Contents of Volume 109, Number 3

D. C. YANG, Some new Hardy spaces $L^2H^q_{\mathrm{R}}(R^2_+\times R^2_+)$ $(0<q\le 1)$ 217–231
J. M. ORTEGA and J. FÀBREGA, Mixed-norm spaces and interpolation 233–254
J. GARCÍA-CUERVA and K. S. KAZARIAN, Calderón–Zygmund operators and unconditional bases of weighted Hardy spaces 255–276
F. D. ANCEL, T. DOBROWOLSKI and J. GRABOWSKI, Closed subgroups in Banach spaces 277–290
C.-P. CHEN and C.-C. LIN, Almost everywhere convergence of Laguerre series 291–301
J.-P. KAHANE et P.-G. LEMARIÉ-RIEUSSET, Remarques sur la formule sommatoire de Poisson 303–316

STUDIA MATHEMATICA

Executive Editors: **Z. Ciesielski, A. Pełczyński, W. Żelazko**

The journal publishes original papers in English, French, German and Russian, mainly in functional analysis, abstract methods of mathematical analysis and probability theory. Usually 3 issues constitute a volume.

Detailed information for authors is given on the inside back cover. Manuscripts and correspondence concerning editorial work should be addressed to

STUDIA MATHEMATICA

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-293997

Correspondence concerning subscription, exchange and back numbers should be addressed to

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
Publications Department

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-293997

© Copyright by Instytut Matematyczny PAN, Warszawa 1994

Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset in TEX at the Institute
Printed and bound by

PRINTED IN POLAND

ISSN 0039-3223

Some new Hardy spaces $L^2H^q_{\mathrm{R}}(\mathbb{R}^2_+\times\mathbb{R}^2_+)$ $(0<q\le 1)$

by

DACHUN YANG (Beijing)

Abstract. For $0<q\le 1$, the author introduces a new Hardy space $L^2H^q_{\mathrm{R}}(\mathbb{R}^2_+\times\mathbb{R}^2_+)$ on the product domain, and gives its generalized Lusin-area characterization. From this characterization, a φ-transform characterization in M. Frazier and B. Jawerth's sense is deduced.

0. Introduction. S. A. Chang and R. Fefferman [1] introduced a Hardy space $H^1_{\mathrm{R}}(\mathbb{R}^2_+\times\mathbb{R}^2_+)$ generated by rectangle atoms. By the inspiration from the papers [3, 4, 7, 8] concerning non-product domains, we consider its "localization" at the origin. More generally, for $0<q\le 1$, we introduce a new Hardy space $L^2H^q_{\mathrm{R}}(\mathbb{R}^2_+\times\mathbb{R}^2_+)$. In §1, we establish its generalized Lusin-area characterization. Applying this, in §2, we give its φ-transform characterizations in M. Frazier and B. Jawerth's sense [5, 6]. It is worth pointing out that our method in §2 differs from the ones in [5, 6, 8]. We find that the generalized Lusin-area characterization of $L^2H^q_{\mathrm{R}}(\mathbb{R}^2_+\times\mathbb{R}^2_+)$ plays a crucial role in establishing its φ-transform characterizations. Further applications of the spaces $L^2H^q_{\mathrm{R}}(\mathbb{R}^2_+\times\mathbb{R}^2_+)$ are under study.

1. The generalized Lusin-area characterization of $L^2H^q_{\mathrm{R}}(\mathbb{R}^2_+\times\mathbb{R}^2_+)$. We first introduce the concept of a center rectangle atom.

DEFINITION 1. Let $0<q\le 1$. A function $a(x_1,x_2)$ on $\mathbb{R}\times\mathbb{R}$ is said to be a *center $(q,2)$-rectangle atom* if

(1) $\operatorname{supp} a\subset R$, where $R=I\times J$ is a rectangle with center at the origin;
(2) $\|a\|_2\le |R|^{1/2-1/q}$;
(3) $\int a(x_1,x_2)x_1^\alpha\,dx_1=0=\int a(x_1,x_2)x_2^\alpha\,dx_2$, for all $\alpha\in\mathbb{N}$ with $0\le\alpha\le 1/q-1$ and all $x_1,x_2\in\mathbb{R}$.

We define the Hardy space $L^2H^q_{\mathrm{R}}(\mathbb{R}^2_+\times\mathbb{R}^2_+)$ to be directly generated by center rectangle atoms:

1991 *Mathematics Subject Classification*: Primary 42B25, 42C15.
Research supported by the National Science Foundation of China.

DEFINITION 2. Let $0 < q \le 1$. A distribution $f(x_1, x_2)$ on $\mathbb{R} \times \mathbb{R}$ belongs to the Hardy space $L^2 H^q_{\mathbb{R}}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ if and only if

$$f = \sum \lambda_j a_j$$

in the distribution sense, where $\sum |\lambda_j|^q < \infty$ and each $a_j(x)$ is a center $(q,2)$-rectangle atom. Moreover, we set

$$\|f\|_{L^2 H^q_{\mathbb{R}}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)} := \inf \Big\{ \Big(\sum |\lambda_j|^q \Big)^{1/q} \Big\},$$

where the infimum is taken over all the decompositions of f as above.

The dual of the Hardy space $H^1(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ is $L^1(\mathbb{R} \times \mathbb{R})$ (see [1, 2]). Our results will indicate that the dual of $L^2 H^q_{\mathbb{R}}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ is the following Herz-type space.

DEFINITION 3. For $k, l \in \mathbb{Z}$, let $C_{k,l} = \{x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R} : 2^{k-1} < |x_1| \le 2^k, 2^{l-1} < |x_2| \le 2^l\}$. Suppose $0 < q \le 1$. A function $f \in L^2_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R} \setminus \{(0,0)\})$ belongs to the space $K^q_2(\mathbb{R} \times \mathbb{R})$ if and only if

$$\|f\|_{K^q_2(\mathbb{R}\times\mathbb{R})} := \Big\{ \sum_{k,l\in\mathbb{Z}} 2^{(k+l)(1-q/2)} \|f\chi_{C_{k,l}}\|_2^q \Big\}^{1/q} < \infty .$$

Obviously, $K^q_2(\mathbb{R} \times \mathbb{R}) \subsetneq L^q(\mathbb{R} \times \mathbb{R})$ for $0 < q \le 1$.

In order to establish the generalized Lusin-area characterization of the Hardy space $L^2 H^q_{\mathbb{R}}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$, we still need some notations. Suppose that $\overline{\psi}(t)$ is a sufficiently smooth function on $\mathbb{R}^1$ with compact support (without loss of generality, we can assume that $\operatorname{supp} \overline{\psi} \subset [-1,1]$), $\overline{\psi}(-t) = \overline{\psi}(t)$, $\int_{-1}^1 \overline{\psi}(t) t^\alpha \, dt = 0$ for all $\alpha \in \mathbb{N}$ with $0 \le \alpha \le 1/q - 1$, and

$$\int_0^\infty |\widehat{\overline{\psi}}(t\xi)|^2 t^{-1} \, dt = 1 \quad \text{for each } \xi \ne 0 .$$

If $y > 0$, we write $\overline{\psi}_y(t) = y^{-1} \overline{\psi}(y^{-1} t)$ and if $y = (y_1, y_2)$, $t = (t_1, t_2) \in \mathbb{R}^2$, we define $\overline{\psi}_y(t) = \overline{\psi}_{y_1}(t_1) \overline{\psi}_{y_2}(t_2)$. For $f \in \mathcal{S}'(\mathbb{R} \times \mathbb{R})$, the *generalized Lusin-area integral* of f is defined as

$$s(f)(x) = \Big\{ \iint_{\Gamma(x)} |(f * \overline{\psi}_y)(t)|^2 y_1^{-2} y_2^{-2} \, dt \, dy \Big\}^{1/2},$$

where $\Gamma(x) = \Gamma(x_1) \times \Gamma(x_2)$ and $\Gamma(x_i) = \{(t_i, y_i) \in \mathbb{R}^2_+ : |x_i - t_i| < y_i\}$, $i = 1, 2$.

The space $L^2 H^q_{\mathbb{R}}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ has the following characterization in terms of the generalized Lusin-area function.

THEOREM 1. *Let $0 < q \le 1$, $f \in \mathcal{S}'(\mathbb{R} \times \mathbb{R})$. Then $f \in L^2 H^q_{\mathbb{R}}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ if and only if $s(f) \in K^q_2(\mathbb{R}\times\mathbb{R})$, and for each $\varphi \in \mathcal{S}(\mathbb{R} \times \mathbb{R})$, $f * \varphi_{t_1,t_2} \to 0$ in the distribution sense as $t_1, t_2 \to \infty$, where $\varphi_{t_1,t_2} = t_1^{-1} t_2^{-1} \varphi(t_1^{-1} x_1, t_2^{-1} x_2)$. Furthermore,*

$$\|f\|_{L^2 H^q_{\mathbb{R}}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)} \sim \|s(f)\|_{K^q_2(\mathbb{R}\times\mathbb{R})} .$$

Proof. For the sufficiency, we only give the center atom decomposition of f. Write $Q_k = \{x \in \mathbb{R} : |x| \le 2^k\}$ for $k \in \mathbb{Z}$. Let $Q_{j,k} = \{x \in \mathbb{R} : 2^j x - k \in [0,1)\}$, where $j, k \in \mathbb{Z}$; $\mathfrak{D}_0 = \{Q_{j,k} : j, k \in \mathbb{Z}\}$ and $\mathfrak{D} = \{I \times J : I, J \in \mathfrak{D}_0\}$. Moreover, let $\mathfrak{D}^l_k = \{Q \in \mathfrak{D} : Q = I \times J,\ I \subset Q_k,\ I \not\subset Q_{k-1};\ J \subset Q_l,\ J \not\subset Q_{l-1}\}$ for $k, l \in \mathbb{Z}$.

For $Q \in \mathfrak{D}^l_k$, write

$$Q_+ = \{(t,y) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+ : t \in Q = I \times J,\ |I| < y_1 \le 2|I|, |J| < y_2 \le 2|J|\} .$$

Then $\bigcup_{k=-\infty}^{\infty} \bigcup_{l=-\infty}^{\infty} \bigcup_{Q \in \mathfrak{D}^l_k} Q_+$ is a disjoint decomposition of $\mathbb{R}^2_+ \times \mathbb{R}^2_+$. By the Calderón representation formula [2], we have

$$\begin{aligned} f(x) &= \iint_{\mathbb{R}^2_+ \times \mathbb{R}^2_+} f(t,y) \overline{\psi}_y(x-t) y_1^{-1} y_2^{-1} \, dt \, dy \\ &= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \sum_{Q \in \mathfrak{D}^l_k} \iint_{Q_+} f(t,y) \overline{\psi}_y(x-t) y_1^{-1} y_2^{-1} \, dt \, dy, \end{aligned}$$

where $f(t,y) = (f * \overline{\psi}_y)(t)$.

Write

$$a^l_k(x) := \lambda_{k,l}^{-1} \sum_{Q \in \mathfrak{D}^l_k} \iint_{Q_+} f(t,y) \overline{\psi}_y(x-t) y_1^{-1} y_2^{-1} \, dt \, dy ,$$

where $\lambda_{k,l}$ is a constant to be determined in the following. We want to verify that $a^l_k(x)$ is a center $(q,2)$-rectangle atom. If $x \in \operatorname{supp} a^l_k$ then we can assume that for some $Q \in \mathfrak{D}^l_k$ and some $(t,y) \in Q_+$ we have $|x_i - t_i| \le y_i$. Thus, $|x_i| \le |t_i| + y_i$. In particular, $|x_1| \le 2^k + 2 \cdot 2^{k+1} \le 2^{k+3}$. Similarly, $|x_2| \le 2^{l+3}$. Thus, $\operatorname{supp} a^l_k \subset Q_{k+2} \times Q_{l+2}$, where $k, l \in \mathbb{Z}$. By the hypothesis about $\overline{\psi}$, we easily deduce that

$$\int a^l_k(x_1, x_2) x_1^\alpha \, dx_1 = 0 = \int a^l_k(x_1, x_2) x_2^\alpha \, dx_2$$

for all $x_1, x_2 \in \mathbb{R}$ and all $\alpha \in \mathbb{N}$ with $0 \le \alpha \le 1/q - 1$. Setting

$$\lambda_{k,l} := (|Q_{k+2}| \cdot |Q_{l+2}|)^{1/q-1/2} \Big(\sum_{Q \in \mathfrak{D}^l_k} \iint_{Q_+} |f(t,y)|^2 y_1^{-1} y_2^{-1} \, dt \, dy \Big)^{1/2},$$

we now estimate $\|a^l_k\|_2$. In fact, we have

$$\|a^l_k\|_2 = \sup_{\|g\|_2 \le 1} \Big| \iint_{\mathbb{R}\times\mathbb{R}} a^l_k(x) g(x) \, dx \Big| ,$$

and

$$\Big|\iint_{\mathbb{R}^2} a_k^l(x)g(x)\,dx\Big| = \lambda_{k,l}^{-1}\Big|\sum_{Q\in\mathfrak{D}_k^l}\iint_{Q_+} f(t,y)g(t,y)y_1^{-1}y_2^{-1}\,dt\,dy\Big|$$
$$\le \lambda_{k,l}^{-1}\Big\{\sum_{Q\in\mathfrak{D}_k^l}\Big(\iint_{Q_+}|f(t,y)|^2y_1^{-1}y_2^{-1}\,dt\,dy\Big)^{1/2}$$
$$\times\Big(\iint_{Q_+}|g(t,y)|^2y_1^{-1}y_2^{-1}\,dt\,dy\Big)^{1/2}\Big\}$$
$$\le (|Q_{k+2}|\cdot|Q_{l+2}|)^{1/2-1/q}\|g\|_2\,.$$

Thus,

$$\|a_k^l\|_2 \le (|Q_{k+2}|\cdot|Q_{l+2}|)^{1/2-1/q}\,.$$

Therefore, $a_k^l(x)$ is a center $(q,2)$-rectangle atom. It remains to estimate $\sum_{k,l\in\mathbb{Z}}|\lambda_{k,l}|^q$. We first have

$$\iint_{C_{k,l}}\{s(f)\}^2\,dx = \iint_{C_{k,l}}\Big\{\iint_{\Gamma(x)}|f(t,y)|^2y_1^{-2}y_2^{-2}\,dt\,dy\Big\}\,dx$$
$$= \iint_{\mathbb{R}_+^2\times\mathbb{R}_+^2}|f(t,y)|^2|\{x\in C_{k,l}:(t,y)\in\Gamma(x)\}|y_1^{-2}y_2^{-2}\,dt\,dy$$
$$\ge C\sum_{Q\in\mathfrak{D}_k^l}\iint_{Q_+}|f(t,y)|^2y_1^{-1}y_2^{-1}\,dt\,dy\,.$$

So,

$$\lambda_{k,l}\le C(|Q_{k+2}|\cdot|Q_{l+2}|)^{1/q-1/2}\|s(f)\chi_{C_{k,l}}\|_2$$
$$= C2^{(k+l)(1/q-1/2)}\|s(f)\chi_{C_{k,l}}\|_2\,.$$

Therefore,

$$\Big(\sum_{k,l\in\mathbb{Z}}|\lambda_{k,l}|^q\Big)^{1/q}\le C\{2^{(k+l)(1-q/2)}\|s(f)\chi_{C_{k,l}}\|_2^q\}^{1/q} = C\|s(f)\|_{K_2^q(\mathbb{R}\times\mathbb{R})}\,.$$

This proves the sufficiency of the theorem. Now, we turn to the proof of the necessity. We only need to show that

$$\|s(a)\|_{K_2^q(\mathbb{R}\times\mathbb{R})}\le C$$

for any center $(q,2)$-rectangle atom a, where C is independent of a.

Suppose that $\operatorname{supp} a\subset I\times J$, $Q_{k_0-1}\subset I\subset Q_{k_0}$ and $Q_{l_0-1}\subset J\subset Q_{l_0}$.

Then

$$\|s(a)\|_{K_2^q(\mathbb{R}\times\mathbb{R})}^q = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}2^{(k+l)(1-q/2)}\|s(a)\chi_{C_{k,l}}\|_2^q$$
$$= \sum_{k=-\infty}^{k_0+10}\sum_{l=-\infty}^{l_0+10}\ldots + \sum_{k=-\infty}^{k_0+10}\sum_{l=l_0+11}^{\infty}\ldots$$
$$+\sum_{k=k_0+11}^{\infty}\sum_{l=-\infty}^{l_0+10}\ldots + \sum_{k=k_0+11}^{\infty}\sum_{l=l_0+11}^{\infty}\ldots$$
$$=: I_1+I_2+I_3+I_4\,.$$

Now,

$$I_1\le C\|a\|_2^q\sum_{k=-\infty}^{k_0+10}\sum_{l=-\infty}^{l_0+10}2^{(k+l)(1-q/2)}$$
$$\le C(|I|\cdot|J|)^{(1/2-1/q)q}2^{(k_0+l_0)(1-q/2)}\le C\,,$$

where C is independent of a. The estimations of I_2 and I_3 are similar; we only compute I_2. First, we have

$$\|s(a)\chi_{C_{k,l}}\|_2^2 = \iint_{C_{k,l}}dx\int_{\Gamma(x)}|(a*\overline{\psi}_y)(t)|^2y_1^{-2}y_2^{-2}\,dt\,dy$$
$$\le C\int_{C_l}dx_2\int_{\Gamma(x_2)}y_2^{-2}\,dt_2\,dy_2\int|(a(\xi_1,\cdot)*\overline{\psi}_{y_2})(t_2)|^2\,d\xi_1\,,$$

where $C_l=\{x_2\in\mathbb{R}:2^{l-1}<|x_2|\le 2^l\}$. Taking $\alpha\in\mathbb{N}$ with $1/q-2<\alpha\le 1/q-1$, from the Taylor formula we deduce that

$$\int|(a(\xi_1,\cdot)*\overline{\psi}_{y_2})(t_2)|^2\,d\xi_1$$
$$= \int\Big|\int a(\xi_1,\xi_2)\Big(\overline{\psi}_{y_2}(t_2-\xi_2)-\overline{\psi}_{y_2}(t_2)-[\overline{\psi}_{y_2}]'(t_2)(-\xi_2)-\ldots$$
$$-\frac{1}{\alpha!}[\overline{\psi}_{y_2}]^{(\alpha)}(t_2)(-\xi_2)^\alpha\Big)\,d\xi_2\Big|^2\,d\xi_1$$
$$\le C|J|^{2(\alpha+1)}y_2^{-2(\alpha+2)}\int\Big(\int|a(\xi_1,\xi_2)|\,d\xi_2\Big)^2\,d\xi_1$$
$$\le C|I|^{1-2/q}|J|^{2(2-1/q+\alpha)}y_2^{-2(\alpha+2)}\,.$$

Noting that $x_2\notin Q_{l_0+10}$ and taking into account the supports of $\overline{\psi}$ and a, from $y_2\ge|x_2-t_2|$ and $y_2\ge|t_2-\xi_2|$ we obtain $y_2\ge|x_2-\xi_2|/2\ge|x_2|/4$.

Therefore,

$$\|s(a)\chi_{C_{k,l}}\|_2^2 \le C \int_{C_l} dx_2 \int_{|x_2|/4}^{\infty} |I|^{1-2/q}|J|^{2(2-1/q+\alpha)} y_2^{-(2\alpha+5)}\, dy_2$$
$$\le C|I|^{1-2/q}|J|^{2(2-1/q+\alpha)}2^{-(2\alpha+3)l}.$$

Thus,

$$I_2 \le C|I|^{q/2-1}|J|^{q(2-1/q+\alpha)} \sum_{k=-\infty}^{k_0+10} \sum_{l=l_0+11}^{\infty} 2^{(k+l)(1-q/2)}2^{-q(\alpha+3/2)l} \le C,$$

where C is independent of a.

We now estimate I_4. Similarly to I_2, we can assume that $y_1 \ge |x_1|/4$, $y_2 \ge |x_2|/4$. Taking $\alpha \in \mathbb{N}$ with $1/q-2 < \alpha \le 1/q-1$ and using the Taylor formula, we have

$$|(a * \overline{\psi}_y)(t)| = \Big| \int\int a(\xi_1,\xi_2)\Big(\overline{\psi}_{y_1}(t_1-\xi_1) - \overline{\psi}_{y_1}(t_1) - [\overline{\psi}_{y_1}]'(t_1)(-\xi_1) - \dots$$
$$- \frac{1}{\alpha!}[\overline{\psi}_{y_1}]^{(\alpha)}(t_1)(-\xi_1)^\alpha\Big)\Big(\overline{\psi}_{y_2}(t_2-\xi_2) - \overline{\psi}_{y_2}(t_2)$$
$$- [\overline{\psi}_{y_2}]'(t_2)(-\xi_2) - \dots - \frac{1}{\alpha!}[\overline{\psi}_{y_2}]^{(\alpha)}(t_2)(-\xi_2)^\alpha\Big)\, d\xi_1\, d\xi_2\Big|$$
$$\le C(|I|\cdot|J|)^{(\alpha+1)}(y_1y_2)^{-(\alpha+2)} \int\int |a(\xi_1,\xi_2)|\, d\xi_1\, d\xi_2$$
$$\le C(|I|\cdot|J|)^{(\alpha+2-1/q)}(y_1y_2)^{-(\alpha+2)}.$$

Thus,

$$\|s(a)\chi_{C_{k,l}}\|_2^2 \le C \int\int_{C_{k,l}} dx \int_{|x_1|/4}^{\infty}\int_{|x_2|/4}^{\infty} (|I|\cdot|J|)^{2(\alpha+2-1/q)}(y_1y_2)^{-2\alpha-5}\, dy_1\, dy_2$$
$$\le C(|I|\cdot|J|)^{2(\alpha+2-1/q)}2^{-(k+l)(2\alpha+3)}.$$

From this, it is easy to verify that $I_4 \le C$, where C is independent of a. This finishes the proof of Theorem 1.

2. The φ-transform characterizations of $L^2H^q_{\mathbb{R}}(\mathbb{R}^2_+\times\mathbb{R}^2_+)$. Now, we give the φ-transform characterizations of $L^2H^q_{\mathbb{R}}(\mathbb{R}^2_+\times\mathbb{R}^2_+)$ in M. Frazier and B. Jawerth's sense [5, 6] by using its generalized Lusin-area characterization. For this, we first introduce some notations. Let $\varphi,\psi \in \mathcal{S}(\mathbb{R})$, supp $\widehat{\varphi}$, supp $\widehat{\psi} \subseteq \{\xi \in \mathbb{R} : 1/2 \le |\xi| \le 2\}$, and $|\widehat{\varphi}(\xi)|, |\widehat{\psi}(\xi)| \ge C > 0$ whenever $3/5 \le |\xi| \le 5/3$. In addition, $\sum_{\nu\in\mathbb{Z}} \overline{\widehat{\varphi}(2^\nu\xi)}\widehat{\psi}(2^\nu\xi) = 1$ for $\xi \ne 0$. Write $\varphi_\nu(x) = 2^\nu\varphi(2^\nu x)$, $\psi_\nu(x) = 2^\nu\psi(2^\nu x)$ for $\nu\in\mathbb{Z}$. Let $Q_{j,k}, \mathfrak{D}_0, \mathfrak{D}$ be as in the proof of

Theorem 1. If $I = Q_{j,k}$, we define

$$\varphi_I(x) = |I|^{-1/2}\varphi(2^\nu x - k) = |I|^{1/2}\varphi_\nu(x - x_I),$$

where $|I| = 2^{-\nu}$ and $x_I = 2^{-\nu}k$. Moreover, for $R = I\times J \in \mathfrak{D}$, let $\varphi_R = \varphi_I \otimes \varphi_J$. Similarly, we define ψ_R. Then, for $f \in \mathcal{S}'(\mathbb{R}\times\mathbb{R})$, we easily obtain

$$f(x_1,x_2) = \sum_{R\in\mathfrak{D}} (f,\varphi_R)\psi_R(x_1,x_2)$$

in the distribution sense (see [5, 6]). The space $L^2H^q_{\mathbb{R}}(\mathbb{R}^2_+\times\mathbb{R}^2_+)$ has the following characterization.

THEOREM 2. *Suppose that $0 < q \le 1$ and φ,ψ are as above. Consider the distribution*

$$f(x) = \sum_{R\in\mathfrak{D}} S(R)\psi_R(x)$$

on $\mathbb{R}\times\mathbb{R}$, where $S(R) = (f,\varphi_R)$. Then the following four statements are equivalent.

(1) $G(f) := (\sum_R |S(R)|^2|\psi_R(x)|^2)^{1/2} \in K^q_2(\mathbb{R}\times\mathbb{R})$;

(2) *There exists a constant $C_0 > 0$ such that for any dyadic rectangle $R\in\mathfrak{D}$, there is a dyadic rectangle $Q(R)\subset R$ such that $|Q(R)| \ge C_0|R|$ and*

$$A(f) := \Big(\sum_R |S(R)|^2|R|^{-1}\chi_{Q(R)}(x)\Big)^{1/2} \in K^q_2(\mathbb{R}\times\mathbb{R});$$

(3) $W(f) := (\sum_R |S(R)|^2|R|^{-1}\chi_R(x))^{1/2} \in K^q_2(\mathbb{R}\times\mathbb{R})$;

(4) $f \in L^2H^q_{\mathbb{R}}(\mathbb{R}^2_+\times\mathbb{R}^2_+)$.

Moreover, the related norms are mutually equivalent.

In order to simplify the proof of Theorem 2, we need to introduce some "tent" space $TK^q_2(\mathbb{R}\times\mathbb{R})$. For this, we define a measurable function

$$S(\alpha)(x) = \Big(\sum_{R\ni x} |\alpha(R)|^2|R|^{-1}\Big)^{1/2}$$

for any sequence of complex numbers $\alpha = \{\alpha(R)\}_{R\in\mathfrak{D}}$, and write

$$\operatorname{supp}\alpha = \bigcup_{\{R:\alpha(R)\ne 0\}} R.$$

DEFINITION 4. Let $0 < q \le 1$. We say that $\alpha = \{\alpha(R)\}_{R\in\mathfrak{D}} \in TK^q_2(\mathbb{R}\times\mathbb{R})$ if $S(\alpha) \in K^q_2(\mathbb{R}\times\mathbb{R})$. Moreover, let $\|\alpha\|_{TK^q_2(\mathbb{R}\times\mathbb{R})} := \|S(\alpha)\|_{K^q_2(\mathbb{R}\times\mathbb{R})}$.

DEFINITION 5. Let $0 < q \le 1$. If there exists a rectangle Q with center at the origin such that $Q \supset \operatorname{supp}\alpha$ and

$$\sum_{R\in\mathfrak{D}} |\alpha(R)|^2 \le |Q|^{1-2/q},$$

then we call $\alpha = \{\alpha(R)\}_{R\in\mathfrak{D}}$ a *center* $(q,2)$-*atom sequence*, and the smallest rectangle Q as above the *base* of α.

The "tent" space $TK_2^q(\mathbb{R}\times\mathbb{R})$ has the following characterization.

THEOREM 3. *Let* $0 < q \le 1$. *The following three statements are equivalent.*

(1) $\alpha \in TK_2^q(\mathbb{R}\times\mathbb{R})$;

(2) *There exists a constant* $C_0 > 0$ *such that for any dyadic rectangle* $R\in\mathfrak{D}$, *there is a dyadic rectangle* $Q(R)\subset R$ *such that* $|Q(R)|\ge C_0|R|$ *and*

$$\sigma(x) := \Big(\sum_R |\alpha(R)|^2|R|^{-1}\chi_{Q(R)}(x)\Big)^{1/2} \in K_2^q(\mathbb{R}\times\mathbb{R});$$

(3) *There exists a constant* $C_1 > 0$, *a sequence of center* $(q,2)$-*atom sequences* $\{\alpha_{j,k}\}_{j,k\in\mathbb{Z}}$ *and a sequence of numbers* $\{\lambda_{j,k}\}_{j,k\in\mathbb{Z}}$ *such that* $\operatorname{supp}\alpha_{j,k}\subset C_1(Q_j\times Q_k)$ *and* $\alpha = \sum_{j,k\in\mathbb{Z}}\lambda_{j,k}\alpha_{j,k}$ *with* $\sum_{j,k\in\mathbb{Z}}|\lambda_{j,k}|^q<\infty$.

In addition, the related norms $\|\alpha\|_{TK_2^q(\mathbb{R}\times\mathbb{R})}$ *and* $\|\sigma\|_{K_2^q(\mathbb{R}\times\mathbb{R})}$ *and* $\inf\{(\sum_{j,k\in\mathbb{Z}}|\lambda_{j,k}|^q)^{1/q}\}$ *are mutually equivalent, where the infimum is taken over all the decompositions of* α *as in* (3).

As the proof of Theorem 3 is, in essence, similar to that for non-product domains [7], we omit the details.

Now, we show Theorem 2 by using Theorems 1 and 3. First of all, we point out that equivalence of (2) and (3) has been proven in Theorem 3, while the proof of (1)⇒(2) is trivial. Thus, we only need to show (3)⇒(1) and (3)⇔(4).

We first prove that (3)⇒(4), which is the crux of the proof of Theorem 2. For this, by Theorems 1 and 3, we only have to prove that if $f(x)=\sum_R S(R)\psi_R(x)$, where $\{S(R)\}_{R\in\mathfrak{D}}$ is a center $(q,2)$-atom sequence supported on $Q_{k_1}\times Q_{l_1}$, then $\|s(f)\|_{K_2^q(\mathbb{R}\times\mathbb{R})}\le C$, where $s(f)$ is the generalized Lusin-area integral of f and C is independent of k_1 and l_1. In fact,

$$\begin{aligned}\|s(f)\|^q_{K_2^q(\mathbb{R}\times\mathbb{R})} &= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}2^{(k+l)(1-q/2)}\|s(f)\chi_{C_{k,l}}\|_2^q\\ &= \sum_{k=-\infty}^{k_1+10}\sum_{l=-\infty}^{l_1+10}\cdots+\sum_{k=-\infty}^{k_1+10}\sum_{l=l_1+11}^{\infty}\cdots\\ &\quad+\sum_{k=k_1+11}^{\infty}\sum_{l=-\infty}^{l_1+10}\cdots+\sum_{k=k_1+11}^{\infty}\sum_{l=l_1+11}^{\infty}\cdots\\ &=: I_1+I_2+I_3+I_4\,.\end{aligned}$$

Now,

$$\begin{aligned}I_1 &\le C\|f\|_2^q\sum_{k=-\infty}^{k_1+10}\sum_{l=-\infty}^{l_1+10}2^{(k+l)(1-q/2)}\\ &\le C\Big(\sum_R|S(R)|^2\Big)^{q/2}2^{(k_1+l_1)(1-q/2)}\le C\,,\end{aligned}$$

where C is independent of k_1 and l_1. The estimations of I_2 and I_3 are similar; we only compute I_2. For this, let $\chi(t)$ be the characteristic function of the interval $(0,1)$. We first have

$$\begin{aligned}&\|s(f)\chi_{C_{k,l}}\|_2^2\\ &\quad=\int\!\!\int\chi_{C_{k,l}}(x)\,dx\int_{\Gamma(x)}|(f*\overline{\psi}_y)(t)|^2y_1^{-2}y_2^{-2}\,dt\,dy\\ &\quad\le C\int_{C_l}dx_2\int_{\Gamma(x_2)}y_2^{-2}\,dt_2dy_2\int|(f(\xi_1,\cdot)*\overline{\psi}_{y_2})(t_2)|^2\,d\xi_1\\ &\quad=C\int_{C_l}dx_2\int_{\Gamma(x_2)}y_2^{-2}\,dt_2\,dy_2\\ &\qquad\times\int\Big|\sum_{R=I\times J\in\mathfrak{D}}\{S(R)(\psi_J*\overline{\psi}_{y_2})(t_2)\}\psi_I(\xi_1)\Big|^2\,d\xi_1\\ &\quad\le\int_{C_l}dx_2\int_{\Gamma(x_2)}\sum_{I\in\mathfrak{D}_0}\Big|\sum_{J\in\mathfrak{D}_0}S(R)(\psi_J*\overline{\psi}_{y_2})(t_2)\Big|^2y_2^{-2}\,dt_2\,dy_2\\ &\quad\le C\sum_I\int_{C_l}dx_2\int_{\Gamma(x_2)}\Big(\sum_{J\in\mathfrak{D}_0}|S(R)|\cdot|(\psi_J*\overline{\psi}_{y_2})(t_2)|\Big)^2y_2^{-2}\,dt_2\,dy_2\\ &\quad\le C\Big(\sum_R|S(R)|^2\Big)\\ &\qquad\times\Big(\sum_{\nu=-l_1}^{\infty}\sum_{l(J)=2^{-\nu}}\int_{C_l}dx_2\int_{\Gamma(x_2)}|(\psi_J*\overline{\psi}_{y_2})(t_2)|^2y_2^{-2}\,dt_2\,dy_2\Big)\\ &\quad\le C2^{(k_1+l_1)(1-2/q)}\sum_{\nu=-l_1}^{\infty}\sum_{l(J)=2^{-\nu}}\int_{C_l}dx_2\\ &\qquad\times\int_0^{\infty}\int_{\mathbb{R}}\chi\Big(\frac{x_2-t_2}{y_2}\Big)|(\psi_J*\overline{\psi}_{y_2})(t_2)|^2y_2^{-2}\,dt_2\,dy_2\,.\end{aligned}$$

For $x_2\in C_l$ with $l\ge l_1+11$, since $J\subset Q_{l_1}$, there exists a geometric constant

$C_0 > 0$ such that $|x_2 - x_J| \geq C_0|x_2|$. Fix any $x_2 \in C_l$ with $l \geq l_1 + 11$. Write

$$\int_0^\infty \int_{\mathbb{R}} \chi\Big(\frac{x_2 - t_2}{y_2}\Big)|(\psi_J * \overline{\psi}_{y_2})(t_2)|^2 y_2^{-2}\, dt_2\, dy_2 = \int_0^{C_0|x_2|/4} \int_{\mathbb{R}} \ldots + \int_{C_0|x_2|/4}^{\infty} \int_{\mathbb{R}} \ldots =: II_1 + II_2 .$$

Using the fact that $\operatorname{supp} \overline{\psi} \subset (0,1)$, the regularity of ψ_J and its vanishing moments, from the mean value theorem, we easily compute that

$$\begin{aligned} II_1 &= \int_0^{C_0|x_2|/4} \int_{\mathbb{R}} \chi\Big(\frac{x_2 - t_2}{y_2}\Big) \\ &\quad\times \Big| \int (\psi_J(\xi_2) - \psi_J(t_2))\overline{\psi}_{y_2}(t_2 - \xi_2)\, d\xi_2\Big|^2 y_2^{-2}\, dt_2\, dy_2 \\ &\leq C_L \int_0^{C_0|x_2|/4} \int_{\mathbb{R}} \chi\Big(\frac{x_2 - t_2}{y_2}\Big)|J|(2^\nu|x_2|)^{-2(L+1)} \\ &\quad\times \Big(\int |t_2 - \xi_2| y_2^{-1}\Big|\overline{\psi}\Big(\frac{t_2 - \xi_2}{y_2}\Big)\Big|\, d\xi_2\Big)^2 y_2^{-2}\, dt_2\, dy_2 \\ &= C_L 2^{-\nu(2L-1)}|x_2|^{-2L}, \end{aligned}$$

where L is a constant to be determined in the following. For II_2, taking $\alpha \in \mathbb{N}$ with $1/q - 2 < \alpha \leq 1/q - 1$, from the Taylor formula we obtain

$$\begin{aligned} II_2 &= \int_{C_0|x_2|/4}^{\infty} \int_{\mathbb{R}} \chi\Big(\frac{x_2 - t_2}{y_2}\Big)\Big| \int \psi_J(\xi_2)\Big(\overline{\psi}_{y_2}(t_2 - \xi_2) \\ &\quad - \overline{\psi}_{y_2}(t_2 - x_J) - [\overline{\psi}_{y_2}]'(t_2 - x_J)(x_J - \xi_2) - \ldots \\ &\quad - \frac{1}{\alpha!}[\overline{\psi}_{y_2}]^{(\alpha)}(t_2 - x_J)(x_J - \xi_2)^\alpha\Big)\, d\xi_2\Big|^2 y_2^{-2}\, dt_2\, dy_2 \\ &\leq C\|\overline{\psi}^{(\alpha+1)}\|_\infty \int_{C_0|x_2|/4}^{\infty} \int_{\mathbb{R}} \chi\Big(\frac{x_2 - t_2}{y_2}\Big)|J|^{-1} \\ &\quad\times \Big(\int |\xi_2 - x_J|^{\alpha+1}(1 + 2^\nu|\xi_2 - x_J|)^{-L}\, d\xi_2\Big)^2 y_2^{-2\alpha-6}\, dt_2\, dy_2 \\ &\leq C2^{-\nu(2\alpha+3)}|x_2|^{-2\alpha-4}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\|s(f)\chi_{C_{k,l}}\|_2^2 \\ &\quad\leq C_L 2^{(k_1+l_1)(1-2/q)} \\ &\qquad\times \Big(\sum_{\nu=-l_1}^{\infty} \sum_{l(J)=2^{-\nu}} \int_{C_l} (2^{-\nu(2L-1)}|x_2|^{-2L} + 2^{-\nu(2\alpha+3)}|x_2|^{-2\alpha-4})\, dx_2\Big) \\ &\quad\leq C_L 2^{(k_1+l_1)(1-2/q)} \\ &\qquad\times \sum_{\nu=-l_1}^{\infty} (2^{-\nu(2L-1)+l_1+\nu-l(2L-1)} + 2^{-\nu(2\alpha+3)+l_1+\nu-l(2\alpha+3)}) \\ &\quad= C_L(2^{k_1(1-2/q)+l_1(2L-2/q)-l(2L-1)} + 2^{k_1(1-2/q)+l_1(2\alpha+4-2/q)-l(2\alpha+3)}). \end{aligned}$$

Selecting $L > 1/q$, it is now easy to verify that $I_2 \leq C$, where C is independent of k_1 and l_1.

Next, we estimate I_4. For this, let $\mathfrak{D}_{k_1,l_1} = \{R : S(R) \neq 0\}$ and let C_0 be the geometric constant as in the estimation of I_2. First,

$$\begin{aligned} &\|s(f)\chi_{C_{k,l}}\|_2^2 \\ &\quad= \int\int_{C_{k,l}} dx \int_{\Gamma(x)} \Big|\sum_R S(R)(\psi_R * \overline{\psi}_y)(t)\Big|^2 y_1^{-2}y_2^{-2}\, dt\, dy \\ &\quad\leq \Big(\sum_R |S(R)|^2\Big) \int\int_{C_{k,l}} dx \int_0^\infty \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} \chi\Big(\frac{x_1 - t_1}{y_1}\Big)\chi\Big(\frac{x_2 - t_2}{y_2}\Big) \\ &\qquad\times \Big(\sum_{R\in\mathfrak{D}_{k_1,l_1}} |(\psi_R * \overline{\psi}_y)(t)|^2\Big) y_1^{-2}y_2^{-2}\, dt\, dy \\ &\quad\leq 2^{(k_1+l_1)(1-2/q)} \int\int_{C_{k,l}} dx\Big\{\int_0^{C_0|x_1|/4} \int_0^{C_0|x_2|/4} \int_{\mathbb{R}} \int_{\mathbb{R}} \ldots \\ &\qquad+ \int_0^{C_0|x_1|/4} \int_{C_0|x_2|/4}^{\infty} \int_{\mathbb{R}} \int_{\mathbb{R}} \ldots \\ &\qquad+ \int_{C_0|x_1|/4}^{\infty} \int_0^{C_0|x_2|/4} \int_{\mathbb{R}} \int_{\mathbb{R}} \ldots + \int_{C_0|x_1|/4}^{\infty} \int_{C_0|x_2|/4}^{\infty} \int_{\mathbb{R}} \int_{\mathbb{R}} \ldots\Big\} \\ &\quad=: 2^{(k_1+l_1)(1-2/q)} \int\int_{C_{k,l}} (III_1 + III_2 + III_3 + III_4)\, dx. \end{aligned}$$

For III_1, noting that $x \in C_{k,l}$ with $k \geq k_1 + 11$ and $l \geq l_1 + 11$, by the regularity of ψ as well as $\operatorname{supp} \overline{\psi} \subset (0,1)$ and their vanishing moments, it is easy to prove that

$$|(\psi_R * \overline{\psi}_y)(t)| = \Big| \int\int (\psi_I(\xi_1) - \psi_I(t_1))(\psi_J(\xi_2) - \psi_J(t_2)) \times \overline{\psi}_{y_1}(t_1 - \xi_1)\overline{\psi}_{y_2}(t_2 - \xi_2)\, d\xi_1\, d\xi_2 \Big| \le C_L(|I|\cdot|J|)^{-3/2} 2^{-(\nu_1+\nu_2)(L+1)}(|x_1|\cdot|x_2|)^{-L-1},$$

where L is a constant to be determined. Therefore,

$$\begin{aligned} III_1 &\le C \sum_{R\in\mathfrak{D}_{k_1,l_1}} \int_0^{C_0|x_1|/4}\int_0^{C_0|x_2|/4} (|I|\cdot|J|)^{-3} \times 2^{-2(\nu_1+\nu_2)(L+1)}(|x_1|\cdot|x_2|)^{-2(L+1)} y_1 y_2\, dy_1\, dy_2 \\ &\le C(|x_1|\cdot|x_2|)^{-2L} \times \Big(\sum_{\nu_1=-k_1}^{\infty} 2^{3\nu_1-2\nu_1(L+1)+k_1+\nu_1}\Big)\Big(\sum_{\nu_2=-l_1}^{\infty} 2^{3\nu_2-2\nu_2(L+1)+l_1+\nu_2}\Big) \\ &= C2^{(k_1+l_1)(2L-1)}(|x_1|\cdot|x_2|)^{-2L}. \end{aligned}$$

The estimations of III_2 and III_3 are similar; we only compute III_2. Take $\alpha\in\mathbb{N}$ with $1/q-2<\alpha\le 1/q-1$. Similarly to the computation of III_1,

$$\begin{aligned} &|(\psi_R * \overline{\psi}_y)(t)| \\ &= \Big| \int\int (\psi_I(\xi_1) - \psi_I(t_1))\overline{\psi}_{y_1}(t_1-\xi_1) \\ &\quad \times \Big(\overline{\psi}_{y_2}(t_2-\xi_2) - \overline{\psi}_{y_2}(t_2-x_J) - [\overline{\psi}_{y_2}]'(t_2-x_J)(x_J-\xi_2) - \ldots \\ &\quad - \frac{1}{\alpha!}[\overline{\psi}_{y_2}]^{(\alpha)}(t_2-x_J)(x_J-\xi_2)^{\alpha}\Big)\psi_J(\xi_2)\, d\xi_1\, d\xi_2 \Big| \\ &\le C_L|I|^{-3/2}(2^{\nu_1}|x_1|)^{-L-1}\int |t_1-\xi_1|\cdot|\overline{\psi}_{y_1}(t_1-\xi_1)|\, d\xi_1 \\ &\quad \times \int |J|^{-1/2}(1+2^{\nu_2}|\xi_2-x_J|)^{-L}|\xi_2-x_J|^{\alpha+1} y_2^{-\alpha-2}\, d\xi_2 \\ &\le C_L|I|^{-3/2}2^{-\nu_1(L+1)-\nu_2(\alpha+3/2)}|x_1|^{-L-1} y_1 y_2^{-\alpha-2}, \end{aligned}$$

where L is a constant to be determined. Therefore,

$$III_2 \le C_L \sum_{R\in\mathfrak{D}_{k_1,l_1}} 2^{-\nu_2(2\alpha+3)} \times \Big(\int_0^{C_0|x_1|/4} y_1|I|^{-3}(2^{\nu_1}|x_1|)^{-2(L+1)}\, dy_1\Big)\int_{C_0|x_2|/4}^{\infty} y_2^{-2\alpha-5}\, dy_2$$

$$\begin{aligned} &\le C_L|x_1|^{-2L}|x_2|^{-2\alpha-4} \\ &\quad \times \Big(\sum_{\nu_1=-k_1}^{\infty} 2^{3\nu_1-2\nu_1(L+1)+k_1+\nu_1}\Big)\Big(\sum_{\nu_2=-l_1}^{\infty} 2^{-\nu_2(2\alpha+3)+l_1+\nu_2}\Big) \\ &= C_L 2^{2k_1(L-1)+k_1+l_1(2\alpha+3)}|x_1|^{-2L}|x_2|^{-2\alpha-4}. \end{aligned}$$

For III_4, we obtain similarly

$$\begin{aligned} &|(\psi_R * \overline{\psi}_y)(t)| \\ &= \Big| \int \psi_I(\xi_1)\Big(\overline{\psi}_{y_1}(t_1-\xi_1) - \overline{\psi}_{y_1}(t_1-x_I) \\ &\quad - [\overline{\psi}_{y_1}]'(t_1-x_I)(x_I-\xi_1) - \ldots - \frac{1}{\alpha!}[\overline{\psi}_{y_1}]^{(\alpha)}(t_1-x_I)(x_I-\xi_1)^{\alpha}\Big)\, d\xi_1 \Big| \\ &\quad \times \Big| \int \psi_J(\xi_2)\Big(\overline{\psi}_{y_2}(t_2-\xi_2) - \overline{\psi}_{y_2}(t_2-x_J) \\ &\quad - [\overline{\psi}_{y_2}]'(t_2-x_J)(x_J-\xi_2) - \ldots - \frac{1}{\alpha!}[\overline{\psi}_{y_2}]^{(\alpha)}(t_2-x_J)(x_J-\xi_2)^{\alpha}\Big)\, d\xi_2 \Big| \\ &\le C2^{-(\nu_1+\nu_2)(\alpha+3/2)}(y_1y_2)^{-2\alpha-2}. \end{aligned}$$

Therefore,

$$\begin{aligned} III_4 &\le C \sum_{R\in\mathfrak{D}_{k_1,l_1}} \int_{C_0|x_1|/4}^{\infty}\int_{C_0|x_2|/4}^{\infty} 2^{-(\nu_1+\nu_2)(2\alpha+3)}(y_1y_2)^{-2\alpha-5}\, dy_1\, dy_2 \\ &\le C(|x_1|\cdot|x_2|)^{-2\alpha-4} \\ &\quad \times \Big(\sum_{\nu_1=-k_1}^{\infty} 2^{-\nu_1(2\alpha+3)+k_1+\nu_1}\Big)\Big(\sum_{\nu_2=-l_1}^{\infty} 2^{-\nu_2(2\alpha+3)+l_1+\nu_2}\Big) \\ &= C2^{(k_1+l_1)(2\alpha+3)}(|x_1|\cdot|x_2|)^{-2\alpha-4}. \end{aligned}$$

Thus,

$$\begin{aligned} \|s(f)\chi_{C_{k,l}}\|_2^2 \le C\{&2^{(k_1+l_1)(2L-2/q)-(k+l)(2L-1)} \\ &+ 2^{k_1(2L-2/q)+l_1(2\alpha+4-2/q)-k(2L-1)-l(2\alpha+3)} \\ &+ 2^{k_1(2\alpha+4-2/q)+l_1(2L-2/q)-k(2\alpha+3)-l(2L-1)} \\ &+ 2^{(k_1+l_1)(2\alpha+4-2/q)-(k+l)(2\alpha+3)}\}. \end{aligned}$$

Taking $L>1/q$, we now easily get $I_4\le C$, where C is independent of k_1 and l_1. We have thus proven (3)⇒(4). The proof of (4)⇒(3), by properly classifying the dyadic rectangles, is essentially similar to that of Proposition 2.1 in [8]; we omit the details so as to limit the length of this paper.

Now, in order to complete the proof of Theorem 2, we only need to show that (3)⇒(1). Similarly to the proof of (3)⇒(4), without loss of generality, we can suppose that $f = \sum_{R\in\mathfrak{D}} S(R)\psi_R$, where $\{S(R)\}_{R\in\mathfrak{D}}$ is a center $(q,2)$-atom sequence supported on $Q_{k_0}\times Q_{l_0}$. We only need to bound

$$\begin{aligned}\|G(f)\|^q_{K_2^q(\mathbb{R}\times\mathbb{R})} &= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} 2^{(k+l)(1-q/2)}\|G(f)\chi_{C_{k,l}}\|_2^q \\ &= \sum_{k=-\infty}^{k_0+3}\sum_{l=-\infty}^{l_0+3}\ldots + \sum_{k=-\infty}^{k_0+3}\sum_{l=l_0+4}^{\infty}\ldots \\ &\quad + \sum_{k=k_0+4}^{\infty}\sum_{l=-\infty}^{l_0+3}\ldots + \sum_{k=k_0+4}^{\infty}\sum_{l=l_0+3}^{\infty}\ldots \\ &=: I_1+I_2+I_3+I_4\,.\end{aligned}$$

For I_1, we can estimate as follows:

$$\begin{aligned}I_1 &= \sum_{k=-\infty}^{k_0+3}\sum_{l=-\infty}^{l_0+3} 2^{(k+l)(1-q/2)}\|G(f)\chi_{C_{k,l}}\|_2^q \\ &\le C\Big(\sum_R |S(R)|^2\Big)^{q/2} 2^{(k_0+l_0)(1-q/2)} \le C\,,\end{aligned}$$

where C is independent of k_0 and l_0. As before, the computations for I_2 and I_3 are similar; we only estimate I_2. Note that if $J\subset Q_{l_0}$ and $x_2\in C_l$ with $l\ge l_0+4$, then $|\psi_J(x_2)|\le C_L 2^{\nu/2-\nu L}|x_2|^{-L}$. We choose $L>1/q$ to deduce that

$$\begin{aligned}\|G(f)\chi_{C_{k,l}}\|_2^2 &= \iint_{C_{k,l}} \sum_R |S(R)|^2|\psi_R(x)|^2\,dx \\ &\le \sum_R |S(R)|^2 \int_{C_l} |\psi_J(x_2)|^2\,dx_2 \\ &\le C2^{k_0(1-2/q)+l_0(2L-2/q)-l(2L-1)}\,.\end{aligned}$$

We can now easily verify that $I_2\le C$, where C is independent of f. Finally, we estimate I_4. Again, we have

$$\begin{aligned}\|G(f)\chi_{C_{k,l}}\|_2^2 &= \sum_R \iint_{C_{k,l}} |S(R)|^2|\psi_R(x)|^2\,dx \\ &\le C\sum_R |S(R)|^2 2^{2(k_0+l_0)(L-1/2)} \int_{C_k} |x_1|^{-2L}\,dx_1 \int_{C_l} |x_2|^{-2L}\,dx_2 \\ &\le C2^{(k_0+l_0)(2L-2/q)-(k+l)(2L-1)}\,,\end{aligned}$$

and it easily follows that $I_4\le C$, where C is independent of f. We have finished the proof of (3)⇒(1). This proves Theorem 2.

Acknowledgements. The author would like to thank Jerzy Trzeciak, copy editor, for making this paper more readable.

References

[1] S. A. Chang and R. Fefferman, *Some recent developments in Fourier analysis and H^p-theory on product domains*, Bull. Amer. Math. Soc. (N.S.) 12 (1985), 1–43.

[2] —, —, *A continuous version of duality of H^1 with BMO on the bidisc*, Ann. of Math. 112 (1980), 179–201.

[3] Y. Z. Chen and K. S. Lau, *On some new classes of Hardy spaces*, J. Funct. Anal. 84 (1989), 255–278.

[4] J. García-Cuerva, *Hardy spaces and Beurling algebras*, J. London Math. Soc. (2) 39 (1989), 499–513.

[5] M. Frazier and B. Jawerth, *The φ-transform and applications to distribution spaces*, in: Function Spaces and Applications, M. Cwikel *et al.* (eds.), Lecture Notes in Math. 1302, Springer, 1989, 223–246.

[6] M. Frazier and B. Jawerth, *A discrete transform and decompositions of distribution spaces*, J. Funct. Anal. 93 (1990), 34–170.

[7] S. Z. Lu and D. C. Yang, *The wavelet characterizations of some new Hardy spaces associated with the Herz spaces*, J. Beijing Normal Univ. (Natur. Sci.) 29 (1993), 10–19 (in Chinese).

[8] —, —, *The Littlewood–Paley function and φ-transform characterizations of a new Hardy space HK_2 associated with the Herz space*, Studia Math. 101 (1992), 285–298.

DEPARTMENT OF MATHEMATICS
BEIJING NORMAL UNIVERSITY
100875 BEIJING, P.R. CHINA

Received April 23, 1992
Revised version September 9, 1993
(2934)