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Mixed-norm spaces and interpolation

by

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Abstract. Let D be a bounded strictly pseudoconvex domain of $\mathbb{C}^n$ with smooth boundary. We consider the weighted mixed-norm spaces $A^{p,q}_{\delta,k}(D)$ of holomorphic functions with norm

$$\|f\|_{p,q,\delta,k}=\Big(\sum_{|\alpha|\le k}\int_0^{r_0}\Big(\int_{\partial D_r}|D^\alpha f|^p\,d\sigma_r\Big)^{q/p}r^{\delta q/p-1}\,dr\Big)^{1/q}.$$

We prove that these spaces can be obtained by real interpolation between Bergman–Sobolev spaces $A^p_{\delta,k}(D)$ and we give results about real and complex interpolation between them. We apply these results to prove that $A^{p,q}_{\delta,k}(D)$ is the intersection of a Besov space $B^{p,q}_s(D)$ with the space of holomorphic functions on D. Further, we obtain several properties of the mixed-norm spaces.

1. Introduction and main results. Let $D=\{z:\varrho(z)<0\}$ be a bounded strictly pseudoconvex domain of $\mathbb{C}^n$ with C^∞ boundary. Thus we can assume that the strictly plurisubharmonic function ϱ is of class C^∞ in a neighbourhood of $\overline{D}$, that $-1\le\varrho(z)<0$ for $z\in D$ and that $|\partial\varrho|\ge c_0>0$ for $|\varrho|\le r_0$.

We denote by D_r the set $\{z:\varrho(z)<-r\}$, by ∂D_r its boundary, by $d\sigma_r$ the normalized surface measure on ∂D_r and by dz the normalized volume element on D.

Now for $0<p<\infty$, $0<q\le\infty$, $\delta>0$ and $k=0,1,\ldots$ we define the weighted mixed-norm spaces

$$A^{p,q}_{\delta,k}(D)=\{f \text{ holomorphic on } D \text{ such that } \|f\|_{p,q,\delta,k}<\infty\}$$

where

$$\|f\|_{p,q,\delta,k}=\Big(\sum_{|\alpha|\le k}\int_0^{r_0}\Big(\int_{\partial D_r}|D^\alpha f|^p\,d\sigma_r\Big)^{q/p}r^{\delta q/p-1}\,dr\Big)^{1/q}$$

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for $0 < q < \infty$ and

$$\|f\|_{p,\infty,\delta,k} = \sup\Big\{\Big(\sum_{|\alpha|\le k} r^\delta \int_{\partial D_r} |D^\alpha f|^p\,d\sigma_r\Big)^{1/p} : 0 < r < r_0\Big\}.$$

(We use the word norm although $\|\ \|_{p,q,\delta,k}$ is only a quasi-norm when $0 < p < 1$ or $0 < q < 1$.)

These spaces have been considered by many authors. For example, spaces of this type can be found in some classical papers of G. M. Hardy and J. E. Littlewood [HA-LIT] or in the book [DU] of P. Duren, and more recently in several papers, like those of S. Gadbois [GA] and Shi Ji-Huai [SH].

Note that if $p = q$ these spaces are a well-known class of weighted Bergman–Sobolev spaces ([BEA], [BEA-BU]). In this case using standard notations we will write $A^p_{\delta,k}$, $\|f\|_{p,\delta,k}$ instead of $A^{p,p}_{\delta,k}$, $\|f\|_{p,p,\delta,k}$.

We recall that the norm

$$|||f|||_{p,\delta,k} = \Big(\sum_{|\alpha|\le k}\int_D |D^\alpha f|^p(-\varrho)^{\delta-1}\,d\zeta\Big)^{1/p}$$

is equivalent to the norm $\|f\|_{p,\delta,k}$ and thus for $1 < p$ the space $A^p_{\delta,k}$ is the space of holomorphic functions on D intersected with the weighted Sobolev space $W^p_k(D,(-\varrho)^{\delta-1}\,d\zeta)$. Also, E. Ligocka [LI2] proves that these spaces are the intersection of some Besov space $B^{p,p}_s$ with the space of holomorphic functions on D.

One of our results shows that for $1 < p < \infty$ and $1 \le q \le \infty$, the space $A^{p,q}_{\delta,k}(D)$ is the intersection of a weighted Besov space $B^{p,q}_{\delta,k}(D)$ and the space of holomorphic functions on D. The method used in this paper to prove this and other properties of the mixed-norm spaces is to show that these spaces can be obtained by real interpolation between weighted Bergman–Sobolev spaces. The following is one of our main results.

THEOREM A. *For $0 < p < \infty$, $0 < q \le \infty$, $0 < \delta_0,\delta_1$, $\delta_0 \ne \delta_1$, $k = 0,1,\ldots$, $0 < \theta < 1$ and $\delta = (1-\theta)\delta_0 + \theta\delta_1$, we have*

$$(A^p_{\delta_0,k}, A^p_{\delta_1,k})_{\theta,q} = A^{p,q}_{\delta,k}.$$

The above theorem for starshaped domains with Lipschitz boundary and $p = q$ has been obtained by E. Straube [STR].

Further, defining

$$A^{p,q}_{\delta,s} = (A^p_{\delta,k}, A^p_{\delta,k+1})_{s-k,q}, \qquad 0 \le k < s < k+1,$$

and using some results of interpolation theory, we obtain the following more complete result:

COROLLARY A. *For $0 < p < \infty$, $0 < q_0 < q_1 \le \infty$, $0 < q \le \infty$, $0 < \delta_0,\delta_1$, $0 \le s_0, s_1$, $s_0 - (n+\delta_0)/p \ne s_1 - (n+\delta_1)/p$, $\delta = (1-\theta)\delta_0 + \theta\delta_1$ and $s = (1-\theta)s_0 + \theta s_1$, we have*

$$(A^{p,q_0}_{\delta_0,s_0}, A^{p,q_1}_{\delta_1,s_1})_{\theta,q} = A^{p,q}_{\delta,s}.$$

Our second result is about complex interpolation.

THEOREM B. *For $0 < p_0,p_1 < \infty$, $0 < q_0 < q_1 \le \infty$, $0 < \delta_0,\delta_1$, $0 \le s_0, s_1$, $0 < \theta < 1$,*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},\quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},\quad \frac{\delta}{p} = (1-\theta)\frac{\delta_0}{p_0} + \theta\frac{\delta_1}{p_1}$$

and $s = (1-\theta)s_0 + \theta s_1$, we have

$$(A^{p_0,q_0}_{\delta_0,k}, A^{p_1,q_1}_{\delta_1,k})_{[\theta]} = A^{p,q}_{\delta,s}.$$

Finally, we generalize some results on Bergman–Sobolev spaces to the mixed-norm spaces. We obtain an atomic decomposition and a theorem on interpolation sequences; we also consider extension problems, trace theorems, the Gleason problem, and a division problem.

The paper is organized as follows. In Section 2 we prove Theorem A and we obtain some properties of the mixed-norm spaces. Moreover, in this section we obtain the above mentioned relation

$$A^{p,q}_{\delta,s}(D) = B^{p,q}_s(D,(-\varrho)^{\delta-1}) \cap H(D)$$

for $1 < p < \infty$, $1 \le q \le \infty$, $1 \le \delta$ and $s \ge 0$.

In Section 3 we prove Theorem B and also we give other results on real interpolation. Finally, in Section 4 we give the above mentioned applications.

We will use the notation $K \approx I$ to mean $c_1K \le I \le c_2K$, and we denote by c all different constants in the inequalities.

2. Real interpolation between Bergman–Sobolev spaces.

The main goal of this section is to prove Theorem A.

We will follow the notations of [BER-LO]. We recall that for every compatible couple of two quasi-normed spaces (A_0,A_1) the interpolation space $(A_0,A_1)_{\theta,q}$, $0 < \theta < 1$, $0 < q \le \infty$, is defined by

$$(A_0,A_1)_{\theta,q} = \{f \in A_0 + A_1 : \|f\|_{\theta,q} < \infty\}$$

where

$$\|f\|_{\theta,q} = \Big(\int_0^\infty (t^{-\theta}K(t,f))^q\frac{dt}{t}\Big)^{1/q} \tag{2.1}$$

and

$$K(t,f) = K(t,f,A_0,A_1) = \inf\{\|f_0\|_{A_0} + t\|f_1\|_{A_1} : f = f_0 + f_1,\ f_0 \in A_0,\ f_1 \in A_1\}.$$

The first step is to compute $K(t,f,A^p_{\delta_0,k},A^p_{\delta_1,k})$. To do this we need some technical results.

For ζ, z in a neighbourhood of $\overline{D}$, we denote by $\Phi(\zeta,z)$ the support function of Henkin and we write $a(\zeta,z)=-\varrho(\zeta)+\Phi(\zeta,z)$. We recall that Φ and a are functions of class $C^\infty(\overline{D}\times\overline{D})$ and holomorphic in z.

Now using a result of Berndtsson–Andersson [B-AN] and integrations by parts (see Lemma 2.3 of [BEA]) we have the following representation formula.

LEMMA 2.1. *Let s be a nonnegative integer. There exist kernels $K_\alpha(\zeta,z)$, $|\alpha|\le k$, each holomorphic in z, such that*

$$f(z)=\sum_{|\alpha|\le k}\int_D D^\alpha f(\zeta)K_\alpha(\zeta,z)(-\varrho(\zeta))^{s+k}\,d\zeta$$

for every function f of class $A^1_{s,k}$ and

$$|D^\beta K_\alpha(\zeta,z)|\le c_\beta|a(\zeta,z)|^{-(n+1+s+|\beta|)}$$

for every multi-index β. ∎

The next lemma gives a well-known estimate for these kernels.

LEMMA 2.2. *For $s>-1$ and $t\in\mathbb{R}$ we have*

$$\int_D(-\varrho(\zeta))^s|a(\zeta,z)|^t\,d\zeta=\begin{cases}1 & \text{if } n+1+s+t>0,\\ \log|\varrho(z)| & \text{if } n+1+s+t=0,\\ |\varrho(z)|^{n+1+s+t} & \text{if } n+1+s+t<0.\end{cases}$$ ∎

The following lemma is a version of Corollary 2.7 of [BEA].

LEMMA 2.3. *For $0<\varepsilon<\varepsilon_0$ small enough, $0<p\le 1$, $s>(1/p-1)(n+1)$ and f holomorphic in D, we have*

$$\text{(i)}\quad\Big(\int_{D\setminus D_\varepsilon}|f|\frac{(-\varrho)^s}{|a|^{n+1+s}}\,d\zeta\Big)^p\le c\int_{D\setminus D_{2\varepsilon}}|f|^p\frac{(-\varrho)^{(n+1+s)p-n-1}}{|a|^{(n+1+s)p}}\,d\zeta,$$

$$\text{(ii)}\quad\Big(\int_{D_\varepsilon}|f|\frac{(-\varrho)^s}{|a|^{n+1+s}}\,d\zeta\Big)^p\le c\int_{D_{\varepsilon/2}}|f|^p\frac{(-\varrho)^{(n+1+s)p-n-1}}{|a|^{(n+1+s)p}}\,d\zeta,$$

uniformly in $z\in D$ and with c independent of ε. ∎

Now using the above lemmas, we give an estimate of $K(t,f)$.

LEMMA 2.4. *If $0<\delta_1<\delta_0$, $\beta=p/(\delta_0-\delta_1)$ and $m(t)=\min(1,t^\beta)$ then* $K(t,f):=K(t,f,A^p_{\delta_0,k},A^p_{\delta_1,k})\approx$

$$\sum_{|\alpha|\le k}\Big(\int_{D\setminus D_{m(t)}}|D^\alpha f|^p(-\varrho)^{\delta_0-1}\,d\zeta+t^p\int_{D_{m(t)}}|D^\alpha f|^p(-\varrho)^{\delta_1-1}\,d\zeta\Big)^{1/p}=:I(t,f).$$

Proof. To prove $K(t,f)\le cI(t,f)$ we need to find a decomposition $f=f_0+f_1$ with $f_0\in A^p_{\delta_0,k}$ and $f_1\in A^p_{\delta_1,k}$ such that

$$\|f_0\|_{p,\delta_0,k}+t\|f_1\|_{p,\delta_1,k}\le cI(t,f)\,. \tag{2.2}$$

We point out that in the following proof of this inequality we do not use the explicit expression of $m(t)$. We only use the fact that $0<m(t)<1$.

Lemma 2.1 gives the representation

$$f(z)=\sum_{|\alpha|\le k}\int_D D^\alpha f(\zeta)K_\alpha(\zeta,z)(-\varrho(\zeta))^{s+k}\,d\zeta\quad\text{for } s>\delta_0/p\,.$$

Define

$$f_0(z)=\sum_{|\alpha|\le k}\int_{D\setminus D_{m(t)}}D^\alpha f(\zeta)K_\alpha(\zeta,z)(-\varrho(\zeta))^{s+k}\,d\zeta\,,$$

$$f_1(z)=\sum_{|\alpha|\le k}\int_{D_{m(t)}}D^\alpha f(\zeta)K_\alpha(\zeta,z)(-\varrho(\zeta))^{s+k}\,d\zeta\,.$$

To prove (2.2) we will consider two cases.

Case 1: $1<p$. Taking $0<\lambda<\delta_0/p$ and using Hölder's inequality, Fubini's theorem and Lemma 2.2 we have

$$\begin{aligned}\|f_0\|^p_{p,\delta_0,k}&\le\sum_{|\alpha|\le k}\int_D\Big(\int_{D\setminus D_{m(t)}}|D^\alpha f(\zeta)|\frac{(-\varrho(\zeta))^{s+k}}{|a(\zeta,z)|^{n+1+s+k}}\,d\zeta\Big)^p(-\varrho(z))^{\delta_0-1}\,dz\\ &\le\sum_{|\alpha|\le k}\int_D\Big(\int_{D\setminus D_{m(t)}}|D^\alpha f(\zeta)|^p\frac{(-\varrho(\zeta))^{(s+k)p}}{|a(\zeta,z)|^{n+1+(s+k-\lambda)p}}\,d\zeta\Big)\\ &\qquad\times\Big(\int_{D\setminus D_{m(t)}}\frac{1}{|a(\zeta,z)|^{n+1+\lambda p'}}\,d\zeta\Big)^{p/p'}(-\varrho(z))^{\delta_0-1}\,dz\\ &\le c\sum_{|\alpha|\le k}\int_D\int_{D\setminus D_{m(t)}}|D^\alpha f(\zeta)|^p\frac{(-\varrho(\zeta))^{(s+k)p}}{|a(\zeta,z)|^{n+1+(s+k-\lambda)p}}\,d\zeta\\ &\qquad\times(-\varrho(z))^{\delta_0-1-\lambda p}\,dz\\ &\le c\sum_{|\alpha|\le k}\int_{D\setminus D_{m(t)}}|D^\alpha f(\zeta)|^p(-\varrho(\zeta))^{\delta_0-1}\,d\zeta\,.\end{aligned}$$

In the same way we obtain

$$\|f_1\|^p_{p,\delta_1,k}\le c\sum_{|\alpha|\le k}\int_{D_{m(t)}}|D^\alpha f|^p(-\varrho)^{\delta_1-1}\,d\zeta$$

and therefore (2.2) is proved.

Case 2: $0 < p \le 1$. As in the above case, using the kernels of Lemma 2.1, we define

$$f_0(z) = \sum_{|\alpha|\le k} \int_{D\setminus D_{2m(t)}} D^\alpha f(\zeta) K_\alpha(\zeta,z)(-\varrho(\zeta))^{s+k}\,d\zeta\,,$$

$$f_1(z) = \sum_{|\alpha|\le k} \int_{D_{2m(t)}} D^\alpha f(\zeta) K_\alpha(\zeta,z)(-\varrho(\zeta))^{s+k}\,d\zeta$$

for $s > \delta_0/p$.

To prove (2.2) we will consider two cases.

In the first case we take $0 < t_0 < 1$ small enough and we assume $0 < t < t_0$. Now, using the estimate of Lemma 2.3 we obtain

$$\begin{aligned}\|f_0\|^p_{p,\delta_0,k} &\le \sum_{|\alpha|\le k} \int_D \Big(\int_{D\setminus D_{2m(t)}} |D^\alpha f(\zeta)| \frac{1}{|a(\zeta,z)|^{n+1+s+k}} (-\varrho(\zeta))^{s+k}\,d\zeta\Big)^p \\ &\qquad\qquad \times (-\varrho(z))^{\delta_0-1}\,dz \\ &\le c \sum_{|\alpha|\le k} \int_D \int_{D\setminus D_{2m(t)}} |D^\alpha f(\zeta)|^p \frac{(-\varrho(\zeta))^{(n+1+s+k)p-n-1}}{|a(\zeta,z)|^{(n+1+s+k)p}}\,d\zeta \\ &\qquad\qquad \times (-\varrho(z))^{\delta_0-1}\,dz \\ &\le c \sum_{|\alpha|\le k} \int_{D\setminus D_{4m(t)}} |D^\alpha f(\zeta)|^p (-\varrho(\zeta))^{\delta_0-1}\,d\zeta \\ &\le c \sum_{|\alpha|\le k} \int_{D\setminus D_{m(t)}} |D^\alpha f(\zeta)|^p (-\varrho(\zeta))^{\delta_0-1}\,d\zeta\,.\end{aligned}$$

In the last inequality we have used the fact that $\int_{\partial D_r} |D^\alpha f|^p\,d\sigma_r$ is a nonincreasing function of r.

In the same way we obtain

$$\|f_1\|^p_{p,\delta_1,k} \le c \sum_{|\alpha|\le k} \int_{D_{m(t)}} |D^\alpha f|^p (-\varrho)^{\delta_1-1}\,d\zeta$$

and therefore (2.2) is proved for $0 < t < t_0$.

If $t_0 \le t$ the estimates of $\|f_0\|_{p,\delta_0,k}$ and $\|f_1\|_{p,\delta_1,k}$ are trivially satisfied. More precisely, in this case we have

$$\|f_0\|_{p,\delta_0,k} + t\|f_1\|_{p,\delta_1,k} \approx I(t,f) \approx \|f\|_{p,\delta_0,k}\,.$$

To complete the proof of the lemma, we will show that $I(t,f) \le cK(t,f)$. To do this, it is sufficient to prove that

$$I(t,f) \le c(\|f_0\|_{p,\delta_0,k} + t\|f_1\|_{p,\delta_1,k})$$

for all decompositions $f = f_0 + f_1$.

First note that $m(t) = \min(1, t^\beta)$, $\beta = p/(\delta_0 - \delta_1)$, implies

$$\text{(2.3)}\qquad \begin{aligned} (-\varrho(z))^{\delta_0-1} &\le t^p(-\varrho(z))^{\delta_1-1} && \text{if } z \in D\setminus D_{m(t)}, \quad \text{and} \\ t^p &\le (-\varrho(z))^{\delta_0-\delta_1} && \text{if } z\in D_{m(t)}\,.\end{aligned}$$

Now from the definition of $I(t,f)$, and (2.3), we have

$$\begin{aligned} I(t,f)^p &\le c\sum_{|\alpha|\le k} \Big(\int_{D\setminus D_{m(t)}} |D^\alpha f_0|^p(-\varrho)^{\delta_0-1}\,d\zeta + t^p \int_{D_{m(t)}} |D^\alpha f_0|^p (-\varrho)^{\delta_1-1}\,d\zeta\Big) \\ &\quad + c\sum_{|\alpha|\le k} \Big(\int_{D\setminus D_{m(t)}} |D^\alpha f_1|^p (-\varrho)^{\delta_0-1}\,d\zeta \\ &\quad + t^p \int_{D_{m(t)}} |D^\alpha f_1|^p(-\varrho)^{\delta_1-1}\,d\zeta\Big) \\ &\le c(\|f_0\|_{p,\delta_0,k} + t\|f_1\|_{p,\delta_1,k})^p\,. \ \blacksquare\end{aligned}$$

The next two lemmas will be used to prove the interpolation theorem.

LEMMA 2.5. *For $0 < \delta < \delta_0$ and f holomorphic in D, we have $\|f\|_{p,\delta_0,k} \le c\|f\|_{p,q,\delta,k}$.*

Proof. It is sufficient to prove the lemma for $k = 0$. We write $h(r) = \int_{\partial D_r} |f|^p\,d\sigma_r$. We consider three cases.

If $p < q$, by Hölder's inequality we have

$$\begin{aligned}\|f\|_{p,\delta_0,0} &\le \Big(\int_0^{r_0} r^{(\delta-\delta_0)q/(q-p)-1}\,dr\Big)^{(q-p)/(qp)} \Big(\int_0^{r_0} h(r)^{q/p}\,d\sigma_r r^{\delta q/p-1}\,dr\Big)^{1/q} \\ &\le c\|f\|_{p,q,\delta,0}\,.\end{aligned}$$

The case $p = q$ is obvious.

If $p > q$, using the fact that $h(r)$ is a nonincreasing function of r we obtain

$$\begin{aligned}\|f\|_{p,\delta_0,0} &\le c\Big(\sum_{j=j_0}^\infty h(2^{-j})2^{-j\delta_0}\Big)^{1/p} \le c\Big(\sum_{j=j_0}^\infty h(2^{-j})^{q/p}2^{-j\delta_0 q/p}\Big)^{1/q} \\ &\le c\Big(\int_0^{r_0} h(r)^{q/p} r^{\delta_0 q/p-1}\,dr\Big)^{1/q} = c\|f\|_{p,q,\delta_0,0} \le c\|f\|_{p,q,\delta,0}\,.\end{aligned}$$

This completes the proof of the lemma. ■

The next lemma is a special version of Hardy's inequalities.

LEMMA 2.6. *Let q be a real number, $r > 0$ and $0 < p < \infty$. Then*

$$\text{(i)}\qquad \int_0^\infty \Big(\int_0^x y^q h(y)\,dy\Big)^p x^{-r-1}\,dx \approx \int_0^\infty (x^{q+1}h(x))^p x^{-r-1}\,dx,$$

(ii) $$\int_0^\infty \Big(\int_x^\infty y^q h(y)\,dy\Big)^p x^{r-1}\,dx \approx \int_0^\infty (x^{q+1}h(x))^p x^{r-1}\,dx,$$

for each positive nonincreasing function $h(x)$.

Proof. The proof of "$\le$" for $1\le p$ is well known (see [ST]). For $0<p<1$ the proof is obtained by discretization of the integrals, as in Lemma 2.5 (see [STR]).

The reverse inequality in (i) follows from

$$x^{q+1}h(x)=\frac{1}{q+1}h(x)\int_0^x y^q\,dy\le\frac{1}{q+1}\int_0^x y^q h(y)\,dy\quad \text{if } q\ge 0,$$

$$x^{q+1}h(x)=x^q h(x)\int_0^x dy\le\int_0^x y^q h(y)\,dy\quad \text{if } q<0.$$

Further, observe that the above reasoning gives

$$x^{q+1}h(x)\le c\int_{x/2}^x y^q h(y)\,dy,$$

and thus, we have

$$\int_0^\infty (x^{q+1}h(x))^p x^{r-1}\,dx\le c\int_0^\infty\Big(\int_{x/2}^x y^q f(y)\,dy\Big)^p x^{r-1}\,dx$$
$$\le c\int_0^\infty\Big(\int_x^\infty y^q f(y)\,dy\Big)^p x^{r-1}\,dx.$$

Hence (ii) is proved. ■

THEOREM 2.7. *For* $0<p<\infty$, $0<q\le\infty$, $0<\delta_0,\delta_1,\delta_0\ne\delta_1$, $k=0,1,\dots$, $0<\theta<1$ *and* $\delta=(1-\theta)\delta_0+\theta\delta_1$, *we have*

$$(A^p_{\delta_0,k},A^p_{\delta_1,k})_{\theta,q}=A^{p,q}_{\delta,k}.$$

Proof. First, we consider the case $0<q<\infty$. We want to show that $\|f\|_{\theta,q}\approx\|f\|_{p,q,\delta,k}$. Using $(A_0,A_1)_{\theta,q}=(A_1,A_0)_{1-\theta,q}$, we can assume $\delta_0>\delta_1$.

To simplify the notations, we write

$$h_\alpha(r)=\begin{cases}\int_{\partial D_r}|D^\alpha f|^p\,d\sigma_r, & 0<r\le r_0,\\ 0, & r_0<r,\end{cases}$$

for each $|\alpha|\le k$. We recall that $h_\alpha(r)$ are positive nonincreasing functions.

Now, using Lemma 2.4, the change of coordinates $s=t^\beta$ and the fact that $K(t,f)\approx\|f\|_{p,\delta_0,k}$ for $t^\beta>r_0$, we have

$$\|f\|^q_{\theta,q}\approx\sum_{|\alpha|\le k}\int_0^{r_0}\Big(\int_0^{t^\beta}h_\alpha(r)r^{\delta_0-1}\,dr\Big)^{q/p}t^{-\theta q-1}\,dt$$
$$+\sum_{|\alpha|\le k}\int_0^{r_0}\Big(\int_{t^\beta}^{r_0}h_\alpha(r)r^{\delta_1-1}\,dr\Big)^{q/p}t^{(1-\theta)q-1}\,dt$$
$$+\sum_{|\alpha|\le k}\int_{r_0}^\infty\|f\|^q_{p,\delta_0,k}t^{-\theta q-1}\,dt$$
$$\approx\sum_{|\alpha|\le k}\int_0^{r_0}\Big(\int_0^s h_\alpha(r)r^{\delta_0-1}\,dr\Big)^{q/p}s^{-\theta(\delta_0-\delta_1)q/p-1}\,ds$$
$$+\sum_{|\alpha|\le k}\int_0^{r_0}\Big(\int_s^{r_0}h_\alpha(r)r^{\delta_1-1}\,dr\Big)^{q/p}s^{(1-\theta)(\delta_0-\delta_1)q/p-1}\,ds+\|f\|^q_{p,\delta_0,k}.$$

Finally, by Lemmas 2.5 and 2.6 we obtain

$$\|f\|^q_{\theta,q}\approx\sum_{|\alpha|\le k}\int_0^{r_0}(h_\alpha(r)r^{\delta_0})^{q/p}r^{-\theta(\delta_0-\delta_1)q/p-1}\,dr$$
$$+\sum_{|\alpha|\le k}\int_0^{r_0}(h_\alpha(r)r^{\delta_1})^{q/p}r^{(1-\theta)(\delta_0-\delta_1)q/p-1}\,dr+\|f\|_{p,\delta_0,k}$$
$$\approx\|f\|^q_{p,q,\delta,k}.$$

Therefore the theorem is proved for $0<q<\infty$.

Now, we consider the case $q=\infty$. First we prove that $\|f\|_{\theta,\infty}\le c\|f\|_{p,\infty,\delta,k}$.

Using the definitions of the corresponding norms and Lemma 2.4, we have

$$\|f\|^p_{\theta,\infty}\le c\sum_{|\alpha|\le k}\sup_{0<t^\beta<r_0}\Big\{t^{-\theta p}\int_0^{t^\beta}h_\alpha(r)r^{\delta_0-1}\,dr\Big\}$$
$$+c\sum_{|\alpha|\le k}\sup_{0<t^\beta<r_0}\Big\{t^{(1-\theta)p}\int_{t^\beta}^{r_0}h_\alpha(r)r^{\delta_1-1}\,dr\Big\}$$
$$+c\sum_{|\alpha|\le k}\sup_{r_0<t^\beta<\infty}\{t^{-\theta p}\|f\|^p_{p,\delta_0,k}\}$$
$$\le c\sum_{|\alpha|\le k}\sup_{0<t^\beta<r_0}\Big\{t^{-\theta p}\int_0^{t^\beta}\|f\|_{p,\infty,\delta,k}r^{\delta_0-\delta-1}\,dr\Big\}$$

$$+c\sum_{|\alpha|\le k}\sup_{0<t^\beta<r_0}\Big\{t^{(1-\theta)p}\int_{t^\beta}^{r_0}\|f\|_{p,\infty,\delta,k}r^{\delta_1-\delta-1}\,dr\Big\}$$
$$+c\sum_{|\alpha|\le k}\sup_{r_0<t^\beta<\infty}\{t^{-\theta p}\|f\|^p_{p,\delta_0,k}\}$$
$$\le c\|f\|^p_{p,\infty,\delta,k}\sup_{0<t^\beta<r_0}\{t^{-\theta p+p(\delta_0-\delta)/(\delta_0-\delta_1)}\}$$
$$+c\|f\|^p_{p,\infty,\delta,k}\sup_{0<t^\beta<r_0}\{t^{(1-\theta)p+p(\delta_1-\delta)/(\delta_0-\delta_1)}\}+c\|f\|^p_{p,\delta_0,k}\,.$$

Now, since $\delta=(1-\theta)\delta_0+\theta\delta_1$, we find that the exponents of t in the last expressions are 0, and therefore by Lemma 2.5 we have $\|f\|_{\theta,\infty}\le c\|f\|_{p,\infty,\delta,k}$.

To finish, we prove the reverse inequality. Since $h_\alpha(r)$ is a nonincreasing function of r, for $0<r<r_0$, the same reasoning used in the proof of Lemma 2.6 gives

$$t^{\delta\beta}h_\alpha(2t^\beta)\le c\int_{t^\beta}^{2t^\beta}h_\alpha(r)r^{\delta-1}\,dr$$
$$\le c\Big(t^{-\theta p}\int_{t^\beta}^{t^{2\beta}}h_\alpha(r)r^{\delta_0-1}\,dr+t^{(1-\theta)p}\int_{t^\beta}^{t^{2\beta}}h_\alpha(r)r^{\delta_1-1}\,dr\Big)$$
$$\le ct^{-\theta p}K(t,f)^p\,.$$

Thus taking the supremum in t and summing over $|\alpha|\le k$, we complete the proof. ∎

The next result summarizes several properties of the mixed-norm spaces. Some of these results can be obtained using the standard methods (see [GA], [SH] for example). In our case they are an easy consequence of the above theorem.

COROLLARY 2.8. *If* $0<p_0\le p_1<\infty$, $0<q_0<q_1\le\infty$, $0<q\le\infty$, $0<\delta_0,\delta_1$, $\delta_0\ne\delta_1$ *and* $k,m=0,1,\dots$, *then*

(1) $A^{p_0,q_0}_{\delta_0,k}\subset A^{p_0,q_1}_{\delta_0,k}$,

(2) $A^{p_0,q_1}_{\delta_0,k}\subset A^{p_1,q_1}_{\delta_0',k}$ *with* $\dfrac{n+\delta_0'}{p_1}=\dfrac{n+\delta_0}{p_0}$,

(3) $A^{p_0,q_1}_{\delta_0,k}=A^{p_0,q_1}_{\delta_0+mp_0,k+m}$,

(4) $A^{p_0,q_0}_{\delta_0,k}\subset\mathrm{VMOA}$ *if* $k-\dfrac{n+\delta_0}{p_0}=0$,

(5) $(A^{p_0,q_0}_{\delta_0,k},A^{p_0,q_1}_{\delta_1,k})_{\theta,q}=A^{p_0,q}_{\delta,k}$ *with* $\delta=(1-\theta)\delta_0+\theta\delta_1$.

Proof. (1) This result follows from the property $(A_0,A_1)_{\theta,q_0}\subset(A_0,A_1)_{\theta,q_1}$ if $q_0\le q_1$ (see Theorem 3.4.1 of [BER-LO]).

(2) Since $A^{p_0}_{\delta,k}$ is continuously contained in $A^{p_1}_{\delta',k}$ with $\delta'=(p_1/p_0)(n+\delta_0)-n$ we have

$$A^{p_0,q_1}_{\delta_0,k}=(A^{p_0}_{\delta_0-\varepsilon,k},A^{p_0}_{\delta_0+\varepsilon,k})_{1/2,q_1}\subset(A^{p_1}_{(\delta_0-\varepsilon)',k},A^{p_1}_{(\delta_0+\varepsilon)',k})_{1/2,q_1}=A^{p_1,q_1}_{\delta_0',k}\,.$$

(3) If $q_1=p_0$ this result is well known. Thus, (3) follows trivially from Theorem 2.7.

(4) As in the above cases this result is well known if $p_0=q_0$. On the other hand, using (1) and (2) we have $A^{p_0,q_0}_{\delta_0,k}\subset A^p_{\delta_0',k}$ with $p=\max(p_0,q_0)$ and δ_0' satisfying $k-(n+\delta_0')/p=0$. Hence (4) is proved.

(5) This result is the well-known reiteration theorem applied to these spaces (see [BER-LO], [HO]). ∎

The next result completes property (4) of Corollary 2.8.

We recall that $\mathrm{BMOA}(D)=\{f\in H^2(D):f^*\in\mathrm{BMO}(\partial D)\}$ where f^* denotes the boundary function and H^2 denotes the classical Hardy space, and that the averages of $\mathrm{BMO}(\partial D)$ functions are taken over Korányi pseudoballs (see [VA] for precise definitions).

PROPOSITION 2.9. *If* $k-(n+\delta)/p=0$ *then* $A^{p,\infty}_{\delta,k}$ *is a subset of* BMOA.

Proof. By property (2) of Corollary 2.8, it will be enough to prove this result for $p>1$.

Using a result of [VA], to show the inclusion $A^{p,q}_{\delta,k}\subset\mathrm{BMOA}$ it is sufficient to see that there exists p_0 such that each function f of $A^{p,\infty}_{\delta,k}$ is in H^{p_0} and that $(|f|+|\partial f|)\,dz$ is a Carleson measure.

First we prove the inclusion of $A^{p,\infty}_{\delta,k}$ in H^{p_0}, for some p_0.

We write $A^{p,\infty}_{\delta,k}=(A^p_{\delta+\varepsilon,k},A^p_{\delta-\varepsilon,k})_{1/2,\infty}$ for an $\varepsilon>0$ small enough.

Now if we take $p_0>p$ and $\varepsilon<np/p_0$, then by Theorem 1.5 of [BEA], we have

$$A^p_{\delta-\varepsilon,k}\subset A^p_{\delta+\varepsilon,k}\subset H^{p_0}_{n/p_0-\varepsilon/p}\subset H^{p_0}$$

and therefore the inclusion is proved.

To prove that $(|f|+|\partial f|)\,dz$ is a Carleson measure we want to show that for every $0<t<t_0$ small enough,

$$I:=\int_{K(w,t)\cap D}(|f|+|\partial f|)\,dz\le c\|f\|_{p,\infty,\delta,k}t^n \tag{2.4}$$

where $K(w,t)=\{z:d(w,z)<t\}$ denotes the Korányi ball with respect to the pseudodistance

$$d(w,z)=\Big|\sum_{i=1}^n\frac{\partial\varrho}{\partial\zeta_i}(w)(w_i-z_i)\Big|+\Big|\sum_{i=1}^n\frac{\partial\varrho}{\partial\zeta_i}(z)(z_i-w_i)\Big|+|z-w|^2\,.$$

We recall that the volume of this ball is of order of t^{n+1}.

We write K_t instead of $K(w,t)\cap D$. By Lemma 2.1 we have

$$I\le c\int_{K_t}\int_0^t\int_{\partial D_r}\sum_{|\alpha|\le k}|D^\alpha f|\frac{1}{|a|^{n+2+s}}\,d\sigma_r r^{s+k}\,dr\,dz$$
$$+c\int_{K_t}\int_t^{t_0}\int_{\partial D_r}\sum_{|\alpha|\le k}|D^\alpha f|\frac{1}{|a|^{n+2+s}}\,d\sigma_r r^{s+k}\,dr\,dz=:I_1+I_2\,.$$

Now, applying Hölder's inequality, taking $0<\lambda<1/p'$, and using well-known estimates, we get

$$I_1\le\Big(\int_{K_t}\int_0^t\int_{\partial D_r}\sum_{|\alpha|\le k}|D^\alpha f|^p\frac{1}{|a|^{n+1+(s+1-\lambda)p}}\,d\sigma_r\, r^{(s+k)p}\,dr\,dz\Big)^{1/p}$$
$$\times\Big(\int_{K_t}\int_0^t\int_{\partial D_r}\frac{1}{|a|^{n+1+\lambda p'}}\,d\sigma_r\,dr\,dz\Big)^{1/p'}$$
$$\le c\|f\|_{p,\infty,\delta,k}\Big(\int_0^t r^{(s+k)p-\delta-(s+1-\lambda)p}dr\Big)^{1/p}\Big(\int_{K_t}(-\varrho)^{-\lambda p'}\,dz\Big)^{1/p'}$$
$$\le c\|f\|_{p,\infty,\delta,k}t^{(sp+kp-\delta-(s+1-\lambda)p+1)/p+(n+1-\lambda p')/p'}\le c\|f\|_{p,\infty,\delta,k}t^n\,.$$

To estimate I_2 we recall that the Lebesgue measure of K_t is of the order of t^{n+1} and therefore taking $0<\lambda<1$, we have

$$I_2\le c\int_{K_t}\Big(\int_t^{t_0}\int_{\partial D_r}\sum_{|\alpha|\le k}|D^\alpha f|^p\frac{1}{|a|^{n+1+(s+1-\lambda)p}}\,d\sigma_r r^{(s+k)p}\,dr\Big)^{1/p}$$
$$\times\Big(\int_t^{t_0}\int_{\partial D_r}\frac{1}{|a|^{n+1+\lambda p'}}\,d\sigma_r\,dr\Big)^{1/p'}dz$$
$$\le c\|f\|_{p,\infty,\delta,k}t^{n+1+(sp+kp-\delta-n-(s+1-\lambda)p)/p-\lambda}\le c\|f\|_{p,\infty,\delta,k}t^n$$

and thus (2.4) is proved. ∎

Remark. In the above proposition we cannot replace BMOA by VMOA. For example, if we consider the unit disk of $\mathbb{C}$, then $\log(1-z)$ belongs to $A^{p,\infty}_{2p-1,2}\setminus\mathcal{B}_0$ where $\mathcal{B}_0$ denotes the little Bloch space. Since VMOA $\subset\mathcal{B}_0$ we find that $\log(1-z)\in A^{p,\infty}_{2p-1,2}\setminus$ VMOA. ∎

Up to now, we have only considered the mixed-norm spaces whose order of derivation is a nonnegative integer k. Next we generalize this definition to the noninteger case.

Definition 2.10. For $0<p<\infty$, $0<q\le\infty$, $\delta>0$ and $k<s<k+1$ we define

$$A^{p,q}_{\delta,s}=(A^p_{\delta,k},A^p_{\delta,k+1})_{\theta,q}\quad\text{ with }\theta=s-k\,.$$

The next lemma generalizes property (3) of Corollary 2.8 and it permits us to reduce the study of the mixed-norm spaces with noninteger order of derivation to the case with integer derivatives.

Lemma 2.11. *For $0<p<\infty$, $0<q\le\infty$, $\delta>0$ and $k\le s<k+1$ we have*

$$A^{p,q}_{\delta,s}=A^{p,q}_{\delta+(k+1-s)p,k+1}\,.$$

Proof. Using property (3) of Corollary 2.8 we have

$$A^{p,q}_{\delta,s}=(A^p_{\delta+p,k+1},A^p_{\delta,k+1})_{s-k,q}=A^{p,q}_{\delta+(k+1-s)p,k+1}\,.\ ∎$$

Now we compare the result of Theorem 2.7 with the well-known interpolation results for nonholomorphic weighted Sobolev spaces.

For $0<p<\infty$, we denote by $L^p_{\delta,k}(D)$ the weighted Sobolev space $L^p_k(D,(-\varrho)^{\delta-1}\,d\zeta)$. Further, we recall the definition of the weighted Besov space $B^{p,q}_{\delta,s}$ given in Section 3.3.3 of [TR1].

Definition 2.12. For $1<p<\infty$, $1\le q\le\infty$, $\delta>1$ and $0\le k<s\le k+1$, the weighted Besov space $B^{p,q}_{\delta,s}$ is defined by

$$B^{p,q}_{\delta,s}(D)=\{f\in L^p_{\delta,0}(D):|||f|||_{p,q,\delta,s}<\infty\}$$

where

$$|||f|||_{p,q,\delta,s}=\|f\|_{p,\delta,0}+\sum_{|\alpha|\le k}\Big(\int_{|h|<\varepsilon}\frac{\|\Delta^2_hD^\alpha f\|^q_{L^p_{\delta,0}(D^h)}}{|h|^{2n+(s-k)q}}\,dh\Big)^{1/q},$$

$$|||f|||_{p,\infty,\delta,s}=\|f\|_{p,\delta,0}+\sup_{|h|<\varepsilon}\Big\{\sum_{|\alpha|\le k}\frac{\|\Delta^2_hD^\alpha f\|_{L^p_{\delta,0}(D^h)}}{|h|^{2n+s-k}}\Big\},$$

$$\Delta^2_hD^\alpha f(\zeta)=D^\alpha f(\zeta+2h)+D^\alpha f(\zeta)-2D^\alpha f(\zeta+h)\,,$$

and $D^h\subset D$ is a certain domain where the function $\Delta^2_hD^\alpha f(\zeta)$ is defined.

Theorem 2.13. *For $1<p<\infty$, $1\le q\le\infty$, $1\le\delta$ and $k_0,k_1=0,1,\ldots$ we have*

$$(L^p_{\delta,k_0},L^p_{\delta,k_1})_{\theta,q}=B^{p,q}_{\delta,s},\qquad s=(1-\theta)k_0+\theta k_1\,.$$

If $p=\delta=1$ then the above result also holds.

Proof. The result for $1<p$ follows from Theorem 3.3.3 of [TR1]. If $p=\delta=1$ then the result follows from the results of Section 2.5.7 of [TR2] and Theorem 3.1.5 of [ST]. ∎

Now, using the above theorem we obtain a new characterization of the $A^{p,q}_{\delta,k}$ spaces.

THEOREM 2.14. *For $1 < p < \infty$, $1 \le q \le \infty$, $1 \le \delta$ and $0 < s$, we have*

$$A^{p,q}_{\delta,s}(D) = B^{p,q}_{\delta,s}(D) \cap H(D)$$

with equivalent norms. If $p = \delta = 1$ the result also holds. Furthermore, $A^{p,q}_{\delta,s}$ is a retract of $B^{p,q}_{\delta,s}$.

Proof. Since the reproducing kernel for holomorphic functions of type

$$K(\zeta, z) = \frac{(-\varrho(\zeta))^t \varphi(\zeta, z)}{a(\zeta, z)^{n+1+t}}, \quad t > \frac{\delta}{p},$$

maps continuously $L^p_{\delta,k}$ onto $A^p_{\delta,k}$, it follows that $A^p_{\delta,k}$ is a retract of $L^p_{\delta,k}$. Thus, Theorem 2.14 follows from Theorems 2.7 and 2.13. ■

3. Complex interpolation between mixed-norm spaces. The aim of this section is to prove the following theorem:

THEOREM 3.1. *For $0 < p_0, p_1 < \infty$, $0 < q_0 < q_1 \le \infty$, $0 < \delta_0, \delta_1$ and $0 \le s_0, s_1$, we have*

$$(A^{p_0,q_0}_{\delta_0,s_0}, A^{p_1,q_1}_{\delta_1,s_1})_{[\theta]} = A^{p,q}_{\delta,s}$$

with

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

$$s = (1-\theta)s_0 + \theta s_1 \quad \textit{and} \quad \frac{\delta}{p} = (1-\theta)\frac{\delta_0}{p_0} + \theta\frac{\delta_1}{p_1}.$$

The result of this theorem for $1 \le p_0, p_1$ and $p_0 = q_0$, $p_1 = q_1$ is due to E. Ligocka [LI1], and for $0 < p_0 = p_1 = q_0 = q_1$ to F. Beatrous [BEA]. Also E. Straube [STR] proves that if D is a starshaped domain with Lipschitz boundary and $0 < p_0 = p_1 = q_0 = q_1$ then the interpolation result of the theorem is true.

We briefly recall the complex method of interpolation (see [BER-LO]). Although this method is not defined for a general couple of quasi-Banach spaces, in our case, as in [BEA] or [STR], the construction is also meaningful for $0 < p < 1$ or $0 < q < 1$ (see [BE-CE]).

Let (A_0, A_1) be a compatible pair of complete quasi-Banach spaces. We will denote by $\| \ \|_0$ and $\| \ \|_1$ the corresponding quasi-norms of A_0 and A_1. We also consider the space $A_0 + A_1$ with quasi-norm given by

$$\|f\| = \inf\{\|f_0\|_0 + \|f_1\|_1 : f_0 \in A_0,\ f_1 \in A_1,\ f = f_0 + f_1\}.$$

Let S be the strip $\{\lambda \in \mathbb{C} : 0 < \operatorname{Re}\lambda < 1\}$. Now, we consider the space $\mathcal{F} = \mathcal{F}(A_0, A_1)$ of all $A_0 + A_1$-valued functions F on S which are bounded and continuous on $\overline{S}$, holomorphic on S and such that $F_j(t) = F(j + it)$ defines a bounded continuous function from $\mathbb{R}$ to A_j, $j = 0, 1$, with the complete quasi-norm

$$\|F\|_{\mathcal{F}} = \max\{\sup_t \|F(it)\|_0, \sup_t \|F(1+it)\|_1\}.$$

For $0 < \theta < 1$ the interpolating space of exponent θ is defined by

$$(A_0, A_1)_{[\theta]} = \{F(\theta) : F \in \mathcal{F}\}$$

with the complete quasi-norm

$$\|f\|_{[\theta]} = \inf\{\|F\|_{\mathcal{F}} : F \in \mathcal{F},\ F(\theta) = f\}.$$

To prove Theorem 3.1 we need some definitions and technical results.

DEFINITION 3.2. For every $0 < \varepsilon$, $0 < p < \infty$, $0 < q \le \infty$ and every real ν, we define $l^{p,q,\varepsilon}_\nu$ to be the space of sequences $c = \{c_{m,j}\}$ such that

$$\|c\|_{p,q,\nu,\varepsilon} = \Big(\sum_{m=0}^{\infty}\Big(\sum_{j=0}^{\infty}|c_{m,j}|^p\Big)^{q/p}\varepsilon^{m\nu q/p}\Big)^{1/q} < \infty, \quad 0 < q < \infty,$$

$$\|c\|_{p,\infty,\nu,\varepsilon} = \Big(\sup_m\Big\{\varepsilon^{m\nu}\sum_{j=0}^{\infty}|c_{m,j}|^p\Big\}\Big)^{1/p} < \infty.$$

Now we prove the following interpolation lemma.

LEMMA 3.3. *Let $0 < p_0, p_1 < \infty$, $0 < q_0 < q_1 \le \infty$, $\nu_0, \nu_1 \in \mathbb{R}$ and $0 < \theta < 1$.*

(i) *If $\nu_0 \ne \nu_1$ and $\nu = (1-\theta)\nu_0 + \theta\nu_1$, then*

$$(l^{p_0,q_0,\varepsilon}_{\nu_0}, l^{p_0,q_1,\varepsilon}_{\nu_1})_{\theta,q} = l^{p_0,q,\varepsilon}_\nu.$$

(ii) *If*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

and

$$\frac{\nu}{p} = (1-\theta)\frac{\nu_0}{p_0} + \theta\frac{\nu_1}{p_1},$$

then

$$(l^{p_0,q_0,\varepsilon}_{\nu_0}, l^{p_1,q_1,\varepsilon}_{\nu_1})_{[\theta]} = l^{p,q,\varepsilon}_\nu.$$

Proof. First, observe that for $0 < \varepsilon_0, \varepsilon_1$ the spaces $l^{p,q,\varepsilon_0}_\nu$ and $l^{p,q,\varepsilon_1}_\nu$ are isomorphic. Further, note that if $\varepsilon = 2$ these spaces are considered in Section 1.18 of [TR1] and in Section 5.6 of [BER-LO]. Thus (i) follows from Theorem 1.18.2 of [TR1].

Now we prove (ii). Without lost of generality we can assume $\varepsilon = 2$. First we want to see that $\|c\|_{[\theta]} \le \|c\|_{p,q,\nu,2}$.

If $c = \{c_{m,j}\} \in l_\nu^{p,q,2}$, then we write $c_m = (\sum_{j=1}^\infty |c_{m,j}|^p)^{1/p}$ and we define

$$F(\lambda, c) = \left\{ c_{m,j} |c_{m,j}|^{\frac{p}{p(\lambda)}-1} c_m^{\frac{q}{q(\lambda)}-\frac{p}{p(\lambda)}} 2^{m(\frac{q}{pq(\lambda)}\nu - \frac{\nu(\lambda)}{p(\lambda)})} \|c\|_{p,q,\nu,2}^{1-\frac{q}{q(\lambda)}} \right\}$$

where $p(\lambda)$, $q(\lambda)$ and $\nu(\lambda)$ are defined by

$$(3.1)\quad \frac{1}{p(\lambda)} = \frac{1-\lambda}{p_0} + \frac{\lambda}{p_1}, \quad \frac{1}{q(\lambda)} = \frac{1-\lambda}{q_0} + \frac{\lambda}{q_1}, \quad \nu(\lambda) = (1-\lambda)\nu_0 + \lambda\nu_1 .$$

It is clear that F satisfies $F(\theta, c) = c$. Further, using an easy computation we obtain $\|F(it)\|_{p_0,q_0,\nu_0,2} \le \|c\|_{p,q,\nu,2}$ and $\|F(1+it)\|_{p_1,q_1,\nu_1,2} \le \|c\|_{p,q,\nu,2}$. Therefore we have $\|c\|_{[\theta]} \le \|c\|_{p,q,\nu,2}$.

The reverse inequality follows using the usual steps as in Theorem 1.18.1 of [TR1] for $p_i, q_i \ge 1$, $i = 0, 1$, or as in Theorem 3 of [BE-CE] for the general case. ∎

Definition 3.4. Let $D = \{\zeta : \varrho(\zeta) < 0\}$ be a bounded strictly convex domain of $\mathbb{C}^n$, with smooth boundary and such that $0 \in D$. For $s, t \ge 0$ we define the kernels

$$K^{s,t}(\zeta, z) = c_s \frac{(-\varrho(\zeta))^s a(\zeta, 0)^{t-s}}{a(\zeta, z)^t} \left(\overline{\partial} \frac{\partial \varrho(\zeta)}{a(\zeta, z)} \right)^n = \frac{(-\varrho(\zeta))^s a(\zeta, 0)^{t-s} \varphi_s(\zeta)}{a(\zeta, z)^{n+1+t}}$$

where

$$a(\zeta, z) = -\varrho(\zeta) + \sum_{i=1}^n \frac{\partial \varrho(\zeta)}{\partial \zeta_i}(\zeta_i - z_i), \qquad c_s = \frac{\Gamma(n+s+1)}{n!\,\Gamma(s+1)} \frac{1}{(2\pi i)^n} .$$

We denote by $K^{s,t}$ the corresponding integral operators.

Note that for $s = t$ the kernels $K^{s,s}$ are reproducing kernels for smooth holomorphic functions on D (see [B-AN]).

We also define the differential operators

$$R_t = I + \frac{1}{n+1+t} R, \quad \text{where} \quad R = \sum_{i=1}^n z_i \frac{\partial}{\partial z_i} .$$

Lemma 3.5. *If $0 < p < \infty$, $\delta > 0$, $k = 0, 1, \ldots$, $s > \delta/p$, and m is a positive integer, then the operator $K^{s+m,s}$ gives an isomorphism from $A^p_{\delta,k}$ to $A^p_{\delta,k+m}$. The inverse operator is given by $R_{s+m-1} \ldots R_s$.* ∎

Proof. First, note that the operator R_t maps $A^p_{\delta,k}$ into $A^p_{\delta,k-1}$ and that if $R_t f = 0$ for a holomorphic function f on D, then $f = 0$.

On the other hand, $K^{s+m,s}$ maps $A^p_{\delta,k}$ into $A^p_{\delta,k+m}$ (see [BEA]). Also, we have the identity $R_t K^{s,t} = K^{s,t+1}$ that follows by direct computation:

$$R_t K^{s,t}(\zeta, z) = \left(I + \frac{1}{n+1+t} R \right) \frac{(-\varrho(\zeta))^s a(\zeta, 0)^{t-s} \varphi_s(\zeta)}{a(\zeta, z)^{n+1+t}}$$

$$= (-\varrho(\zeta))^s a(\zeta, 0)^{t-s} \varphi_s(\zeta) \left(\frac{1}{a(\zeta, z)^{n+1+t}} + \frac{\sum_{i=1}^n \frac{\partial \varrho}{\partial \zeta_i} z_i}{a(\zeta, z)^{n+2+t}} \right)$$

$$= \frac{(-\varrho(\zeta))^s a(\zeta, 0)^{t+1-s} \varphi_s(\zeta)}{a(\zeta, z)^{n+2+t}} = K^{s,t+1}(\zeta, z) .$$

Thus, since $K^{s,s}$ is the identity operator on $A^p_{\delta,k}$, we have $R_{s+m-1} \ldots R_s K^{s+m,s} f = f$ and the lemma is proved. ∎

Lemma 3.6. *Let D be a strictly pseudoconvex domain of $\mathbb{C}^n$, with smooth boundary,*

$$d(\zeta, z) = \left| \sum_{i=1}^n \frac{\partial \varrho(\zeta)}{\partial \zeta_i}(\zeta_i - z_i) \right| + \left| \sum_{i=1}^n \frac{\partial \varrho(z)}{\partial z_i}(z_i - \zeta_i) \right| + |\zeta - z|^2$$

the Korányi pseudodistance, $K(\zeta, \varepsilon)$ the Korányi ball centered at ζ of radius ε, and $d(z) = d(z, \partial D)$. Then, for each $\eta > 0$ and $0 < \varepsilon < 1$, there exist $0 < \eta_0 < \eta_1 < \eta$ and a sequence $\{a_{m,j}\}$ of points of D which satisfies

(1) $\bigcup_{m,j} K(a_{m,j}, \eta_1 d(a_{m,j})) = D$,
(2) $K(a_{m,j}, \eta_0 d(a_{m,j})) \cap K(a_{m',j'}, \eta_0 d(a_{m,j})) \ne \emptyset$ *iff* $m = m'$, $j = j'$,
(3) $-\varrho(a_{m,j}) \approx \varepsilon^m$ *for each* $m \ge 0$, *and each* $j = 0, \ldots, j_m$.

Proof. Consider a Whitney covering of D by Korányi balls (see Chapter 3 of [CO-WE]) such that for some $0 < \eta_0 < \eta_1 < \eta$,

$$\bigcup_{m,j} K(b_s, \eta_1 d(b_s)) = D ,$$

$$K(b_s, \eta_0 d(b_s)) \cap K(b_{s'}, \eta_0 d(b_{s'})) \ne \emptyset \quad \text{iff} \quad s = s' .$$

We conclude the proof by renumbering the points $\{b_s\}$ such that $\varepsilon^{m+1} < d(b_s) \le \varepsilon^m$ as $a_{m,0}, \ldots, a_{m,j_m}$. ∎

As usual, a set $\{a_{m,j}\}$ which satisfies properties (1) and (2) will be called an *η_0-η_1-lattice*.

Proof of Theorem 3.1. By property (3) of Corollary 2.8 and Lemma 2.11 we can assume that $s_0 = s_1 = k$ and δ_0, δ_1 are large enough.

Now using Fornæss' embedding theorem [FOR] and the extension theorems of [BEA], we find there is a bounded strictly convex domain M of $\mathbb{C}^{n+n'}$ with smooth boundary and continuous linear operators

$$E_0 : A^p_{\delta,k}(D) \to A^p_{\delta-n',k}(M), \qquad E_1 : A^p_{\delta-n',k}(M) \to A^p_{\delta,k}(D)$$

such that $E_1 E_0$ is the identity on $A^p_{\delta,k}(D)$. Thus, by Lemma 3.4 the operators

$$S_0 := R_{s+k-1} \ldots R_s E_0 : A^p_{\delta,k}(D) \to A^p_{\delta-n',0}(M) ,$$

$$T_0 := E_1 K^{s+k,s} : A^p_{\delta-n',0}(M) \to A^p_{\delta,k}(D)$$

are continuous and satisfy $T_0 S_0 = \mathrm{Id}$.

On the other hand, using Lemma 3.6 and Theorem 1 of [COU] we see that for $0<\varepsilon<\varepsilon_0$ small enough, there exists a sequence of points $\{\zeta_{m,j}\}$, $m=0,1,\ldots,\ j=0,1,\ldots,j_m$, of M and continuous linear operators

$$S_1 : A^p_{\delta-n',0}(M)\to l^{p,p,\varepsilon}_{n+\delta-n'},\qquad T_1 : l^{p,p,\varepsilon}_{n+\delta-n'}\to A^p_{\delta-n',0}(M)$$

such that:

(i) $S_1(f)=\{f(\zeta_{m,j})\}$,

(ii) $T_1(\{c_{m,j}\})=\sum_{m=1}^{\infty}\sum_{j=0}^{j_m}c_{m,j}K^{n+1+s,s}(\zeta_{m,j},z)$,

(iii) T_1S_1 is bijective,

where the kernel $K^{n+1+s,s}$ is

$$K^{n+1+s,s}(\zeta,z)=\frac{(-\varrho_M(\zeta))^{n+1+s}\varphi_s(\zeta)}{\left(-\varrho_M(\zeta)+\sum_{j=1}^{n+n'}\frac{\partial\varrho_M(\zeta)}{\partial\zeta_j}(\zeta_j-z_j)\right)^{n+1+s}}$$

and where ϱ_M denotes the defining function for M.

Therefore, using the operators $S=S_1S_0$ and $T=T_0(T_1S_1)^{-1}T_1$, we see that $A^p_{\delta,k}(D)$ is a retract of $l^{p,p,\varepsilon}_{n+\delta-n'}$. Thus, by Lemma 3.3(i) and Theorem 2.7 we find that $A^{p,q}_{\delta,k}(D)$ is also a retract of $l^{p,q,\varepsilon}_{n+\delta-n'}$. Finally, using Lemma 3.3(ii) together with Theorem 4.2.1 of [BER-LO] in the cases $p_i,q_i\ge 1$, or Theorem 2 of [BE-CE] in the general case, we obtain

$$(A^{p_0,q_0}_{\delta_0,k},A^{p_1,q_1}_{\delta_1,k})_{[\theta]}=T((l^{p_0,q_0,\varepsilon}_{n+\delta_0-n'},l^{p_1,q_1,\varepsilon}_{n+\delta_1-n'})_{[\theta]})=T(l^{p,q,\varepsilon}_{n+\delta-n'})=A^{p,q}_{\delta,k}.$$

Hence the theorem is proved. ∎

Observe that the same method gives the following real interpolation result.

THEOREM 3.7. *Let $0<p_0,p_1<\infty$, $0<q_0<q_1\le\infty$ and $0<\delta_0,\delta_1$. Then*

$$(A^{p_0,q_0}_{\delta_0,k},A^{p_1,q_1}_{\delta_1,k})_{\theta,q}=T((l^{p_0,q_0,\varepsilon}_{n+\delta_0-n'},l^{p_1,q_1,\varepsilon}_{n+\delta_1-n'})_{\theta,q}).\ ∎$$

4. Applications. In the same way as in Corollary 2.8, Theorem 2.7 gives an easy method to obtain some properties of the mixed-norm spaces from the properties of the Bergman–Sobolev spaces. Roughly speaking, in many cases the results on Bergman–Sobolev spaces cited in the literature can be extended to the mixed-norm spaces upon replacing $A^p_{\delta,k}$ by $A^{p,q}_{\delta,k}$. In this section we state some of these properties.

A. *Atomic decomposition.* The main result of this section is a generalization of those obtained by R. Coifman and R. Rochberg [CO-RO], E. Amar [AM], D. Luecking [LU], S. Gadbois [GA] and B. Coupet [COU].

THEOREM 4.1. *Let D be a bounded strictly convex domain of $\mathbb{C}^n$ with smooth boundary and let $\{a_{m,j}\}$ be an η_0-η_1-lattice in D, like the one of Lemma 3.6. Then for all $0<p<\infty$, $0<q\le\infty$, $\delta>0$, $k=0,1,\ldots$ and $s>\delta/p$ the operator defined by*

$$T_s(c)=\sum_{m=0}^{\infty}\sum_{j=0}^{j_m}c_{m,j}K^{n+1+s,s}(a_{m,j},z),$$

where $K^{n+1+s,s}(\zeta,z)$ is the kernel of Definition 3.4, is continuous and onto from $l^{p,q,\varepsilon}_{n+\delta-kp}$ to $A^{p,q}_{\delta,k}(B)$. If D is the unit ball, then

$$T_s(c)=\sum_{m=0}^{\infty}\sum_{j=0}^{j_m}c_{m,j}\frac{(1-|a_{m,j}|^2)^{n+1+s}}{(1-\overline{a}_{m,j}z)^{n+1+s}}.$$

Proof. As shown in the proof of Theorem 3.1, using Lemma 3.6 and Theorem 1 of [COU], the operator T_{s+k} is continuous and onto from $l^{p,p,\varepsilon}_{n+\delta}$ to $A^p_{\delta,0}$ and there exists a continuous linear operator S_{s+k} from $A^p_{\delta,0}$ to $l^{p,p,\varepsilon}_{n+\delta}$ such that $T_{s+k}S_{s+k}$ is the identity on $A^p_{\delta,0}$. Thus, by Lemma 3.5 the operator $K^{s+k,s}T_{s+k}$ is also continuous and onto from $l^{p,p,\varepsilon}_{n+\delta}$ to $A^p_{\delta,k}$. But using the explicit formula for the inverse of $K^{s+k,s}$ obtained in Lemma 3.5, we find that

$$K^{s+k,s}T_{s+k}(c)=\sum_{m=0}^{\infty}\sum_{j=0}^{j_m}c_{m,j}K^{n+1+s+k,s}(a_{m,j},z).$$

Note that $T_s=K^{s+k,s}T_{s+k}U_k$, where U_k is the isomorphism from $l^{p,p,\varepsilon}_{n+\delta-kp}$ to $l^{p,p,\varepsilon}_{n+\delta}$ defined by $U_k(c)=\{c_{m,j}(-\varrho(a_{m,j}))^{-k}\}$. Thus, defining $S_k=U_k^{-1}(K^{s+k,s})^{-1}S_{s+k}$ we find that T_sS_s is the identity operator on $A^p_{\delta,k}$. Finally, using Theorem 2.7 and Lemma 3.3(i) we conclude that T_sS_s is the identity on $A^{p,q}_{\delta,k}$ and hence the first part of the theorem is proved.

If D is the unit ball of $\mathbb{C}^n$ it is well known that the reproducing kernel $K^{s,s}$ of Definition 3.4 is

$$K^{s,s}(\zeta,z)=c_s\frac{(1-|\zeta|^2)^s}{(1-\overline{\zeta}z)^{n+1+s}}$$

and thus, by the same Definition 3.4, we have

$$K^{n+1+s,s}(\zeta,z)=c_s\frac{(1-|\zeta|^2)^{n+1+s}}{(1-\overline{\zeta}z)^{n+1+s}}.$$

Hence the theorem is proved. ∎

B. *Interpolation sequences.* The next result follows from the interpolation result of Theorem 2.7 and the results obtained by E. Amar [AM] and R. Rochberg [RO] for the Bergman–Sobolev spaces in the unit ball of $\mathbb{C}^n$.

THEOREM 4.2. *If* $\{a_{m,j}\}$ *is an* η_0-η_1*-lattice on the unit ball* B *of* $\mathbb{C}^n$ *satisfying properties* (1)–(3) *of Lemma* 3.6 *and with* η_0 *large enough, then there exist continuous linear operators*

$$S_0 : A^{p,q}_{\delta,0} \to l^{p,q,\varepsilon}_{n+\delta} \quad \text{and} \quad S_1 : l^{p,q,\varepsilon}_{n+\delta} \to A^{p,q}_{\delta,0}$$

such that $S_0(f) = \{f(a_{m,j})\}$ *and* $S_0S_1 = \mathrm{Id}$. ∎

Proof. This follows from the result of [RO] and Theorem 2.7. ∎

C. *Extension theorems from holomorphic submanifolds*. First, note that if D is a bounded strictly pseudoconvex domain in a Stein submanifold of $\mathbb{C}^n$, with smooth boundary, then we can define the spaces $A^{p,q}_{\delta,k}(D)$ in the same way. Moreover, the same method gives Theorem 2.7 also in this case. Thus using the results of F. Beatrous (Theorems 1.5 and 1.6 of [BEA]), we obtain:

THEOREM 4.3. *Let* D *be a bounded strictly pseudoconvex domain of* $\mathbb{C}^n$ *with smooth boundary and* Y *a Stein submanifold defined in a neighbourhood of* $\overline{D}$, *of codimension* l *and transversal to the boundary of* D. *Let* $M = D\cap Y$. *Let* $0 < p < \infty$, $0 < q \le \infty$, $0 < \delta$ *and* $k = 0, 1, \ldots$ *Then*

$$A^{p,q}_{\delta,k}(D)|_M = A^{p,q}_{\delta+l,k}(M)\,. \ ∎$$

D. *Trace theorems*. The result of this section is an extension of the trace theorems on curves in the boundary of B, obtained by J. Bruna and J. M. Ortega [BR-OR1], [BR-OR2].

THEOREM 4.4. *Let* B *be the unit ball of* $\mathbb{C}^n$ *and let* Γ *be a simple closed smooth curve in the boundary of* B. *For* $1 \le p < \infty$, $1 \le q \le \infty$, $0 < \delta$ *and* $k > (n+\delta)/p$ *if* $p > 1$, *or* $k \ge n+\delta$ *if* $p = 1$, *we have*

$$A^{p,q}_{\delta,k}(B)|_\Gamma \subset B^{p,q}_s(\Gamma), \qquad s = k - \frac{n+\delta}{p} + \frac{1}{p}\,.$$

Moreover, if Γ *is complex-tangential, then*

$$A^{p,q}_{\delta,k}(B)|_\Gamma = B^{p,q}_s(\Gamma), \qquad s = 2\left(k - \frac{n+\delta}{p}\right) + \frac{1}{p}\,.$$

Proof. This result follows from the theorems of Section 5.1 of [BR-OR1] and Theorems 2.2 and 3.1 of [BR-OR2], the interpolation theorems between Besov spaces (Section 3.3.6 of [TR2]) and Theorem 2.7. ∎

E. *The Gleason problem*. The next result is a generalization of that obtained by J. M. Ortega (Theorem 1 of [OR]).

THEOREM 4.5. *Let* D *be a bounded strictly pseudoconvex domain of* $\mathbb{C}^n$ *with* C^∞ *boundary, and* $1 \le p < \infty$, $0 < q \le \infty$, $0 < \delta$ *and* $k = 0, 1, \ldots$ *Then, for every* ζ *in* D, *there exist continuous linear operators* T_j, $j = 1, \ldots, n$, *from* $A^{p,q}_{\delta,k}$ *to* $A^{p,q}_{\delta,k}$ *such that*

$$f(z) - f(\zeta) = \sum_{j=1}^{n} T_j f(z)(z_j - \zeta_j)\,. \ ∎$$

F. *A division problem*. This result is a generalization of that obtained by ourselves (Theorem 1.1 of [OR-FA]).

THEOREM 4.6. *Let* D *be a bounded strictly pseudoconvex domain with* C^∞ *boundary. Let* $Y = \{z : u_1 = \ldots = u_l = 0\}$ *be a holomorphic submanifold defined in a neighbourhood of* $\overline{D}$ *such that*

$$(\partial u_1 \wedge \ldots \wedge \partial u_l)(z) \ne 0 \quad \text{for } z \in Y\,,$$
$$(\partial u_1 \wedge \ldots \wedge \partial u_l \wedge \partial\varrho)(z) \ne 0 \quad \text{for } z \in Y \cap \partial D\,.$$

Then there exist continuous linear operators T_j, $j = 1, \ldots, l$, *from* $A^{p,q}_{\delta,k}$ *to* $A^{p,q}_{\delta+p/2,k}$, $1 \le p < \infty$, $1 \le q \le \infty$, $0 < \delta$, $k = 0, 1, \ldots$, *such that if* f *vanishes on* $M = D \cap Y$, *then*

$$f(z) = \sum_{j=1}^{l} u_l(z) T_j f(z)\,. \ ∎$$

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Calderón–Zygmund operators and unconditional bases of weighted Hardy spaces

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Abstract. We study sufficient conditions on the weight w, in terms of membership in the A_p classes, for the spline wavelet systems to be unconditional bases of the weighted space $H^p(w)$. The main tool to obtain these results is a very simple theory of regular Calderón–Zygmund operators.

0. Introduction. The purpose of this article is twofold. First of all, we present (in Section 2) a very simple theory of regular Calderón–Zygmund operators, based upon the notion of weighted atom and a general extrapolation principle. The whole theory develops almost immediately from the basic estimate in Theorem 2.3 below. This estimate contains almost all the information about the boundedness properties of the operator.

Secondly, as an illustration and an extension of the theory, we find (in Section 3) sufficient conditions on the weight w, in terms of membership in the A_p classes, for the systems of m-splines to be unconditional bases of $H^p(w)$. Only the unweighted case has been treated so far in the literature. For this problem, the operators to be studied are different from the Calderón–Zygmund operators of Section 2, but the basic estimates they satisfy turn out to be the same. This unity makes the theory transparent. The first estimates for the basic m-splines appear in the work of Z. Ciesielski. We improve the estimates which were obtained in [St] to deal with the unweighted case. Moreover, we show that the high-dimensional case, which is treated in [St] in a way far from satisfactory, is not essentially different from the one-dimensional case.

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