) = \left[n\left(\frac{q_w}{p} - 1\right)\right],$$

i.e. the largest integer $\leq n(q_w/p - 1)$.

DEFINITION 1.2. Given a weight $w \geq 0$ on $\mathbb{R}^n$, and a number p with $0 < p \leq 1$, a *p-atom* with respect to w will be a function a supported in a cube Q, and such that

$$(1.3) \qquad \|a\|_\infty \leq w(Q)^{-1/p}$$

and

$$(1.4) \qquad \int_{\mathbb{R}^n} x^\alpha a(x)\,dx = 0 \quad \text{for every multi-index } \alpha \text{ with } |\alpha| \leq N_p(w).$$

These p-atoms with respect to w are the basic building blocks of $H^p(w)$, as stated in the next proposition, whose proof can be seen in [G] and [S-T]. In those references our p-atoms are called (p,∞)-atoms, since other (p,q)-atoms are also considered there, which we shall not need in the present paper.

PROPOSITION 1.5. *Let $w \in A_\infty$ be a weight in $\mathbb{R}^n$, and let $0 < p \leq 1$. A tempered distribution f on $\mathbb{R}^n$ belongs to $H^p(w)$ if and only if f can be written as a series*

$$(1.6) \qquad f = \sum_j \lambda_j a_j$$

convergent in the sense of distributions, where each a_j is a p-atom with respect to w and the coefficients λ_j satisfy

$$(1.7) \qquad \sum_j |\lambda_j|^p < \infty.$$

Moreover, the infimum of the sums (1.7) *over all decompositions* (1.6) *is equivalent to the p-norm $\|f\|^p_{H^p(w)}$.*

Next, we define m-splines on $\mathbb{R}$. Let m be an integer ≥ 0. Let $V_0 = \{f \in L^2(\mathbb{R}) \cap C^{m-1}(\mathbb{R})$: the restriction of f to each interval $]n, n+1[$ is a

polynomial of degree $\leq m\}$, where we denote by $C^r(\mathbb{R})$ the class of functions on $\mathbb{R}$ whose derivatives of order r are continuous and by $C^{-1}(\mathbb{R})$ the class of piecewise continuous functions on $\mathbb{R}$.

Then we get a multi-scale analysis $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ in the sense of S. Mallat [Ma] and Y. Meyer [Me], simply by defining $V_j \subset L^2(\mathbb{R})$ in this way:

$$(1.8) \qquad f(2x) \in V_{j+1} \Leftrightarrow f(x) \in V_j.$$

That $\{V_j\}_{j\in\mathbb{Z}}$ is a multi-scale analysis means that the V_j's are an increasing sequence of closed subspaces of $L^2(\mathbb{R})$ satisfying (1.8) and also the following properties:

$$(1.9) \qquad \bigcap_{j\in\mathbb{Z}} V_j = \{0\} \quad \text{and} \quad \bigcup_{j\in\mathbb{Z}} V_j \text{ is dense in } L^2(\mathbb{R});$$

$$(1.10) \qquad f(x) \in V_0 \Leftrightarrow f(x-k) \in V_0 \ \forall k \in \mathbb{Z};$$

(1.11) there is $g \in V_0$ such that $\{g(x-k)\}_{k\in\mathbb{Z}}$ is a Riesz basis of V_0, i.e. for some C,

$$C^{-1}\Big(\sum_{k\in\mathbb{Z}} |\alpha_k|^2\Big)^{1/2} \leq \Big\|\sum_{k\in\mathbb{Z}} \alpha_k g(x-k)\Big\|_{V_0} \leq C\Big(\sum_{k\in\mathbb{Z}} |\alpha_k|^2\Big)^{1/2}.$$

Actually, g can be taken to be

$$(1.12) \qquad g = \chi * \ldots * \chi \quad (m+1 \text{ times}),$$

where χ is the characteristic function of $[0,1]$. This multi-scale analysis is *m-regular*, in the sense that, with the choice made in (1.12),

$$(1.13) \qquad |D^\alpha g(x)| \leq C_{N,\alpha}(1+|x|)^{-N} \quad \text{for all } \alpha \text{ with } 0 \leq \alpha \leq m \text{ and all } N \in \mathbb{N}.$$

In (1.13), the derivatives are to be understood in the sense of distributions. Note that, actually, the one of order m is only L^∞.

Every time we have an m-regular multi-scale analysis, we can find an analyzing wavelet, that is, a function $\psi \in V_1$, $\psi \perp V_0$, such that $\{\psi(x-k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis of W_0, the orthogonal complement of V_0 in V_1. See [Me] and [D]. Since

$$L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j,$$

it turns out that the system

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), \qquad j,k \in \mathbb{Z},$$

is an orthonormal basis of $L^2(\mathbb{R})$. Also, when the multi-scale analysis is m-regular, it is possible to choose ψ such that it satisfies the same conditions

(1.13) as g above. Then from (1.13) for ψ, one obtains

$$\int_{\mathbb{R}} x^\alpha \psi(x)\,dx = 0\,, \qquad 0 \le \alpha \le m. \tag{1.14}$$

In $\mathbb{R}^n$ one can define m-splines by starting with the space

$$V_0(\mathbb{R}^n) = V_0 \otimes \ldots \otimes V_0 \qquad (n \text{ times})\,.$$

The only difference is that when $n > 1$, one analyzing wavelet is not enough; but we can always find $2^n - 1$ functions ψ_η, $\eta \in E = \{0,1\}^n \setminus \{0\}$, such that each ψ_η satisfies

$$|\partial^\alpha \psi_\eta(x)| \le C_{N,\alpha}(1+|x|)^{-N} \quad \text{for all } \alpha \text{ with } 0 \le |\alpha| \le m \text{ and all } N \in \mathbb{N}\,, \tag{1.15}$$

and the system

$$\psi_{\eta,j,k}(x) = 2^{nj/2}\psi_\eta(2^j x - k), \qquad \eta \in E,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^n\,,$$

is an orthonormal basis of $L^2(\mathbb{R}^n)$. The details are given in [Me] and [D].

For the particular case of m-splines, the construction of a wavelet basis was first done by J. O. Strömberg [St], some five years earlier than the systematic development of wavelets, by Y. Meyer [Me], and can also be found in [D].

To present the main problem, let us go back to the one-dimensional case; the case of higher dimensions is essentially the same but requires a different notation.

Given a function f, its wavelet expansion will be

$$\sum_{j,k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle \psi_{j,k}\,.$$

We are actually interested in the partial sum operators

$$T_\Omega f(x) = \sum_{(j,k)\in\Omega} \langle f, \psi_{j,k}\rangle \psi_{j,k}(x)$$

for $\Omega \subset \mathbb{Z}^2$ finite; or even in the operators

$$T_{\Omega,\varepsilon} f(x) = \sum_{(j,k)\in\Omega} \varepsilon_{j,k}\langle f, \psi_{j,k}\rangle \psi_{j,k}(x)\,, \tag{1.16}$$

where $\varepsilon_{j,k} = \pm 1$. Since

$$T_{\Omega,\varepsilon} f(x) = \int_{\mathbb{R}} K_{\Omega,\varepsilon}(x,y) f(y)\,dy\,,$$

where

$$K_{\Omega,\varepsilon}(x,y) = \sum_{(j,k)\in\Omega} \varepsilon_{j,k}\psi_{j,k}(x)\psi_{j,k}(y)\,,$$

by (1.13) we obtain

PROPOSITION 1.17. *The kernels $K_{\Omega,\varepsilon}(x,y)$ associated with the multi-scale analysis by m-splines (or with any m-regular multi-scale analysis) in $\mathbb{R}$ satisfy the estimates*

$$|D_y^\alpha K_{\Omega,\varepsilon}(x,y)| \le |x-y|^{-1-\alpha}\,, \qquad 0 \le \alpha \le m\,.$$

Proof. We simply estimate $K_{\Omega,\varepsilon}$ by using the estimates for $\psi_{j,k}$ which we obtain from (1.13) by dilation and translation. We get

$$|K_{\Omega,\varepsilon}(x,y)| \le \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \frac{2^{j/2}}{2^{jN}(2^{-j}+|x-k/2^j|)^N}\,\frac{2^{j/2}}{2^{jN}(2^{-j}+|y-k/2^j|)^N}\,.$$

In order to estimate this sum we fix $x,y\in\mathbb{R}$, $x \ne y$. First we consider those j's such that $|x-y| < 2^{-j-1}$. For each such j we have

$$\sum_{k\in\mathbb{Z}} \ldots \le C2^j\Big(1 + \frac{1}{2^{2N}} + \frac{1}{3^{2N}} + \ldots\Big) = C2^j\,.$$

Then, summing over all such j's, we get

$$\sum_{|x-y|<2^{-j-1}}\ \sum_{k\in\mathbb{Z}} \ldots \le C \sum_{|x-y|<2^{-j-1}} 2^j \le C|x-y|^{-1}\,.$$

If $|x-y| > 2^{-j-1}$, we first sum over k as before, obtaining

$$\sum_{k\in\mathbb{Z}} \ldots \le C\frac{2^j}{2^{jN}|x-y|^N}\Big(1+\frac{1}{2^N}+\frac{1}{3^N}+\ldots\Big) = C\frac{2^j}{2^{jN}|x-y|^N}$$

and summing over j we again arrive at

$$\sum_{|x-y|>2^{-j-1}} \ldots \le C|x-y|^{-1}\,.$$

This is what we wanted for $\alpha = 0$. For the derivatives we proceed in the same fashion starting from the estimate

$$|D_y^\alpha K_{\Omega,\varepsilon}(x,y)| \le \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} 2^{j\alpha}\frac{2^{j/2}}{2^{jN}(2^{-j}+|x-k/2^j|)^N}\,\frac{2^{j/2}}{2^{jN}(2^{-j}+|y-k/2^j|)^N}\,.$$

We obtain

$$|D_y^\alpha K_{\Omega,\varepsilon}(x,y)| \le C|x-y|^{-1-\alpha}\,,$$

as desired. ■

Of course we also have the corresponding result in higher dimensions, which is obtained in the same way.

PROPOSITION 1.18. *The kernels $K_{\Omega,\varepsilon}(x,y)$ associated with the multi-scale analysis by m-splines (or with any m-regular multi-scale analysis) in*

$\mathbb{R}^n$ *satisfy the estimates*

$$|\partial_y^\alpha K_{\Omega,\varepsilon}(x,y)| \le C|x-y|^{-n-|\alpha|}, \quad 0 \le |\alpha| \le m.$$

2. Calderón–Zygmund theory for regular singular integrals. Motivated by the estimates obtained in the previous section, we make the following

DEFINITION 2.1. Given a kernel $K(x,y)$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, $x \ne y$, and given $\gamma \in \mathbb{R}$, $\gamma > 0$, we say that K is *γ-regular* with respect to y if:

(1) K has continuous derivatives $\partial_y^\alpha K(x,y)$ for every multi-index α with $|\alpha| < \gamma$ and they satisfy

$$|\partial_y^\alpha K(x,y)| \le C|x-y|^{-n-|\alpha|}.$$

(2) For highest order derivatives, that is, those corresponding to $\gamma - 1 \le |\alpha| < \gamma$, we have

$$|\partial_y^\alpha K(x,y) - \partial_y^\alpha K(x,y')| \le C\frac{|y-y'|^{\gamma-|\alpha|}}{|x-y|^{n+\gamma}},$$

provided $2|y-y'| < |x-y|$.

Of course, there is a parallel notion of *γ-regularity* with respect to x which amounts to saying that the dual kernel $K^*(x,y) = K(y,x)$ satisfies the conditions above.

Observation. If $\gamma < \gamma'$ and $K(x,y)$ is γ'-regular with respect to y, then $K(x,y)$ is also γ-regular with respect to y.

DEFINITION 2.2. We shall say that T is a *singular integral operator* with kernel $K(x,y)$ if for every L^∞ function f with bounded support, and for almost every x in the complement of that support,

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)\,dy.$$

For example, the operators $T_{\Omega,\varepsilon}$ associated with the wavelet expansion by m-splines, or in general, with the expansion in wavelets corresponding to an m-regular multi-scale analysis, are singular integral operators with m-regular symmetric kernels.

The starting point for the Calderón–Zygmund theory is the following simple estimate.

THEOREM 2.3. *Let T be a singular integral operator with kernel $K(x,y)$ γ-regular with respect to y. Let f be an L^∞ function supported in a cube Q with center y_0 such that*

$$\int_{\mathbb{R}^n} f(x)x^\alpha\,dx = 0 \quad \textit{for every multi-index } \alpha \textit{ with } |\alpha| < \gamma.$$

Then for almost every $x \notin \widetilde{Q}$, the $2\sqrt{n}$-dilate of Q, we have

$$|Tf(x)| \le C\left(\frac{|Q|}{|x-y_0|^n}\right)^{1+\gamma/n}\|f\|_\infty.$$

Proof. We have

$$\begin{aligned} Tf(x) &= \int_Q K(x,y)f(y)\,dy \\ &= \int_Q \Big\{K(x,y) - \sum_{|\alpha|<\gamma}\frac{1}{\alpha!}\partial_y^\alpha K(x,y_0)(y-y_0)^\alpha\Big\} f(y)\,dy. \end{aligned}$$

Using Taylor's theorem and condition (2) in Definition 2.1, we get

$$|Tf(x)| \le C\frac{(|Q|^{1/n})^\gamma|Q|}{|x-y_0|^{n+\gamma}}\|f\|_\infty. \ \blacksquare$$

COROLLARY 2.4. *Let T be a singular integral operator with kernel $K(x,y)$ γ-regular with respect to y. Let $f \in L_N^\infty(Q)$ (that is, f is an essentially bounded function supported in the cube Q and with moments vanishing up to order N). Then for a.e. $x \notin \widetilde{Q}$,*

$$|Tf(x)| \le C\left(\frac{|Q|}{|x-y_0|^n}\right)^{1+\min(N+1,\gamma)/n}\|f\|_\infty.$$

Proof. If $\gamma \le N+1$, we have

$$\int_{\mathbb{R}^n} f(x)x^\alpha\,dx = 0 \quad \text{for all } \alpha \text{ with } |\alpha| < \gamma$$

and the conclusion of the theorem holds.

If $N+1 \le \gamma$ then K is $N+1$-regular and the conclusion holds with $N+1$ in place of γ. $\blacksquare$

COROLLARY 2.5. *Let T be a singular integral operator with kernel $K(x,y)$ γ-regular with respect to y. Let $w \in A_\infty$ and let f be a p-atom with respect to w, supported in a cube Q. Suppose $0 < p \le 1$ is such that $q_w/p < 1+\gamma/n$. Then*

$$\int_{\mathbb{R}^n\setminus\widetilde{Q}} |Tf(x)|^p w(x)\,dx \le C$$

with C a constant independent of f.

Proof. The atom has vanishing moments up to order $N = N_p(w) = [n(q_w/p-1)]$ at least, so that $N+1 > n(q_w/p-1)$. Also, by hypothesis, this inequality holds with γ in place of $N+1$. Thus $1+\min(N+1,\gamma)/n > q_w/p$. By setting $q = p(1+\min(N+1,\gamma)/n) > q_w$, we obtain from the previous corollary

$$|Tf(x)|^p \le C\left(\frac{|Q|}{|x-y_0|^n}\right)^q \|f\|_\infty^p \quad \text{for a.e. } x \notin \widetilde{Q}\,.$$

But, since $w \in A_q$, we have

$$\int_{\mathbb{R}^n\setminus\widetilde{Q}} |Tf(x)|^p w(x)\,dx \le C \int_{\mathbb{R}^n\setminus\widetilde{Q}} \left(\frac{|Q|}{|x-y_0|^n}\right)^q w(x)\,dx\,\|f\|_\infty^p$$
$$\le Cw(Q)\|f\|_\infty^p\,. \ \blacksquare$$

DEFINITION 2.6. A singular integral operator which is bounded in $L^{p_0}(\mathbb{R}^n)$ for some p_0 such that $1 < p_0 < \infty$ will be called a *Calderón–Zygmund operator*.

PROPOSITION 2.7. *Let T be a Calderón–Zygmund operator bounded in $L^{p_0}(\mathbb{R}^n)$ for some p_0 such that $1 < p_0 < \infty$. Suppose that T has a kernel which is γ-regular in y. Then for every $w \in A_1 \cap A_\infty^{1/p_0'}$, T maps $H^1(w)$ boundedly into $L^1(w)$.*

Proof. Let f be a 1-atom with respect to w. By applying Corollary 2.5 with $q_w = 1$ and $p = 1$, we have

$$\int_{\mathbb{R}^n\setminus\widetilde{Q}} |Tf(x)|w(x)\,dx \le C\,.$$

We need a similar estimate over $\widetilde{Q}$. We shall get it by using the boundedness of T on $L^{p_0}(\mathbb{R}^n)$ plus the fact that $w^{p_0'} \in A_\infty$. Note that this is equivalent to saying that w satisfies a reverse Hölder inequality with exponent p_0' (see [S-W] or [J-N]). Then

$$\int_{\widetilde{Q}} |Tf(x)|w(x)\,dx \le \Big(\int_{\widetilde{Q}} |Tf(x)|^{p_0}\,dx\Big)^{1/p_0}\Big(\int_{\widetilde{Q}} w(x)^{p_0'}\,dx\Big)^{1/p_0'}$$
$$\le Cw(Q)^{-1}|Q|\left(\frac{1}{|\widetilde{Q}|}\int_{\widetilde{Q}} w(x)^{p_0'}\,dx\right)^{1/p_0'} \le C\,. \ \blacksquare$$

Now we can put together in a single statement all the boundedness properties of regular Calderón–Zygmund operators. We shall use the following extrapolation theorem of [G1].

THEOREM (Extrapolation). *Let T be an operator, either linear or sublinear; in the second case suppose that $Tf \ge 0$ for every f.*

(1) *If T is bounded from wL^∞ to* BMO(w) *for every $w \in A_1^\alpha$ and some α such that $0 < \alpha \le 1$, with norm depending only on the A_1 constant of $w^{1/\alpha}$, then T is bounded in $L^q(w)$ for every $q > 1/\alpha$ and every $w \in A_{\alpha q}$.*

(2) *If T is bounded from $H^1(w)$ to $L^1(w)$ for every $w \in A_1$, with norm depending only on the A_1 constant of w, then T is bounded in $L^p(w)$ for every p such that $1 < p < \infty$ and every $w \in A_p$.*

Part (1) is Corollary 1.15b) of [G1] and part (2) is Corollary 2.7 of [G1].

THEOREM 2.8. *Let T be a Calderón–Zygmund operator with kernel γ-regular in y and ε-regular in x.*

(1) *If $1 < p < \infty$ and $w \in A_p$, then T is bounded in $L^p(w)$.*

(2) *If $0 < p \le 1$ and $w \in A_\infty$ are such that $q_w/p < 1 + \gamma/n$, then T is bounded from $H^p(w)$ to $L^p(w)$.*

In all cases the norm of T depends only on the constants for the kernel and the weight.

Proof. By applying Proposition 2.7 to the dual operator T^*, we see that T^* is bounded from $H^1(w)$ to $L^1(w)$ for every $w \in A_1 \cap A_\infty^\alpha$ and some α, $0 < \alpha < 1$. In particular, T^* maps $H^1(w)$ boundedly into $L^1(w)$ for every $w \in A_1^\alpha$. This implies that T is bounded from wL^∞ to BMO(w) for every $w \in A_1^\alpha$. Then part (1) of the extrapolation theorem above can be used to conclude that T is bounded in $L^q(w)$ for every $q > 1/\alpha$ and every $w \in A_{\alpha q}$.

Let us prove (2) first. We assume $0 < p \le 1$ and $w \in A_\infty$ such that $q_w/p < 1 + \gamma/n$. If f is a p-atom with respect to w having support in a cube Q, we know that

$$\int_{\mathbb{R}^n\setminus\widetilde{Q}} |Tf(x)|^p w(x)\,dx \le C\,.$$

We need a similar estimate over $\widetilde{Q}$. We shall get it by taking q so large that $\alpha q > q_w$. Then $w \in A_{\alpha q}$ and, consequently, T is bounded in $L^q(w)$. Thus

$$\int_{\widetilde{Q}} |Tf(x)|^p w(x)\,dx$$
$$= \int_{\widetilde{Q}} |Tf(x)|^p w(x)^{p/q} w(x)^{1-p/q}\,dx$$
$$\le \Big(\int_{\widetilde{Q}} |Tf(x)|^q w(x)\,dx\Big)^{p/q}\Big(\int_{\widetilde{Q}} w(x)^{(1-p/q)(q/p)'}\,dx\Big)^{1/(q/p)'}$$
$$\le Cw(Q)^{-1}w(\widetilde{Q})^{p/q}w(\widetilde{Q})^{1-p/q} = C\,.$$

This completes the proof of (2).

Note that this implies, in particular, that T is bounded from $H^1(w)$ to $L^1(w)$ for every $w \in A_1$. By part (2) of the extrapolation theorem, this implies (1). ■

3. $H^p \to H^p$ estimates. As a consequence of the Calderón–Zygmund theory developed in Section 2, we see that the modified partial sum operators $T_{\Omega,\varepsilon}$ associated with the expansion in m-spline wavelets are uniformly bounded from $H^p(w)$ to $L^p(w)$ provided $q_w/p < 1 + m/n$. We are going to see that they are actually bounded in $H^p(w)$ with the same restriction on the indices. We shall explain the details for the 1-dimensional case using the specific wavelet basis constructed by J. O. Strömberg in [St]. Now instead of looking at the modified partial sum operators

$$T_{\Omega,\varepsilon} f(x) = \sum_{(j,k)\in\Omega} \pm\langle f, \psi_{j,k}\rangle \psi_{j,k}(x),$$

we have to look at the operators

$$\phi^*(T_{\Omega,\varepsilon} f)(x) \le \sum_{(j,k)\in\Omega} |\langle f, \psi_{j,k}\rangle| \phi^*(\psi_{j,k})(x).$$

Consequently, we shall estimate $\phi^*(\psi_{j,k})$ and also the wavelet coefficients $\langle f, \psi_{j,k}\rangle$ of suitable functions f, say atoms. Since ϕ^* commutes with translations and dilations, we start by looking at $\phi^*(\psi)$, where ψ is the analyzing wavelet associated with m-splines in $\mathbb{R}$.

PROPOSITION 3.1.

$$\phi^*(\psi)(x) \le \frac{C}{(1+|x|)^{m+2}}.$$

Proof.

$$|\phi_t * \psi(x)| = \Big|\int_{\mathbb{R}} \psi(x-y)\phi_t(y)\,dy\Big| \le \sup|\psi| \int_{\mathbb{R}} |\phi_t|$$
$$\le C \le \frac{C}{(1+|x|)^{m+2}} \quad \text{if } |x| \le 1.$$

So, we just need to consider $|x| > 1$ and to prove, in that case,

$$|\phi_t * \psi(x)| \le \frac{C}{|x|^{m+2}}.$$

For $0 < t \le 1$, the estimate is trivial:

$$|\phi_t * \psi(x)| \le \int_{\mathbb{R}} |\psi(x-y)|\,|\phi_t(y)|\,dy = \int_{|y|<|x|/2} \ldots dy + \int_{|y|>|x|/2} \ldots dy.$$

But

$$\int_{|y|<|x|/2} |\psi(x-y)|\,|\phi_t(y)|\,dy \le \sup_{B(x,|x|/2)} |\psi| \cdot \int_{\mathbb{R}} |\phi_t| \le \frac{C(N)}{|x|^N},$$

and we can take $N = m+2$. Also,

$$\int_{|y|>|x|/2} |\psi(x-y)|\,|\phi_t(y)|\,dy \le \int_{\mathbb{R}} |\psi(y)|\,dy\, \frac{1}{t}\frac{C_N}{|x/(2t)|^N}$$
$$= Ct^{N-1}\frac{C_N}{|x|^N} \le C_N\frac{1}{|x|^N},$$

and again we can take $N = m+2$.

Now we assume $t > 1$. It is in this case that we use the vanishing moments of ψ in (1.14) and we get an estimate depending essentially on m. We have

$$|\phi_t * \psi(x)| = \Big|\int_{\mathbb{R}} \psi(x-y)\Big\{\phi_t(y) - \sum_{\alpha=0}^{m} \frac{1}{\alpha!}\phi_t^{(\alpha)}(x)(y-x)^\alpha\Big\}\,dy\Big|$$
$$= \Big|\int_{\mathbb{R}} \psi(x-y)\frac{1}{(m+1)!}\phi_t^{(m+1)}(x+\theta_y(x-y))(y-x)^{m+1}\,dy\Big|$$
$$\le C\int_{\mathbb{R}} |\psi(x-y)|\frac{1}{t^{m+2}}\Big|\phi^{(m+1)}\Big(\frac{x+\theta_y(x-y)}{t}\Big)\Big|\,|y-x|^{m+1}\,dy$$
$$\le C\Big(\int_{|x-y|<|x|/2} \ldots dy + \int_{|x-y|>|x|/2} \ldots dy\Big),$$

with $0 \le \theta_y \le 1$. But $\int_{|x-y|>|x|/2} \ldots dy \le C/|x|^N$ and we can take $N = m+2$. Also,

$$\int_{|x-y|<|x|/2} \ldots dy \le \int_{|x-y|<|x|/2} |\psi(x-y)|\,|y-x|^{m+1} \times \frac{1}{t^{m+2}}\frac{C}{\Big(1+\Big|\frac{x+\theta_y(y-x)}{t}\Big|\Big)^{m+2}}\,dy \le \frac{C}{|x|^{m+2}}. \ ■$$

After translating and dilating we get

COROLLARY 3.2.

$$\phi^*(\psi_{j,k})(x) \le \frac{C2^{j/2}}{2^{j(m+2)}(2^{-j}+|x-k/2^j|)^{m+2}}.$$

Our next objective is to estimate the coefficients $\langle f, \psi_{j,k}\rangle$ for appropriate f.

PROPOSITION 3.3. *Let f be an L^∞ function supported in an interval I centered at x_0 and such that*

$$\int_{\mathbb{R}} f(x)x^\alpha\,dx = 0 \quad \text{for all } \alpha \text{ with } 0 \le \alpha \le m-1\,.$$

Then

$$|\langle f, \psi_{j,k}\rangle| \le \frac{\|f\|_\infty C_N 2^{j/2}|I|}{2^{jN}(2^{-j} + (|x_0 - k/2^j| - |I|/2)_+)^N}\,,$$

and, when $2^{-j} > |I|$,

$$|\langle f, \psi_{j,k}\rangle| \le \frac{\|f\|_\infty C_N 2^{j(m+1/2)}|I|^{m+1}}{2^{jN}(2^{-j} + |x_0 - k/2^j|)^N}\,.$$

Proof. The first estimate is almost immediate:

$$|\langle f, \psi_{j,k}\rangle| = \Big|\int_I f(x)\psi_{j,k}(x)\,dx\Big| \le \|f\|_\infty \int_I |\psi_{j,k}|$$
$$\le \frac{\|f\|_\infty C_N 2^{j/2}|I|}{2^{jN}(2^{-j} + (|x_0 - k/2^j| - |I|/2)_+)^N}\,.$$

Now if $2^{-j} > |I|$ we can assume that $|k/2^j - x_0| > |I|$. We can also assume that there is a node x_1 of the spline $\psi_{j,k}$ such that $x_1 \in I$. Of course, there can be at most one such node. Then

$$|\langle f, \psi_{j,k}\rangle| = \Big|\int_I f(x)\psi_{j,k}(x)\,dx\Big|$$
$$= \Big|\int_I f(x)\Big\{\psi_{j,k}(x) - \sum_{\alpha=0}^{m-1}\frac{1}{\alpha!}\Big(\frac{d}{dx}\Big)^\alpha \psi_{j,k}(x_1)(x-x_1)^\alpha\Big\}\,dx\Big|$$
$$\le \int_I |f(x)|\,\Big|\Big(\frac{d}{dx}\Big)^m \psi_{j,k}(x_1 + \theta_x(x-x_1))(x-x_1)^m\Big|\,dx$$
$$\le \|f\|_\infty |I|^{m+1}\frac{C_N 2^{j(m+1/2)}}{2^{jN}(2^{-j} + |x_0 - k/2^j|)^N}\,,$$

where $0 < \theta_x < 1$. ∎

COROLLARY 3.4. *Under the conditions of Proposition 3.3,*

$$\sum_{j,k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|\phi^*(\psi_{j,k})(x) \le C\Big(\frac{|I|}{|x-x_0|}\Big)^{m+1}\|f\|_\infty\,.$$

Proof. We shall split the sum into three pieces:

$$\sum_{j,k\in\mathbb{Z}}\ldots = \underbrace{\sum_{j,k:2^{-j}>|x-x_0|/2}\ldots}_{\sum_I} + \underbrace{\sum_{j,k:|I|<2^{-j}<|x-x_0|/2}\ldots}_{\sum_{II}} + \underbrace{\sum_{j,k:2^{-j}<|I|}\ldots}_{\sum_{III}}$$

Now we estimate each of these sums separately as in the proof of Proposition 1.17:

$$\sum\nolimits_I \le \sum_{j:2^{-j}>|x-x_0|/2}\ \sum_{k\in\mathbb{Z}} \frac{\|f\|_\infty C_N 2^{j(m+1/2)}|I|^{m+1}}{2^{jN}(2^{-j} + |x_0 - k/2^j|)^N}$$
$$\times \frac{C2^{j/2}}{2^{j(m+2)}(2^{-j} + |x - k/2^j|)^{m+2}}$$
$$\le \sum_{j:2^{-j}>|x-x_0|/2} C\Big\{1 + \frac{1}{2^N} + \frac{1}{3^N} + \ldots\Big\}2^{j(m+1)}|I|^{m+1}\|f\|_\infty$$
$$\le C\Big(\frac{|I|}{|x-x_0|}\Big)^{m+1}\|f\|_\infty\,.$$

Next,

$$\sum\nolimits_{II} \le \sum_{j:|I|<2^{-j}<|x-x_0|/2}\ \sum_{k\in\mathbb{Z}} \frac{\|f\|_\infty C_N 2^{j(m+1/2)}|I|^{m+1}}{2^{jN}(2^{-j} + |x_0 - k/2^j|)^N}$$
$$\times \frac{C2^{j/2}}{2^{j(m+2)}(2^{-j} + |x - k/2^j|)^{m+2}}\,.$$

As usual, we sum first over k for j fixed. We split the sum in two. For the nodes $k2^{-j}$ which are closer to x_0 than to x, we get

$$\sum_k \ldots \le C\|f\|_\infty |I|^{m+1}\frac{2^{j(m+1)}}{2^{j(m+2)}|x-x_0|^{m+2}}\Big\{1 + \frac{1}{2^N} + \frac{1}{3^N} + \ldots\Big\}$$
$$= C\|f\|_\infty \frac{|I|^{m+1}}{|x-x_0|^{m+2}}2^{-j}\,,$$

and for those nodes closer to x than to x_0 we just get the same estimate if we take $N = m+2$. Therefore

$$\sum\nolimits_{II} \le C\|f\|_\infty |I|^{m+1}\frac{1}{|x-x_0|^{m+2}}\sum_{j:|I|<2^{-j}<|x-x_0|/2} 2^{-j}$$
$$\le C\|f\|_\infty\Big(\frac{|I|}{|x-x_0|}\Big)^{m+2} \le C\Big(\frac{|I|}{|x-x_0|}\Big)^{m+1}\|f\|_\infty\,.$$

Finally,

$$\sum\nolimits_{III} \le \sum_{j:2^{-j}<|I|}\ \sum_{k\in\mathbb{Z}} \frac{\|f\|_\infty C_N 2^{j/2}|I|}{2^{jN}(2^{-j} + (|x_0 - k/2^j| - |I|/2)_+)^N}$$
$$\times \frac{C2^{j/2}}{2^{j(m+2)}(2^{-j} + |x - k/2^j|)^{m+2}}\,.$$

Let us sum first over k over those nodes belonging to $\widetilde{I}$, the 2-dilate of I. For fixed j, there will be no more than $2|I|/2^{-j}$ of them. We get

$$\sum_k \cdots \le \frac{\|f\|_\infty C 2^j |I|}{2^{j(m+2)}|x-x_0|^{m+2}} 2^j |I| = \frac{C\|f\|_\infty |I|^2}{2^{jm}|x-x_0|^{m+2}}.$$

The remaining nodes will be either closer to x_0 or to x. In any case, we obtain

$$\sum_k \cdots \le \frac{\|f\|_\infty C 2^j |I|}{2^{j(m+2)}|x-x_0|^{m+2}}\left\{1+\frac{1}{2^{m+2}}+\frac{1}{3^{m+2}}+\cdots\right\},$$

which is an even better estimate. Now

$$\sum\nolimits_{III} \le \sum_{j:2^{-j}<|I|} \frac{C\|f\|_\infty |I|^2}{2^{jm}|x-x_0|^{m+2}}$$
$$\le C\|f\|_\infty \left(\frac{|I|}{|x-x_0|}\right)^{m+2} \le C\|f\|_\infty \left(\frac{|I|}{|x-x_0|}\right)^{m+1}. \blacksquare$$

The corresponding estimates in higher dimensions are as follows:

PROPOSITION 3.5. *Denote by ψ any of the 2^n-1 analyzing wavelets associated with m-splines in $\mathbb{R}^n$. Specifically, consider the system obtained by tensor product from the one-dimensional wavelets constructed by Strömberg in* [St] (*see also* [Me]). *Then*

$$\phi^*(\psi)(x) \le \frac{C}{(1+|x|)^{m+1+n}}.$$

Next write $\nu=(j,k_1,\ldots,k_n)=(j,k)\in\mathbb{Z}\times\mathbb{Z}^n$ and set $|\nu|=2^{-j}$. Define

$$\psi_\nu(x)=2^{nj/2}\psi(2^jx-k).$$

Then, if f is an L^∞ function supported in a cube Q centered at x_0 and such that

$$\int_{\mathbb{R}^n} f(x)x^\alpha\,dx = 0 \quad \textit{for all } \alpha \textit{ with } 0\le|\alpha|\le m-1,$$

we have

$$|\langle f,\psi_\nu\rangle| \le \frac{\|f\|_\infty |\nu|^{-n/2}|Q|C_N}{|\nu|^{-N}(|\nu|+(|x_0-2^{-j}k|-c_n|Q|^{1/n})_+)^N},$$

and if $|\nu|>|Q|^{1/n}$,

$$|\langle f,\psi_\nu\rangle| \le \frac{\|f\|_\infty |\nu|^{-(m+n/2)}|Q|^{1+m/n}C_N}{|\nu|^{-N}(|\nu|+|x_0-2^{-j}k|)^N}.$$

This leads to the following estimates for the maximal operator of the modified partial sums of f:

$$\phi^*(T_{\Omega,\varepsilon}f)(x) \le C\left(\frac{|Q|}{|x-x_0|^n}\right)^{1+m/n}\|f\|_\infty.$$

This is exactly the same estimate as that in Theorem 2.3. By arguing as in the proofs of Corollaries 2.4 and 2.5, we get

COROLLARY 3.6. *Let $w\in A_\infty$ in $\mathbb{R}^n$ and let f be a p-atom with respect to w, where $0<p\le 1$. Denote by $T_{\Omega,\varepsilon}$ the modified partial sum operators associated with the wavelet expansion in m-splines, $m\ge 1$. Then if*

$$\frac{q_w}{p} < 1+\frac{m}{n}$$

we have the bound

$$\int_{\mathbb{R}^n\setminus\widetilde{Q}} |\phi^*(T_{\Omega,\varepsilon}f)(x)|^p w(x)\,dx \le C \quad \textit{independent of } f.$$

Note that we also have an estimate on $\widetilde{Q}$ simply by using the fact that $\phi^*\circ T_{\Omega,\varepsilon}$ is (uniformly) bounded in $L^q(w)$ for $q>q_w$:

$$\int_{\widetilde{Q}} |\phi^*(T_{\Omega,\varepsilon}f)(x)|^p w(x)\,dx$$
$$\le \Big(\int_{\widetilde{Q}} |\phi^*(T_{\Omega,\varepsilon}f)(x)|^q w(x)\,dx\Big)^{p/q} w(\widetilde{Q})^{1-p/q} \le C.$$

In order to obtain the main result, we need the characterization of the dual spaces. The atomic decomposition immediately leads to the following characterization of the duals (see [G], [S-T]).

PROPOSITION 3.7. *Suppose $w\in A_r$, $1<r<\infty$ and $0<p\le 1$. Then for every $\Lambda\in(H^p(w))^*$ we can find a function g such that*

$$\left(\frac{1}{w(Q)}\int_Q \left|\frac{g(x)-g_Q}{w(x)}\right|^{r'} w(x)\,dx\right)^{1/r'} \le Cw(Q)^{1/p-1}, \tag{3.8}$$

uniformly for each cube Q where g_Q is the unique polynomial of degree $\le N_p(w)$ having over Q the same moments as g up to order $N_p(w)$ and so that for any linear combination f of atoms,

$$\Lambda(f) = \int_{\mathbb{R}^n} f(x)g(x)\,dx. \tag{3.9}$$

Conversely, any function g satisfying (3.8) *gives rise by means of* (3.9) *to a continuous linear functional $\Lambda\in(H^p(w))^*$.*

We thus arrive at the main result:

THEOREM 3.10. *For $m \geq 1$, m-spline wavelets in $\mathbb{R}^n$ constitute an unconditional basis of*:

(1) $L^p(w)$ *for* $1 < p < \infty$ *and* $w \in A_p$,
(2) $H^p(w)$ *for* $0 < p \leq 1$ *and* $w \in A_\infty$ *such that* $q_w/p < 1 + m/n$.

For example, Strömberg's modified Franklin system ($m = 1$, $n = 1$) is an unconditional basis of $H^1(w)$ provided $q_w < 2$, that is, provided $w \in A_2$.

The history of the discovery by B. Maurey [Mau] of the fact that H^1 has an unconditional basis, and the subsequent construction of concrete bases by L. Carleson [Ca] and P. Wojtaszczyk [W], can be found in [Me]. For the spline bases in the unweighted case, after the research of Z. Ciesielski (see [C1] and [C2]) and S. V. Bochkarev (see [Bo]), we have to cite [St] and further work by A. Chang and Z. Ciesielski [C-C] and also by P. Sjölin and J. O. Strömberg [S-S].

4. The Haar system and dyadic martingales. In the previous part we have only discussed the case $m \geq 1$. When $m = 0$ we get completely different results. We treat this case in terms of dyadic martingales and dyadic martingale H^p spaces. Denote by Δ_0 the collection of cubes of side length 1 in $\mathbb{R}^n$ whose vertices have integer coordinates and let Δ_k be the collection of cubes obtained from those of Δ_0 by dilation with center at the origin and ratio 2^{-k}. The *conditional expectation* $\mathcal{E}_k f$ is the function constant on each cube of Δ_k, whose value on $I \in \Delta_k$ is the mean value of f on I. A *dyadic martingale* is a sequence $F = \{f_k\}$ of step functions with each f_k being almost everywhere constant on the cubes of Δ_k and $\mathcal{E}_k f_l = f_k$ whenever $k \leq l$.

Now we are able to define weighted dyadic Hardy spaces $H^p_{\mathrm{dy}}(w)$ for weights $w \in A^{\mathrm{dy}}_\infty$. The class A^{dy}_∞ is the union of all the classes A^{dy}_p, $1 \leq p < \infty$, the latter being obtained by considering only dyadic cubes in the conditions (A_p) or (A_1) given at the beginning of Section 1. Denote by $MF(x) = \sup_{k\in\mathbb{Z}} |f_k(x)|$ the dyadic maximal function. Defining $H^p_{\mathrm{dy}}(w)$ as the space of dyadic martingales $F = \{f_k\}$ such that $MF \in L^p(w)$ one can similarly obtain the analogue of Proposition 1.5. Atoms in this context are defined as follows.

DEFINITION 4.1. A *dyadic p-atom a* with respect to $w \in A^{\mathrm{dy}}_\infty$ is a function with support contained in some dyadic cube I such that $\|a\|_\infty \leq w(I)^{-1/p}$ and $\int a(x)\,dx = 0$.

For a given locally integrable function f one can always form the martingale $\{f_k\}$ with $f_k = \mathcal{E}_k f$. Now one can formulate the analogue of Proposition 1.5 for a martingale $F \in H^p_{\mathrm{dy}}(w)$ and $w \in A^{\mathrm{dy}}_\infty$. In this formulation dyadic atoms appear instead of ordinary ones and convergence in the sense of distributions is replaced by convergence of a sequence of martingales in the sense that for every fixed k the functions standing in the kth place converge in the L^∞ metric.

The atomic decomposition immediately leads to the following characterization of the duals similar to Proposition 3.7.

PROPOSITION 4.2. *Suppose $w \in A^{\mathrm{dy}}_r$, $1 < r < \infty$ and $0 < p \leq 1$. Then for every $\Lambda \in (H^p_{\mathrm{dy}}(w))^*$ we can find a function g such that*

$$\left(\frac{1}{w(I)}\int_I \left|\frac{g(x) - g_I}{w(x)}\right|^{r'} w(x)\,dx\right)^{1/r'} \leq Cw(I)^{1/p-1} \tag{4.3}$$

uniformly for each dyadic cube I where $g_I = |I|^{-1}\int_I g(x)\,dx$, and so that for any linear combination f of atoms,

$$\Lambda(f) = \int_{\mathbb{R}^n} f(x)g(x)\,dx. \tag{4.4}$$

Conversely, any function g satisfying (4.3) *gives rise by means of* (4.4) *to a continuous linear functional $\Lambda \in (H^p_{\mathrm{dy}}(w))^*$.*

Define

$$\psi^{(0)}(x) = \begin{cases} 1 & \text{for } 0 \leq x < 1/2, \\ -1 & \text{for } 1/2 < x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

This is an analyzing wavelet for the multi-scale analysis by 0-splines on $\mathbb{R}$. The corresponding wavelet basis is the Haar system $\Psi = \{\psi^{(0)}_{j,k}(x) = 2^{j/2}\psi^{(0)}(2^j x - k)\}_{j,k\in\mathbb{Z}}$ on the real line. Similarly on $\mathbb{R}^n$ we have the n-dimensional Haar system Ψ^n generated by dilation and translation from $2^n - 1$ analyzing wavelets which are tensor products of some $\psi^{(0)}$ (at least one) and χ (defined as in (1.12)) in the remaining coordinates (see [Me] or [D]). Then the following theorem holds.

THEOREM 4.5. *The n-dimensional Haar system is an unconditional basis of $H^p_{\mathrm{dy}}(w)$ for every $0 < p \leq 1$ and every $w \in A^{\mathrm{dy}}_\infty$. It is also an unconditional basis of $L^p(w)$ for $1 < p < \infty$ and $w \in A^{\mathrm{dy}}_p$.*

Proof. The proof is simpler than that of Theorem 3.10, although it comes from the same idea. In order to keep the notation as simple as possible, we give the proof only for $n = 1$.

Let $T^0_{\Omega,\varepsilon}$ be the operator defined by (1.16) for the Haar system. We start with $p = 1$. Let f be a dyadic 1-atom with respect to w and let us show that $\|T^0_{\Omega,\varepsilon} f\|_{H^1_{\mathrm{dy}}(w)} \leq C$ with C a constant independent of f, Ω, ε, or equivalently $\|M(T^0_{\Omega,\varepsilon} f)\|_{L^1(w)} \leq C$.

Now a very interesting feature of this situation is that if f is supported in a dyadic interval I, the same is true for $T^0_{\Omega,\varepsilon}f$ and $M(T^0_{\Omega,\varepsilon}f)$. We first assume that $w \in (A^{\rm dy}_\infty)^{1/2}$, that is, $w^2 \in A^{\rm dy}_\infty$. As mentioned in the proof of Proposition 2.7, this is equivalent to saying that w satisfies the reverse Hölder inequality

$$\left(\frac{1}{|I|}\int_I w(x)^2\,dx\right)^{1/2} \le C\frac{1}{|I|}\int_I w(x)\,dx$$

with C independent of the dyadic interval I. Consequently, we have

$$\begin{aligned}\|M(T^0_{\Omega,\varepsilon}f)\|_{L^1(w)} &= \int_I |M(T^0_{\Omega,\varepsilon}f)(x)|w(x)\,dx\\ &\le \Big(\int_I |M(T^0_{\Omega,\varepsilon}f)(x)|^2\,dx\Big)^{1/2}\Big(\int_I w(x)^2\,dx\Big)^{1/2}\\ &\le C\Big(\int_I |f(x)|^2\,dx\Big)^{1/2}|I|^{-1/2}w(I)\,.\end{aligned}$$

We used the boundedness of the maximal function in L^2, the orthogonality of the Haar system and the reverse Hölder inequality. Next we use the condition $|f(x)| \le w(I)^{-1}$ to get the desired inequality $\|M(T^0_{\Omega,\varepsilon}f)\|_{L^1(w)} \le C$ with C independent of f,Ω,ε. Thus we have shown that the operators $T^0_{\Omega,\varepsilon}$ are uniformly bounded in $H^1_{\rm dy}(w)$ provided $w \in (A^{\rm dy}_\infty)^{1/2}$. The operators $T^0_{\Omega,\varepsilon}$ are symmetric. Therefore, they are uniformly bounded from wL^∞ to $\mathrm{BMO}_{\rm dy}(w)$ provided $w \in (A^{\rm dy}_\infty)^{1/2}$. In particular, this holds for $w \in (A^{\rm dy}_1)^{1/2}$, and we can use the dyadic version of the extrapolation theorem (see Section 2) to conclude that the $T^0_{\Omega,\varepsilon}$ are uniformly bounded in $L^q(w)$ for $2 < q < \infty$ provided $w \in A^{\rm dy}_{q/2}$.

Now we shall use this fact to prove the general result for $0 < p \le 1$. Take $w \in A^{\rm dy}_\infty$. For q large enough, we have $w \in A^{\rm dy}_{q/2}$ and by what we have already proved, the $T^0_{\Omega,\varepsilon}$ are uniformly bounded in $L^q(w)$. Now we fix p, $0 < p \le 1$, and show that the operators $T^0_{\Omega,\varepsilon}$ are uniformly bounded in $H^p_{\rm dy}(w)$. In order to do that, we take a dyadic p-atom f with respect to w and prove that $\|T^0_{\Omega,\varepsilon}f\|_{H^p_{\rm dy}(w)} \le C$ with C independent of f,Ω,ε, or equivalently $\|M(T^0_{\Omega,\varepsilon}f)\|_{L^p(w)} \le C$.

We proceed as before with two differences. First, instead of using the Cauchy–Schwarz inequality, we write $w(x) = w(x)^{p/q}w(x)^{1-p/q}$ and use Hölder's inequality. Second, instead of using an L^2 estimate we use an $L^q(w)$ estimate. Thus we deduce that the $T^0_{\Omega,\varepsilon}$ are uniformly bounded in $H^p_{\rm dy}(w)$ for every $0 < p \le 1$ and every $w \in A^{\rm dy}_\infty$. In particular, this holds for $H^1_{\rm dy}(w)$ and every $w \in A^{\rm dy}_1$. This implies, by extrapolation, that the $T^0_{\Omega,\varepsilon}$ are uniformly bounded in $L^p(w)$ for $1 < p < \infty$ and $w \in A^{\rm dy}_p$. ■

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Closed subgroups in Banach spaces

by

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Abstract. We show that zero-dimensional nondiscrete closed subgroups do exist in Banach spaces E. This happens exactly when E contains an isomorphic copy of c_0. Other results on subgroups of linear spaces are obtained.

1. Introduction. For a topological vector space E, we are interested in closed additive subgroups G of E. In case E is finite-dimensional, the structure of G is well known; namely, G is a product of a linear subspace of E and a discrete subgroup. The case when E is infinite-dimensional, in general, is far from being so simple.

Obviously, a (topological-group) isomorphism classification of groups G would provide, in particular, a classification of closed linear subspaces of E; hence, in general, it is out of our reach. Therefore, to avoid dealing with linear spaces, we shall mostly consider subgroups G which contain no nontrivial linear space. Such groups we shall call *line-free*. Note that the maximal linear subspace V contained in a group G is closed and the quotient space G/V is a line-free group.

If E is a Banach space, then the topological classification of G reduces to the line-free case as follows. Write $\kappa : E \to E/V$ for the quotient (linear) map. By a result of Bartle and Graves (see [BP2, p. 86]), there exists an (in general, nonlinear) map $\alpha : E/V \to E$ such that $\alpha \circ \kappa = \mathrm{id}_E$. It follows (see [BP2, p. 86]) that $h(x) = (\kappa(x), x - \alpha \circ \kappa(x))$, $x \in E$, establishes a

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