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Reflexive Orlicz spaces have uniformly normal structure

by

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Abstract. We prove that an Orlicz space equipped with the Luxemburg norm has uniformly normal structure if and only if it is reflexive.

Let X be a Banach space. We say that X has (*weak*) *normal structure* if for any nonsingleton (weakly compact) bounded closed convex subset C of X, there exists $x \in C$ such that

$$r_C(x) = \sup\{\|x-y\| : y \in C\} < \operatorname{diam} C = \sup\{\|u-v\| : u, v \in C\}.$$

If, moreover, there exists $h < 1$ such that for each nonsingleton bounded closed convex subset C, there exists $x \in C$ such that $r_C(x) \le h \operatorname{diam} C$, then X is said to have *uniform normal structure* and the infimum of such h is denoted by $N(X)$.

The above concepts are introduced as powerful tools in fixed point theory, for instance, if X has weak normal structure, then it has fixed point property, i.e., any nonexpansive self-mapping defined on a weakly compact convex subset of X has a fixed point. Moreover, if X has uniformly normal structure, then for $k < N(X)^{-1/2}$, every k-Lipschitzian self-mapping of a bounded closed convex subset of X has a fixed point (see [1], [2] and [3]).

In 1984, T. Landes [7] found a criterion for sequence Orlicz spaces endowed with the Luxemburg norm to have (weakly) normal structure. Using it, it is easy to establish the corresponding results for function Orlicz spaces. It was not until 1991 that Tingfu Wang and Baoxiang Wang [9] obtained necessary and sufficient conditions for sequence Orlicz spaces endowed with the Orlicz norm to have (weakly) normal structure. Shutao Chen and Yanzheng Duan [4] then solved the problem for function Orlicz spaces. In this paper, we establish a criterion for sequence and function Orlicz spaces endowed with the Luxemburg norm to have uniformly normal structure.

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We first recall the concept of Orlicz spaces.

Let (G,Σ,μ) be a complete measure space, and let $M:\mathbb{R}\to\mathbb{R}$ satisfy (i) M is even, convex and continuous; (ii) $M(u)=0$ iff $u=0$ and (iii) $M(u)\to\infty$ as $u\to\infty$. For a μ-measurable function $x(t)$ on G, we define its modular by

$$\varrho_M(x)=\int\limits_G M(x(t))\,dt\,.$$

Then the *Orlicz space*

$$L_M=\{x:\varrho_M(\lambda x)<\infty \text{ for some } \lambda>0\}$$

endowed with the *Luxemburg norm*

$$\|x\|=\inf\{\lambda>0:\varrho_M(x/\lambda)\le 1\}$$

is a Banach space.

In the following, we only consider the case that G is nonatomic and $\mu G<\infty$. For the cases (i) G is nonatomic and $\mu G=\infty$, or (ii) $G=\mathbb{N}=\{1,2,3,\ldots\}$ and $\mu\{n\}=1$, $n=1,2,\ldots$, our main result still holds and the proof will be a little easier.

We say that M satisfies *condition* Δ_2, or simply, $M\in\Delta_2$, if there exist $u'>0$ and $K>1$ such that $M(2u)\le KM(u)$ for all $u\ge u'$.

The condition Δ_2 plays an important role in the theory of Orlicz spaces. For instance, an Orlicz space L_M is reflexive iff $M\in\Delta_2$ and $M^*\in\Delta_2$, where $M^*(v)=\sup_{u\in\mathbb{R}}\{uv-M(u)\}$ is the *complementary function* of M.

THEOREM 1. *L_M has uniformly normal structure iff it is reflexive.*

To prove the theorem, we need the following lemmas.

LEMMA 2. *The following are equivalent:*

(a) $M\in\Delta_2$,

(b) *for any $u'>0$ and $l>1$, there exists $\varepsilon>0$ such that*

$$M((1+\varepsilon)u)\le lM(u)\quad (u\ge u'),$$

(c) *for any $v'>0$ and $\alpha\in(0,1)$, there exists $\delta>0$ such that*

$$(*)\qquad M^*(\alpha v)\le\alpha(1-\delta)M^*(v)\quad (v\ge v').$$

Proof. (a)⇒(b). By (a) and [6], there exists $K>l$ such that $M(2u)\le KM(u)$ for all $u\ge u'$. Let $\varepsilon=(l-1)/(K-1)$. Then by the convexity of M, for all $u\ge u'$, we have

$$\begin{aligned}M((1+\varepsilon)u)&=M((1-\varepsilon)u+2\varepsilon u)\le(1-\varepsilon)M(u)+\varepsilon M(2u)\\&\le(1-\varepsilon)M(u)+\varepsilon KM(u)=lM(u)\,.\end{aligned}$$

(b)⇒(c). By [6], we have (i) if $M_1(u)=aM(bu)$ $(a,b>0)$, then $M_1^*(v)=aM^*(v/(ab))$ and (ii) if $M_1(u)\ge M_2(u)$ $(u\ge u'>0)$, then there exists $v'=v'(u')>0$ such that $M_1^*(v)\le M_2^*(v)$ $(v\ge v')$ and $v'\to0$ as $u'\to0$. Hence, to verify $(*)$, it suffices to find some $\delta>0$ such that

$$\frac{1}{\alpha(1-\delta)}M((1-\delta)u)\ge M(u)\quad (u\ge u')$$

for the previously given $u'>0$.

Pick ε in $(0,1/2)$ such that

$$M((1+\varepsilon)u)\le\alpha^{-1}M(u)\quad (u\ge u'/2)$$

and let $\delta=1-1/(1+\varepsilon)\in(0,1/2)$. Then

$$M\left(\frac{1}{1-\delta}u\right)\le\frac{1}{\alpha}M(u)\le\frac{1}{\alpha(1-\varepsilon)}M(u)\quad (u\ge u'/2)\,.$$

Replacing u by $(1-\delta)u$ in the above inequalities and observing that $1-\delta>1/2$, we find

$$M(u)\le\frac{1}{\alpha(1-\delta)}M((1-\delta)u)\quad (u\ge u')\,.$$

(c)⇒(a). By the same reason as in (b)⇒(c), it suffices to find $K\ge1$ such that

$$\frac{1}{K}M^*\left(\frac{K}{2}v\right)\ge M^*(v)\quad (v\ge v')$$

for given $v'>0$.

By (c), there exists $\delta>0$ such that

$$M^*(v/2)\le\tfrac12(1-\delta)M^*(v)\quad (v\ge v')\,,$$

i.e.,

$$\tfrac12(1-\delta)M^*(2v)\ge M^*(v)\quad (v\ge v'/2)\,.$$

Taking an integer $n\ge1$ such that $(1-\delta)^{-n}\ge2$ and using the above inequality repeatedly, we derive

$$2^{-n-1}M^*(2^nv)\ge2^{-n}(1-\delta)^nM^*(2^nv)\ge M^*(v)\quad (v\ge v')\,.$$

By setting $K=2^{n+1}$, we complete the proof. ∎

LEMMA 3. *Suppose $M\in\Delta_2$. Then for any $\beta>1$ and $\varepsilon>0$, there exists $K\ge2$ such that for all $x\in L_M$,*

$$\varrho_M(\beta x)\le K\varrho_M(x)+\varepsilon\,.$$

Proof. Let $\alpha>0$ satisfy $M(\beta\alpha)\mu G<\varepsilon$. Then since $M\in\Delta_2$, there exists $K\ge2$ such that $M(\beta u)\le KM(u)$ for all $u\ge\alpha$. Given $x\in L_M$, set $F=\{t\in G:|x(t)|\ge\alpha\}$. Then

$$\begin{aligned}\varrho_M(\beta x)&=\varrho_M(\beta x|_F)+\varrho_M(\beta x|_{G\setminus F})\\&\le K\varrho_M(x|_F)+M(\beta\alpha)\mu(G\setminus F)\le K\varrho_M(x)+\varepsilon\end{aligned}$$

where $x|_A(t)=x(t)$ for $t\in A$ and $=0$ when $t\in G\setminus A$. ∎

LEMMA 4. *Assume $M \in \Delta_2$ and $M^* \in \Delta_2$. Then for any $\alpha > 0$, there exist $c > 1$ and $\delta > 0$ such that*

$$M\left(\frac{u+v}{2}\right) \le \frac{1-\delta}{2}[M(u)+M(v)]$$

whenever $|u| \ge \alpha$ and either $|u| \ge c|v|$ or $uv \le 0$.

Proof. By Lemma 2, there exist $\tau > 0$ and $\varepsilon \in (0, 1/2)$ such that

$$M\left(\frac{w}{2}\right) \le \frac{1-\tau}{2} M(w) \quad (|w| \ge \alpha)$$

and

$$M((1+\varepsilon)w) \le \frac{2}{2-\tau} M(w) \quad (|w| \ge \alpha).$$

Set $c = 1/\varepsilon$ and $\delta = 1-(2-2\tau)/(2-\tau)$. Then if $|u| \ge \alpha$ and either $|u| \ge c|v|$ or $uv \le 0$, we have

$$M\left(\frac{u+v}{2}\right) \le M\left(\frac{1+c^{-1}}{2}u\right) \le \frac{1-\tau}{2} M\left(\left(1+\frac{1}{c}\right)u\right) \le \frac{1-\tau}{2}\cdot\frac{2}{2-\tau}M(u) \le \frac{1-\delta}{2}[M(u)+M(v)]. \ \blacksquare$$

LEMMA 5 ([5], [10]). *If $M \in \Delta_2$, then*

(a) $\varrho_M(x_n) \to 0 \Leftrightarrow \|x_n\| \to 0$,
(b) $\varrho_M(x_n) \to 1 \Leftrightarrow \|x_n\| \to 1$ $(n \to \infty)$.

LEMMA 6. *If a Banach space X does not have uniformly normal structure, then for each $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists $\{x_i : 1 \le i \le n+1\}$ in X such that*

$$\|x_j\| \le 1, \quad \|x_i - x_j\| \le 1 \quad (1 \le i \le j \le n+1)$$

and

$$\left\|x_{m+1} - \frac{1}{m}\sum_{i=1}^{m} x_i\right\| > 1-\varepsilon \quad (m = 1, \ldots, n).$$

Proof. By the assumption, there exists a bounded closed convex subset C of X such that for each $z \in C$, there exists $x \in C$ satisfying $\|z - x\| > (1-\varepsilon)\operatorname{diam} C$. Without loss of generality, we may assume $0 \in C$ and $\operatorname{diam} C = 1$, i.e., $\|x\| \le 1$ and $\|x-y\| \le 1$ for all $x, y \in C$.

Pick any $x_1 \in C$. Then by the hypothesis, there exists $x_2 \in C$ such that $\|x_2 - x_1\| > 1-\varepsilon$. Since C is convex, $2^{-1}(x_1+x_2) \in C$, therefore, there exists $x_3 \in C$ such that $\|x_3 - 2^{-1}(x_1+x_2)\| > 1-\varepsilon$. By induction, we finish the proof. $\blacksquare$

Proof of Theorem 1. Since all Banach spaces with uniformly normal structure are reflexive (see [1]), we only need to show the sufficiency.

By Lemmas 3 and 4, there exist $K \ge 2$, $b > 0$, $c > 1$ and $\delta > 0$ such that

$$(1) \qquad \varrho_M(2x) \le K\varrho_M(x) + 1/8 \quad (x \in L_M),$$

$$(2) \qquad M(b)\mu G \le 1/(8K),$$

and

$$(3) \qquad M\left(\frac{u+v}{2}\right) \le \frac{1-\delta}{2}[M(u)+M(v)] \quad (|u| \ge b, \text{ and } |u| \ge c|v| \text{ or } uv \le 0).$$

Select an integer $p > 16c^2K^2$ and let $n = 8p$. If L_M does not have uniformly normal structure, then Lemma 5 and $M \in \Delta_2$ yield the existence of $\{x_i\}$ in L_M such that

$$(4) \qquad \varrho_M(x_i) \le 1, \quad \varrho_M(x_i - x_j) \le 1 \quad (1 \le i \le j \le n+1)$$

and

$$(5) \qquad 1 \ge \varrho_M\left(x_{m+1} - \frac{1}{m}\sum_{i=1}^{m} x_i\right) > 1 - \frac{\delta}{4n^2K}.$$

We first introduce some notations. Set

$$u_i(t) = x_{n+1}(t) - x_i(t) \quad (i \le n)$$

and for each $t \in G$, rearrange $\{u_i(t)\}_{i\le n}$ into $\{y_s(t) = u_{i_s(t)}(t)\}_{s\le n}$ such that $y_1(t) \le \ldots \le y_n(t)$. It is not difficult to check that each $y_s(t)$ is μ-measurable. Moreover, define

$$x(t) = 2^{-1}[y_{4p}(t) + y_{4p+1}(t)], \quad x_0(t) = \frac{2}{n}\sum_{i=1}^{n}|u_i(t)|,$$

$$I(t) = \{i \le n : |u_i(t)| > c|x(t)| \text{ or } c|u_i(t)| < |x(t)| \text{ or } u_i(t)x(t) \le 0\},$$

$$A = \{t \in G : I(t) \text{ contains at least } 4p \text{ elements}\}, \quad B = G \setminus A.$$

Then

$$(6) \qquad |x(t)| \le \max\{|y_s(t)|, |y_{4p+s}(t)|\} \le x_0(t).$$

Moreover, (1), (4) and the convexity of M imply

$$(7) \qquad \varrho_M(x_0) \le K\varrho_M\left(\frac{1}{n}\sum_{i=1}^{n}|u_i|\right) + \frac{1}{8} \le \frac{K}{n}\sum_{i=1}^{n}\varrho_M(u_i) + \frac{1}{8} < K + \frac{1}{8}.$$

For the first step, we show that

$$(8) \qquad \int_B M\left(\frac{x_1(t)-x_2(t)}{2}\right)dt > \frac{1}{2K}.$$

Since (4) and (1) give

$$\frac{7}{8} < 1 - \frac{\delta}{4n^2K} < \varrho_M(x_1 - x_2) \le K\varrho_M\left(\frac{x_1 - x_2}{2}\right) + \frac{1}{8},$$

i.e., $\varrho_M((x_1 - x_2)/2) > 3/(4K)$, to verify (8) it suffices to show

$$\int_A M\left(\frac{x_1(t) - x_2(t)}{2}\right) dt < \frac{1}{4K}.$$

For this purpose, we first check that $t \in A$ implies

$$|y_s(t)| > c|y_{4p+s}(t)| \text{ or } c|y_s(t)| < |y_{4p+s}(t)| \quad \text{or} \quad y_s(t)y_{4p+s}(t) \le 0$$

for each $s \le 4p$. In fact, if there exist some $j \le 4p$ and $t \in A$ such that none of the above three inequalities holds, then either

$$c^{-1}y_{4p+j}(t) \le y_j(t) \le cy_{4p+j}(t) \quad \text{or} \quad c^{-1}y_{4p+j}(t) \ge y_j(t) \ge cy_{4p+j}(t).$$

Since $x(t)$ is between $y_j(t)$ and $y_{4p+j}(t)$, we derive

$$c^{-1}x(t) \le y_s(t) \le cx(t) \quad \text{or} \quad c^{-1}x(t) \ge y_s(t) \ge cx(t)$$

for all $s = j, j+1, \ldots, 4p+j$, which contradicts the definition of A.

Hence, if we define, for each $s \le 4p$,

$$A(s) = \{t \in A : \max\{|y_s(t)|, |y_{4p+s}(t)|\} > b\}$$

then (3) and the convexity of M imply

$$\begin{aligned}
1 - \frac{\delta}{4n^2K} &< \varrho_M\left(x_{n+1} - \frac{1}{n}\sum_{i=1}^{n} x_i\right) \\
&= \varrho_M\left(\frac{2}{n}\sum_{s=1}^{4p} \frac{y_s + y_{4p+s}}{2}\right) \le \frac{2}{n}\sum_{s=1}^{4p} \varrho_M\left(\frac{y_s + y_{4p+s}}{2}\right) \\
&\le \frac{2}{n}\sum_{s=1}^{4p} \frac{1}{2} \int_{G\setminus A(s)} [M(y_s(t)) + M(y_{4p+s}(t))]\,dt \\
&\quad + \frac{2}{n}\sum_{i=1}^{4p} \frac{1}{2}(1-\delta) \int_{A(s)} [M(y_s(t)) + M(y_{4p+s}(t))]\,dt \\
&= \frac{1}{n}\sum_{i=1}^{n} \varrho_M(u_i) - \frac{\delta}{n}\sum_{s=1}^{4p} \int_{A(s)} [M(y_s(t)) + M(y_{4p+s}(t))]\,dt.
\end{aligned}$$

It follows from (4) that

(9) $$\sum_{s=1}^{4p} \int_{A(s)} [M(y_s(t)) + M(y_{4p+s}(t))]\,dt < \frac{1}{4nK} < \frac{1}{8K}.$$

Now, we define

$$D_i = \{t \in A : |u_i(t)| > b\} \quad (i = 1, 2),$$
$$B_i(s) = \{t \in A : u_i(t) = y_s(t) \text{ or } y_{4p+s}(t)\} \quad (i = 1, 2).$$

Then from (2), (9) and the fact that

$$\bigcup_{s=1}^{4p} B_i(s) = A, \quad D_i \cap b_i(s) \subset A(s) \quad (i = 1, 2),$$

we derive

$$\begin{aligned}
&\int_A M\left(\frac{x_1(t) - x_2(t)}{2}\right) dt \\
&\quad = \int_A M\left(\frac{u_1(t) - u_2(t)}{2}\right) dt \le \frac{1}{2}\sum_{i=1}^{2} \int_A M(u_i(t))\,dt \\
&\quad \le \frac{1}{2}\sum_{i=1}^{2} \int_{D_i} M(u_i(t))\,dt + M(b)\mu A \\
&\quad = \frac{1}{2}\sum_{i=1}^{2}\sum_{s=1}^{4p} \int_{D_i\cap B_i(s)} M(u_i(t))\,dt + M(b)\mu A \\
&\quad \le \frac{1}{2}\sum_{i=1}^{2}\sum_{s=1}^{4p} \int_{D\cap B_i(s)} [M(y_s(t)) + M(y_{4p+s}(t))]\,dt + M(b)\mu A \\
&\quad < \frac{1}{8K} + \frac{1}{8K} = \frac{1}{4K}.
\end{aligned}$$

This ends the proof of inequality (8).

For the second step, we set, for each $i = 3, \ldots, n-1$,

$$G(i) = \left\{t \in B : |x_s(t) - x_i(t)| \le \frac{c}{p}|x(t)| \text{ for some } s \text{ with } i < s \le n\right\}.$$

Then

(10) $$\bigcup_{i=3}^{n-1} G(i) = B.$$

In fact, for any $t \in B = G \setminus A$, by the definition of A, there exist at least five $u_i(t)$ such that their distance from each other is no more than $c|x(t)|/p$, and thus, there exist i, j, $3 \le i < j \le n$, such that

$$|u_i(t) - u_j(t)| \le c|x(t)|/p,$$

i.e., $t \in G(i)$. This proves (10).

Now, we define

$$D(3)=G(3)\,,\quad D(i)=G(i)\setminus\bigcup_{k=3}^{i-1}G(k)\,,\quad i=4,\ldots,n-1\,.$$

Then $\{D(i)\}_i$ are disjoint and $\bigcup_{i=3}^{n-1}D(i)=B$.

Let, for each $i=3,\ldots,n-1$ and each $t\in D(i)$,

$$i'(t)=i\,,\quad i''(t)=\max\{k\le n:|x_k(t)-x_i(t)|\le c|x(t)|/p\}\,.$$

Then $i'(t)$ and $i''(t)$ are well defined, by the definition of $G(i)$, and $i'(t)<i''(t)$.

Next, we construct two μ-measurable functions as follows:

$$x'(t)=\sum_{i=3}^{n-1}x_{i'(t)}(t)\chi_{D(i)}(t)\,,\quad x''(t)=\sum_{i=3}^{n-1}x_{i''(t)}(t)\chi_{D(i)}(t)\,.$$

Then by (6) and the definition of $i'(t)$ and $i''(t)$,

$$|x'(t)-x''(t)|\le c|x(t)|/p\le cx_0(t)/p\,. \tag{11}$$

Since (8) and the convexity of M imply

$$\frac12\int_B[M(x''(t)-x_1(t))+M(x''(t)-x_2(t))]\,dt \ge \int_B M\Big(\frac{x_1(t)-x_2(t)}{2}\Big)\,dt>\frac{1}{2K}\,,$$

without loss of generality we assume

$$\int_B M(x''(t)-x_1(t))\,dt>\frac{1}{2K}\,. \tag{12}$$

Finally, let

$$E=\{t\in B:|x''(t)-x_1(t)|\ge\max\{b,c^2x_0(t)/p\}\}\,.$$

Then by (11), $t\in E$ implies

$$|x''(t)-x_1(t)|\ge c^2x_0(t)/p\ge|x'(t)-x''(t)|\,.$$

It follows from (3) that $t\in E$ implies

$$M\Big(\frac{x''(t)-x'(t)+x''(t)-x_1(t)}{2}\Big) \le\frac{1-\delta}{2}[M(x''(t)-x'(t))+M(x''(t)-x_1(t))] \tag{13}$$

and moreover, (12), (3), (1), (7) and the inequalities $c^2/p<1/(16K^2)$ and $K>1$ imply that

$$\begin{aligned}\int_E M(x''(t)-x_1(t))\,dt &=\int_B M(x''(t)-x_1(t))\,dt-\int_{B\setminus E}M(x''(t)-x_1(t))\,dt\\ &>\frac{1}{2K}-\Big[\int_{B\setminus E}M\Big(\frac{c^2x_0(t)}{p}\Big)\,dt+M(b)\mu(B\setminus E)\Big]\\ &\ge\frac{1}{2K}-\Big[\frac{c^2}{p}\int_G M(x_0(t))\,dt+\frac{1}{8K}\Big]\\ &\ge\frac{1}{2K}-\Big[\frac{c^2}{p}\Big(K+\frac18\Big)+\frac{1}{8K}\Big]>\frac{1}{2K}-\frac{1}{8K}-\frac{1}{8K}=\frac{1}{4K}\,.\end{aligned} \tag{14}$$

In view of (13) and the convexity of M, for all $t\in E$, we have

$$\begin{aligned}\sum_{m=2}^{n}M\Big(\frac{1}{m-1}\sum_{k=1}^{m-1}(x_m(t)-x_k(t))\Big) &=\sum_{\substack{2\le m\le n\\ m\ne i''(t)}}M\Big(\frac{1}{m-1}\sum_{k=1}^{m-1}(x_m(t)-x_k(t))\Big)\\ &\quad+M\Big(\frac{1}{i''(t)-1}\Big[\sum_{\substack{2\le k\le i''(t)-1\\ k\ne i'(t)}}(x''(t)-x_k(t))\\ &\qquad+2\frac{x''(t)-x'(t)+x''(t)-x_1(t)}{2}\Big]\Big)\\ &\le\sum_{\substack{2\le m\le n\\ m\ne i''(t)}}M\Big(\frac{1}{m-1}\sum_{k=1}^{m-1}(x_m(t)-x_k(t))\Big)\\ &\quad+\frac{1}{i''(t)-1}\Big[\sum_{\substack{2\le k\le i''(t)-1\\ k\ne i'(t)}}M(x''(t)-x_k(t))\\ &\qquad+2M\Big(\frac{x''(t)-x'(t)+x''(t)-x_1(t)}{2}\Big)\Big]\end{aligned}$$

$$\le \sum_{m=2}^{n} \frac{1}{m-1} \sum_{k=1}^{m-1} M(x''(t)-x_k(t))$$
$$-\frac{\delta}{i''(t)-1}[M(x''(t)-x'(t))+M(x''(t)-x_1(t))].$$

Combining this with (5), (14) and $i''(t)-1\le n-1$, we find

$$1-\frac{\delta}{4n^2K} < \frac{1}{n-1}\sum_{m=2}^{n} \varrho_M\Big(x_m-\frac{1}{m-1}\sum_{k=1}^{m-1}x_k\Big)$$
$$\le \frac{1}{n-1}\sum_{m=2}^{n}\frac{1}{m-1}\sum_{k=1}^{m-1}\varrho_M(x_m-x_k)$$
$$-\frac{\delta}{n(n-1)}\int_E [M(x''(t)-x'(t))+M(x''(t)-x_1(t))]\,dt$$
$$< 1-\frac{\delta}{4n^2K}.$$

This contradiction completes the proof. ■

By Lemmas 3 and 4, if $M\in\Delta_2$ and $M^*\in\Delta_2$, then there exist $K\ge 2$, $b>0$, $c>1$ and $\delta>0$ such that

$$M(2u)\le KM(u)\quad (u>b)\,,\quad 8KM(b)\mu G<1\,,$$

and

$$M\Big(\frac{u+v}{2}\Big)\le\frac{1-\delta}{2}[M(u)+M(v)]$$

whenever $|u|\ge b$ and either $|u|\ge c|v|$ or $uv\le 0$.

For such K, c and δ, we have

COROLLARY 7. *If L_M is reflexive, then*

$$N(L_M)\le 1-\frac{\delta}{1+2^9\cdot 17c^2K^4}=1-\delta'.$$

Proof. From the proof of Theorem 1, we see that for any convex set C in L_M with $\operatorname{diam} C=1$,

$$(15)\qquad \inf_{x\in C}\sup_{y\in C}\varrho_M(x-y)<1-\frac{\delta}{2^8\cdot 17c^2K^3}=1-\delta''$$

and that

$$\varrho_M(2x)\le K\varrho_M(x)+1/8\le 2K$$

provided that $\varrho_M(x)\le 1$. Hence, $\varrho_M(x)\le 1-\delta''$ and $\|x\|\ge 1/2$ imply

$$1=\varrho_M\Big(\frac{x}{\|x\|}\Big)=\varrho_M\Big(\frac{1-\|x\|}{\|x\|}2x+\frac{2\|x\|-1}{\|x\|}x\Big)$$
$$\le\Big(\frac{1}{\|x\|}-1\Big)\varrho_M(2x)+\Big(2-\frac{1}{\|x\|}\Big)\varrho_M(x)$$
$$\le\Big(\frac{1}{\|x\|}-1\Big)2K+1-\delta''\,,$$

which yields $\|x\|\le 1-\delta'$ and hence, $N(L_M)\le 1-\delta'$ follows from (15). ■

To end this paper, we provide an application of our main theorem to the Hammerstein integral operators.

Let G be a bounded closed subset of $\mathbb{R}^n$. The operator P defined by

$$Px(t)=\int_G K(t,s)g(t,x(s))\,ds$$

is called a *Hammerstein operator*, where $K(t,s)$ is a measurable function on $G\times G$ and $g:G\times\mathbb{R}^n\to\mathbb{R}$ satisfies the Carathéodory condition, i.e., for almost all $t\in G$, $g(t,s)$ is a continuous function of s on $\mathbb{R}^n$, and it is a measurable function of t on G for every $s\in\mathbb{R}^n$.

Define

$$Hx(t)=g(t,x(t))\,,\qquad Ax(t)=\int_G K(t,s)x(s)\,ds\,.$$

Then, clearly, A is a linear integral operator and $Px=AHx$. Moreover, from [8], we have

LEMMA 8. *Let $M\in\Delta_2$ and $H:L_M\to L_M$. Then*

$$H:\{x\in L_M:\|x\|\le r\}\to L_M$$

is a k-Lipschitzian operator provided that there exists $k>0$ such that for all $\mu>1/r$, we have

$$(16)\qquad M\Big(\frac{3\mu}{k}g(t,s+l)-g(t,s)\Big)\le M\Big(\frac{1}{r}s\Big)+M(\mu l)+\alpha_\mu(t)$$

where $\alpha_\mu(t)$ is a nonnegative measurable function satisfying $\int_G\alpha_\mu(t)\,dt\le 1$.

By Corollary 7, Lemma 8 and [3], we have

COROLLARY 9. *If L_M is reflexive, the Hammerstein integral operator $P:L_M\to L_M$ is a self-mapping on a bounded closed convex subset of L_M, H is a k-Lipschitzian operator satisfying (16), and $k\|A\|<(1-\delta')^{-1/2}$, then P has a fixed point in the subset.*

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Pointwise ergodic theorems for functions in Lorentz spaces L_{pq} with $p \neq \infty$

by

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Abstract. Let τ be a null preserving point transformation on a finite measure space. Assuming τ is invertible, P. Ortega Salvador has recently obtained sufficient conditions for the almost everywhere convergence of the ergodic averages in L_{pq} with $1 < p < \infty$, $1 < q < \infty$. In this paper we obtain necessary and sufficient conditions for the almost everywhere convergence, without assuming that τ is invertible and only assuming that $p \neq \infty$.

1. Introduction. If τ is an invertible null preserving transformation on a σ-finite measure space $(X, \mathcal{F}, \mu)$, then $A_{n,m}$ and M will denote the ergodic averages and the maximal operator, respectively, defined by

$$A_{n,m}f(x) = \frac{1}{n+m+1} \sum_{i=-n}^{m} f(\tau^i x)$$

and

$$Mf = \sup_{n,m \geq 0} A_{n,m}|f|.$$

In [6], Ortega studied the good weights W for M to be bounded in $L_{pq}(W d\mu)$ $(1 < p < \infty,\ 1 < q \leq \infty)$, under the additional assumption that τ is measure preserving. Among other things, he proved that $\|Mf\|_{pq;W d\mu} \leq C\|f\|_{pq;W d\mu}$ if and only if $\sup_{n,m\geq 0} \|A_{n,m}f\|_{p\infty;W d\mu} \leq C\|f\|_{pq;W d\mu}$, C being a positive constant, not necessarily the same at each occurrence. Applying this result he then considered a null preserving τ on a finite measure space and proved that if $\sup_{n,m\geq 0} \|A_{n,m}f\|_{p\infty} \leq C\|f\|_{pq}$, where $1 < p < \infty$ and $1 < q < \infty$, then for any f in $L_{pq}(\mu)$ the ergodic averages $A_{0,n}f$ converge almost everywhere as $n \to \infty$. It seems to the author that this condition for the validity of the pointwise ergodic theorem is too strong. In fact, as

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