[2] Z. Daróczy and Z. Páles, *Convexity with given infinite weight sequences*, Stochastica 11 (1987), 5–12.

[3] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge Univ. Press, 1952.

[4] M. Kuczma, *An introduction to the theory of functional equations and inequalities, Cauchy's equation and Jensen's inequality*, Prace Nauk. Uniw. Śl. 489, Polish Scientific Publishers, Warszawa–Kraków–Katowice, 1985.

[5] N. Kuhn, *A note on t-convex functions*, in: General Inequalities 4, Internat. Ser. Numer. Math. 71, Birkhäuser, Basel, 1984, 269–276.

[6] J. Matkowski, *Cauchy functional equation on a restricted domain and commuting functions*, in: Iteration Theory and Its Functional Equations (Proc. Schloss Hofen, 1984), Lecture Notes in Math. 1163, Springer, Berlin, 1985, 101–106.

[7] —, *Functional inequality characterizing convex functions, conjugacy and a generalization of Hölder's and Minkowski's inequalities*, Aequationes Math. 40 (1990), 168–180.

[8] —, *The converse of the Minkowski's inequality theorem and its generalization*, Proc. Amer. Math. Soc. 109 (1990), 663–675.

[9] —, *A generalization of Holder's and Minkowski's inequalities and conjugate functions*, in: Constantin Carathéodory: An International Tribute, Vol. II, World Scientific, Singapore, 1991, 819–827.

[10] —, *Functional inequality characterizing concave functions in* $(0,\infty)^k$, Aequationes Math. 43 (1992), 219–224.

DEPARTMENT OF MATHEMATICS
TECHNICAL UNIVERSITY
WILLOWA 2
43-309 BIELSKO-BIAŁA, POLAND

Received March 10, 1993
Revised version September 9, 1993 (3072)

STUDIA MATHEMATICA 109 (2) (1994)

A localization property for B^s_{pq} and F^s_{pq} spaces

by

HANS TRIEBEL (Jena)

Abstract. Let $f^j = \sum_k a_k f(2^{j+1}x - 2k)$, where the sum is taken over the lattice of all points k in $\mathbb{R}^n$ having integer-valued components, $j \in \mathbb{N}$ and $a_k \in \mathbb{C}$. Let A^s_{pq} be either B^s_{pq} or F^s_{pq} ($s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$) on $\mathbb{R}^n$. The aim of the paper is to clarify under what conditions $\|f^j \mid A^s_{pq}\|$ is equivalent to $2^{j(s-n/p)}(\sum_k |a_k|^p)^{1/p}\|f \mid A^s_{pq}\|$.

1. Introduction and theorem. The spaces B^s_{pq} and F^s_{pq} with $s \in \mathbb{R}$, $0 < p \le \infty$ ($p < \infty$ for the F-scale), $0 < q \le \infty$, on $\mathbb{R}^n$ cover many well-known classical function spaces, such as the Sobolev spaces $W^k_p = F^k_{p,2}$ (with $k \in \mathbb{N}_0$, $1 < p < \infty$), the fractional Sobolev spaces $H^s_p = F^s_{p,2}$ (with $s \in \mathbb{R}$, $1 < p < \infty$), the Hölder–Zygmund spaces $\mathcal{C}^s = B^s_{\infty,\infty}$ (with $s > 0$), the (inhomogeneous) Hardy spaces $h_p = F^0_{p,2}$ (with $0 < p < \infty$) and the classical Besov spaces B^s_{pq} (with $s > 0$, $1 < p < \infty, 1 \le q \le \infty$). The theory of these spaces has been developed in [8, 9]. The aim of this paper is to prove a localization property for all these spaces which in this generality and in its almost final form is unexpected and rather surprising.

Let $\mathbb{Z}^n$ be the lattice of all points in $\mathbb{R}^n$ having integer-valued components. Let $x^{k,j} = 2^{-j}k$ with $k \in \mathbb{Z}^n$ and $j \in \mathbb{N}$. Let $f \in S'$ with $\operatorname{supp} f \subset Q_d = \{x \in \mathbb{R}^n : |x_l| < d \text{ if } l = 1, \ldots, n\}$, where $d > 0$ is assumed to be small, at least $d \le 1/2$, and let

$$f^j(x) = \sum_{k\in\mathbb{Z}^n} a_k f(2^{j+1}(x - x^{k,j})), \qquad a_k \in \mathbb{C}. \tag{1}$$

Of course, the terms in (1) have mutually disjoint supports. Let $\sigma_p = \max(0, n(1/p - 1))$ and let $[a]$ be the largest integer less than or equal to $a \in \mathbb{R}$.

THEOREM. *Let $s \in \mathbb{R}$, $0 < p \le \infty$ ($p < \infty$ for the F-scale), $0 < q \le \infty$. Let A^s_{pq} be either B^s_{pq} or F^s_{pq} and let $0 < d \le 1/4$.*

1991 *Mathematics Subject Classification*: Primary 46E35.

(i) *There exist two constants* $c_1 > 0$ *and* $c_2 > 0$ *such that for all* $f \in A^s_{pq}$ *with* $\operatorname{supp} f \subset Q_d$ *and*

$$\int x^\beta f(x)\,dx = 0 \quad \text{for } |\beta| \le L = \max([\sigma_p - s], -1)\,, \tag{2}$$

all $j \in \mathbb{N}$, *and all* f^j *given by* (1),

$$c_1 \|f^j \mid A^s_{pq}\| \le 2^{j(s-n/p)} \Big(\sum_k |a_k|^p\Big)^{1/p} \|f \mid A^s_{pq}\| \le c_2 \|f^j \mid A^s_{pq}\|\,. \tag{3}$$

(ii) *Let* $\sigma_p - s > [\sigma_p - s] = L \in \mathbb{N}_0$ *and let* $0 < c_1 \le c_2 < \infty$. *Then there exists an* $f \in A^s_{pq}$ *with* $\operatorname{supp} f \subset Q_d$ *and*

$$\int x^\beta f(x)\,dx = 0 \quad \text{for } |\beta| \le L - 1 \tag{4}$$

such that (3) *fails for some* j, *with* f^j *given by* (1).

(iii) *Let* $\sigma_p - s = L \in \mathbb{N}$ *and let* $0 < c_1 \le c_2 < \infty$. *Then there exists an* $f \in A^s_{pq}$ *with* $\operatorname{supp} f \subset Q_d$ *and*

$$\int x^\beta f(x)\,dx = 0 \quad \text{for } |\beta| \le L - 2 \tag{5}$$

such that (3) *fails for some* j, *with* f^j *given by* (1).

Remark 1.1. We add a few technical explanations. Of course, A^s_{pq} in (3) is always the same space, that is, either B^s_{pq} or F^s_{pq} for all three occurrences. The integrals in (2), (4) and (5) are over $\mathbb{R}^n$. Furthermore, these three moment conditions must be understood in the distributional sense, i.e. $D^\beta \hat{f}(0) = 0$, where $\hat{f}$ is the Fourier transform of f, and $x^\beta = x_1^{\beta_1} \ldots x_n^{\beta_n}$ has the usual meaning for the multi-index $\beta = (\beta_1, \ldots, \beta_n)$. Of course, $L = -1$ in (2) means that no moment conditions are required. In the same way, $L - 1 = -1$ in (4) and $L - 2 = -1$ in (5) indicates that there exist counterexamples to (3) with no moment conditions. Finally, $\mathbb{N}$ and $\mathbb{N}_0$ stand for the natural numbers and the non-negative integers, respectively.

Remark 1.2. If $\sigma_p - s$ is not an integer then (ii) shows that condition (2) is sharp. There are no counterexamples in the delicate limiting case $\sigma_p = s$. If $\sigma_p - s \in \mathbb{N}$ then there is a gap of length 1 between (2) and (5).

Remark 1.3. Constructions of type (1) are now rather fashionable: a generating function which is dyadically dilated and translated. This is a typical procedure in connection with wavelets, spline bases, and, in a more qualitative version, atomic representations of elements of some function spaces. We refer to [3, 4, 5, 1] and [9; 1.9.2, 1.9.4, 3.2].

Remark 1.4. Of course, (3) is obvious for L_p and, more generally, for the Sobolev spaces W^k_p with $k \in \mathbb{N}_0$ and $1 < p < \infty$. On that basis we proved (3) in [10; 3.1.1] via interpolation, duality and some atomic representations for the fractional Sobolev spaces H^s_p and the special Besov spaces $B^s_p = B^s_{pp}$ with $s \in \mathbb{R}$ and $1 < p < \infty$. In [10; 4.4.3] we used this result to obtain estimates from below for approximation numbers and entropy numbers of compact embeddings between function spaces of the above type defined on bounded domains in $\mathbb{R}^n$. In other words, the equivalence relation (3) is useful in proving "only if" parts (estimates from below) in related theorems by reducing these problems to l_p (or better, to their finite-dimensional counterparts l^N_p). In turn, these sharp estimates, especially for entropy numbers, proved to be a decisive instrument to obtain rather sharp assertions for the distributions of eigenvalues of some degenerate elliptic differential operators (see [2]). A second useful application of (3) is connected with sharp Hölder inequalities of the type

$$A^s_{p_1,q_1} A^s_{p_2,q_2} \subset A^s_{pq} \tag{6}$$

where $s > 0$ is given and where one asks for sharp conditions on the p's and q's for (6) to hold. Again we proved in [6; 4.2, 5.5] the "only if" parts of corresponding results on the basis of a forerunner of part (i) of the above theorem (in an unpublished preprint version of this paper). In other words, the above theorem is not only of interest for its own sake, but it is also a powerful tool to reduce "only if" parts of theorems of the sketched type to the l_p-level.

The plan of the paper is simple. In Section 2 we recall very briefly the definition of B^s_{pq} and F^s_{pq}, and we prove a proposition about homogeneity properties of these inhomogeneous spaces which are of independent interest. The proof of the theorem is then given in Section 3.

2. Preliminaries

2.1. *Definitions*. Let $\mathbb{R}^n$ be the Euclidean n-space. The Schwartz space $S(\mathbb{R}^n)$ and its dual space $S'(\mathbb{R}^n)$ of all complex-valued tempered distributions have the usual meaning here. All spaces in this paper are defined on $\mathbb{R}^n$, so we omit "$\mathbb{R}^n$" in the sequel and write simply S, S' etc. Furthermore, L_p with $0 < p \le \infty$ is the usual quasi-Banach space with respect to the Lebesgue measure, quasi-normed by $\|\cdot \mid L_p\|$. Let $\varphi_0 \in S$ be such that

$$\operatorname{supp}\varphi_0 \subset \{y \in \mathbb{R}^n : |y| < 2\} \quad \text{and} \quad \varphi_0(x) = 1 \quad \text{if } |x| \le 1\,, \tag{7}$$

and let $\varphi_j(x) = \varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x)$ for each $j \in \mathbb{N}$. Then since $1 = \sum_{j=0}^\infty \varphi_j(x)$ for all $x \in \mathbb{R}^n$, the φ_j form a dyadic resolution of unity. Let $\hat{f}$ and $\check{f}$ be the Fourier transform and its inverse, respectively, of $f \in S'$. Then $(\varphi_j \hat{f})^\vee$ is an entire analytic function on $\mathbb{R}^n$ for any $f \in S'$.

DEFINITION. (i) Let $s \in \mathbb{R}$, $0 < p \le \infty$ and $0 < q \le \infty$. Then B^s_{pq} is the collection of all $f \in S'$ such that

(8) $$\|f \mid B^s_{pq}\|_\varphi = \Big(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j \hat{f})^\vee \mid L_p\|^q\Big)^{1/q}$$

(with the usual modification if $q=\infty$) is finite.

(ii) Let $s \in \mathbb{R}$, $0<p<\infty$ and $0<q\le\infty$. Then F^s_{pq} is the collection of all $f \in S'$ such that

(9) $$\|f \mid F^s_{pq}\|_\varphi = \Big\|\Big(\sum_{j=0}^{\infty} 2^{jsq} |(\varphi_j \hat{f})^\vee(\cdot)|^q\Big)^{1/q} \Big| L_p\Big\|$$

(with the usual modification if $q=\infty$) is finite.

Remark 2.1. The theory of these spaces has been developed systematically in [8, 9]. In particular, both B^s_{pq} and F^s_{pq} are quasi-Banach spaces which are independent of $\varphi_0 \in S$ chosen according to (7). This justifies our omission of the subscript φ in (8) and (9) in what follows. If $p \ge 1$ and $q \ge 1$, then both B^s_{pq} and F^s_{pq} are Banach spaces. As mentioned in the introduction, these two scales cover many well-known classical spaces.

2.2. *Homogeneity properties*. As in the introduction and in the formulation of the theorem, A^s_{pq} stands either for B^s_{pq} or for F^s_{pq}.

PROPOSITION. (i) *Let $0<p\le\infty$ ($p<\infty$ in the F-case), $0<q\le\infty$ and $s>\sigma_p = \max(0, n(1/p-1))$. There exists a constant $c>0$ such that for all $f \in A^s_{pq}$ and all $R \ge 1$,*

(10) $$\|f(R\cdot) \mid A^s_{pq}\| \le cR^{s-n/p}\|f \mid A^s_{pq}\|.$$

(ii) *Let $0<p\le\infty$ ($p<\infty$ in the F-case), $0<q\le\infty$ and $s<0$. Then there exists a constant $c>0$ such that for all $f\in A^s_{pq}$ and all $0<R\le 1$,*

(11) $$\|f(R\cdot) \mid A^s_{pq}\| \le cR^{s-n/p}\|f \mid A^s_{pq}\|.$$

Remark 2.2. Of course, A^s_{pq} in (10) and (11) stands either for B^s_{pq} on both sides or for F^s_{pq} on both sides. The restrictions $R\ge 1$ and $0<R\le 1$ in (i) and (ii), respectively, come from the inhomogeneity of the spaces B^s_{pq} and F^s_{pq} given by the terms with $j=0$ in (8) and (9). Furthermore, one can ask whether $s>\sigma_p$ in (i) and $s<0$ in (ii) are natural. We shall not discuss this point in detail. However, in the course of the proof of parts (ii) and (iii) of the Theorem in 3.9, formula (62), we disprove

(12) $$\|f(R\cdot) \mid A^s_{pq}\| \le cR^{s-n/p}\|f \mid A^s_{pq}\| \quad \text{for all } R\ge 1 \text{ and all } f \in A^s_{pq}$$

if $0<p<1$ and $s<n(1/p-1)$. But this makes it clear that at least the most suspicious restriction in (i) is natural. By similar arguments one can see that the remaining restrictions $s>0$ in (i) and $s<0$ in (ii) are also natural.

Proof. Step 1. We prove (i) for $A^s_{pq} = F^s_{pq}$. The proof for $A^s_{pq} = B^s_{pq}$ is the same. Let $\varphi(x) = \varphi_1(x)$ where φ_1 has the same meaning as in Definition 2.1. Since $s>\sigma_p$,

(13) $$\|f \mid L_p\| + \Big\|\Big(\int_0^\infty t^{-sq}|(\varphi(t\cdot)\hat{f})^\vee(\cdot)|^q \frac{dt}{t}\Big)^{1/q} \Big| L_p\Big\|$$

is an equivalent quasi-norm in F^s_{pq} (see [9; 2.3.3, p. 99]). Furthermore, by elementary calculations we have

(14) $$(\varphi(t\cdot) f(R\cdot)^\wedge(\cdot))^\vee(x) = (\varphi(t\cdot)R^{-n}\hat{f}(R^{-1}\cdot))^\vee(x) = (\varphi(Rt\cdot)\hat{f}(\cdot))^\vee(Rx).$$

We use (13) with $f(Rx)$ in place of $f(x)$, insert (14), and obtain

(15) $$\|f(R\cdot) \mid F^s_{pq}\| \le cR^{-n/p}\|f \mid L_p\| + cR^{s-n/p}\Big\|\Big(\int_0^\infty t^{-sq}|(\varphi(t\cdot)\hat{f})^\vee(\cdot)|^q \frac{dt}{t}\Big)^{1/q} \Big| L_p\Big\|.$$

Then (10) with F^s_{pq} follows from $s>0$, $R\ge 1$ and the equivalent quasi-norm (13).

Step 2. We prove (ii) for $A^s_{pq} = F^s_{pq}$. The proof for $A^s_{pq} = B^s_{pq}$ is the same. By [9; 2.4.1, p. 100],

(16) $$\|(\varphi_0\hat{f})^\vee \mid L_p\| + \Big\|\Big(\int_0^1 t^{-sq}|(\varphi(t\cdot)\hat{f})^\vee(\cdot)|^q \frac{dt}{t}\Big)^{1/q} \Big| L_p\Big\|$$

is an equivalent quasi-norm in F^s_{pq}, where φ_0 and φ have the above meaning. By (14) the second term in (16) with $f(R\cdot)$ in place of f equals

(17) $$R^{s-n/p}\Big\|\Big(\int_0^R t^{-sq}|(\varphi(t\cdot)\hat{f})^\vee(\cdot)|^q \frac{dt}{t}\Big)^{1/q} \Big| L_p\Big\|.$$

Since $R\le 1$, the integral over $(0,R)$ can be estimated from above by the integral over $(0,1)$ and hence by the second term in (16) multiplied with $R^{s-n/p}$. We estimate the first term in (16) with $f(R\cdot)$ in place of f. Let $2^{k-1} \le R^{-1} \le 2^k$ for some $k \in \mathbb{N}$. Then we have $\varphi_0(Rx) = \sum_{j=0}^{k+2}\varphi_0(Rx)\varphi_j(x)$ and by (14) with φ_0 in place of $\varphi(t\cdot)$,

(18) $$\|(\varphi_0 f(R\cdot)^\wedge)^\vee \mid L_p\| \le R^{-n/p}\Big\|\sum_{j=0}^{k+2}|(\varphi_0(R\cdot)\varphi_j(\cdot)\hat{f})^\vee(\cdot)| \Big| L_p\Big\|.$$

By the Fourier multiplier theorem in [8; 1.6.3, p. 31] the right-hand side of

(18) can be estimated from above by

$$cR^{-n/p}\Big\| \sum_{j=0}^{k+2} |(\varphi_j \hat{f})^\vee(\cdot)| \,\Big|\, L_p \Big\|, \tag{19}$$

which, in turn, since $s<0$, can be estimated from above by

$$cR^{-n/p}2^{-ks}\| \sup_{0\le j\le k} 2^{js}|(\varphi_j\hat{f})^\vee(\cdot)| \mid L_p\| \le cR^{s-n/p}\|f \mid F^s_{pq}\|. \tag{20}$$

Now (11) with $A^s_{pq}=F^s_{pq}$ follows from (18)–(20) and from what was said after (17).

3. Proof of the Theorem

3.1. *A preparation.* First we prove the right-hand inequality of (3). For this purpose we need a preparation. By (1) we have

$$f^j(2^{-j-1}x) = \sum_{k\in\mathbb{Z}^n} a_k f(x-2k), \qquad a_k\in\mathbb{C}, \tag{21}$$

with $f\in A^s_{pq}$ and $\operatorname{supp} f\subset Q_d$ in accordance with the theorem. We claim that

$$\Big\| \sum_{k\in\mathbb{Z}^n} a_k f(\cdot-2k) \,\Big|\, A^s_{pq}\Big\| \sim \Big(\sum_{k\in\mathbb{Z}^n}|a_k|^p\Big)^{1/p}\|f \mid A^s_{pq}\|, \tag{22}$$

where the equivalence relation "$\sim$" means that each side of (22) can be estimated (from above) by the other side times a constant which is independent of f and $\{a_k\}$. Since the $f(x-2k)$ have disjoint supports in unit cubes centered at $2k$, the relation (22) with $A^s_{pq}=F^s_{pq}$ follows immediately from the localization property of the spaces F^s_{pq} (see [9; 2.4.7, p. 124]). To prove (22) for $A^s_{pq}=B^s_{pq}$ we use the characterization of B^s_{pq} via local means (see [9; 2.5.3, p. 138]). Let K_0 be a C^∞-function in $\mathbb{R}^n$ with

$$\operatorname{supp} K_0\subset\{y\in\mathbb{R}^n : |y|<c\}, \qquad \hat{K}_0(0)\ne 0, \tag{23}$$

for some $c>0$, let $K(x)=(\sum_{l=1}^n \partial^2/\partial x_l^2)^N K_0(x)$ for some $N\in\mathbb{N}$ and let

$$K(t,g)(x) = \int_{\mathbb{R}^n} K(y)g(x+ty)\,dy, \qquad 0<t\le 1, \tag{24}$$

with its obvious counterpart $K_0(1,f)$ (local means). If $2N>\max(s,\sigma_p)$ then

$$\|g \mid B^s_{pq}\| \sim \|K_0(1,g) \mid L_p\| + \Big(\int_0^1 t^{-sq}\|K(t,g) \mid L_p\|^q \frac{dt}{t}\Big)^{1/q} \tag{25}$$

in the sense of equivalent quasi-norms. The proof in [9] shows that we may assume that $c>0$ in (23) is small. We insert (21) in (24) and obtain, by the

support properties of $f(x-2k)$,

$$K\Big(t,\sum_k a_k f(\cdot-2k)\Big)(x) = \sum_k a_k K(t,f)(x-2k), \tag{26}$$

$$\Big\|K\Big(t,\sum_k a_k f(\cdot-2k)\Big) \,\Big|\, L_p\Big\|^p = \|K(t,f) \mid L_p\|^p \sum_k |a_k|^p \tag{27}$$

and finally (22) with $A^s_{pq}=B^s_{pq}$.

3.2. *The right-hand inequality of* (3). Let $s<0$. Using (22), (21), and (11) we obtain

$$\begin{aligned}\Big(\sum_k |a_k|^p\Big)^{1/p}\|f \mid A^s_{pq}\| &\le c\|f^j(2^{-j-1}\cdot) \mid A^s_{pq}\| \\ &\le c' 2^{-j(s-n/p)}\|f^j \mid A^s_{pq}\|.\end{aligned} \tag{28}$$

This is the right-hand inequality of (3) if $s<0$. Let now $s\ge 0$ and $s-m<0$ for some $m\in\mathbb{N}$. We use

$$\|f^j \mid A^s_{pq}\| \sim \|f^j \mid A^{s-m}_{pq}\| + \sum_{|\alpha|=m}\|D^\alpha f^j \mid A^{s-m}_{pq}\| \tag{29}$$

(see [8; 2.3.8, p. 59]). By (28) with $s-m$ in place of s and $(D^\alpha f^j)(x) = (D^\alpha f)^j(x)2^{(j+1)m}$ we have

$$\begin{aligned}\|f^j \mid A^s_{pq}\| &\ge c\Big(\sum_k |a_k|^p\Big)^{1/p}\Big(2^{j(s-m-n/p)}\|f \mid A^{s-m}_{pq}\| \\ &\qquad + \sum_{|\alpha|>m} 2^{jm}2^{j(s-m-n/p)}\|D^\alpha f \mid A^{s-m}_{pq}\|\Big) \\ &\ge c\Big(\sum_k |a_k|^p\Big)^{1/p} 2^{j(s-n/p)}\sum_{|\alpha|=m}\|D^\alpha f \mid A^{s-m}_{pq}\|.\end{aligned} \tag{30}$$

If $\operatorname{supp} g\subset Q$, the unit cube, then

$$\|g \mid A^{s-m}_{pq}\| \le c\sum_{|\alpha|=m}\|D^\alpha g \mid A^{s-m}_{pq}\|, \tag{31}$$

where c is independent of g. The proof of (31) is standard. Assume there does not exist a constant $c>0$ such that (31) holds for all g with $\operatorname{supp} g\subset Q$. Then we find a sequence $\{g_l\}_{l=1}^\infty$ with

$$\begin{gathered}\operatorname{supp} g_l\subset Q, \qquad \|g_l \mid A^{s-m}_{pq}\| = 1, \\ \sum_{|\alpha|=m}\|D^\alpha g_l \mid A^{s-m}_{pq}\| \to 0 \quad \text{as } l\to\infty.\end{gathered} \tag{32}$$

Then $\{g_l\}$ is bounded in A^s_{pq} and hence pre-compact in A^{s-m}_{pq}. By the obvious counterpart of (29) and the last part of (32) it follows that $\{g_l\}$ is also pre-compact in A^s_{pq}. We may assume $g_l \to g$ in A^s_{pq}. By (32) we have

(33) $$\operatorname{supp} g \subset Q\,, \quad \|g \mid A^{s-m}_{pq}\| = 1 \quad \text{and} \quad D^\alpha g = 0 \text{ if } |\alpha| = m\,.$$

By the last part of (33), g must be a polynomial, which contradicts the first and second parts of (33). This justifies (31). Hence in the last factor in (30) we can add the term $\|f \mid A^{s-m}_{pq}\|$ (with a different constant c in (30)). Then (28) follows from (29) and this modification of (30).

3.3. *The left-hand inequality of* (3): *the case* $s > \sigma_p$. Let $s > \sigma_p$. Then we have $L = -1$ in (2), which means that no moment conditions for f are necessary. By (1) and (10) we have

(34) $$\|f^j \mid A^s_{pq}\| \le c2^{j(s-n/p)} \Big\| \sum_k a_k f(\cdot - 2k) \Big| A^s_{pq} \Big\|\,.$$

Now the left-hand inequality of (3) follows from (34) and (22).

3.4. *Atoms.* To prove the left-hand inequality of (3) also for $s < \sigma_p$ we need atomic representations of B^s_{pq} and F^s_{pq}. We recall the necessary notations and results in a form which is convenient for us. Let $\nu \in \mathbb{N}_0$ and $k \in \mathbb{Z}^n$. Then $Q_{\nu k}$ stands for the cube in $\mathbb{R}^n$ centered at $2^{-\nu}k$ with side-length $2^{-\nu}$. Let $\chi_{\nu k}(x)$ be the characteristic function of $Q_{\nu k}$ and let

(35) $$\chi^{(p)}_{\nu k}(x) = 2^{\nu n/p}\chi_{\nu k}(x)\,, \qquad 0 < p \le \infty\,,$$

be the L_p-normalized characteristic function of $Q_{\nu k}$. Let $\lambda = \{\lambda_{\nu k} \in \mathbb{C} : \nu \in \mathbb{N}_0, k \in \mathbb{Z}^n\}$. We introduce the sequence spaces

(36) $$\|\lambda \mid b_{pq}\| = \Big(\sum_\nu \Big(\sum_k |\lambda_{\nu k}|^p\Big)^{q/p}\Big)^{1/q}, \qquad 0 < p \le \infty\,,\ 0 < q \le \infty\,,$$

and

(37) $$\|\lambda \mid f_{pq}\| = \Big\| \Big(\sum_{\nu,k} |\lambda_{\nu k}\chi^{(p)}_{\nu k}(\cdot)|^q\Big)^{1/q} \Big| L_p \Big\|\,,$$
$$0 < p < \infty\,,\ 0 < q \le \infty\,,$$

with obvious modifications if $p = \infty$ and/or $q = \infty$. Let $rQ_{\nu k}$ be the cube centered at $2^{-\nu}k$ with side-length $r2^{-\nu}$.

1_K-*atoms.* Let $K \in \mathbb{N}_0$. Then a function $b(x)$ is called a 1_K-*atom* if $\operatorname{supp} b \subset 5Q_{0k}$ for some $k \in \mathbb{Z}^n$ and

(38) $$|D^\alpha b(x)| \le 1 \quad \text{if } |\alpha| \le K\,.$$

$(s,p)_{K,L}$-*atoms.* Let $K \in \mathbb{N}_0$, $L+1 \in \mathbb{N}_0$, $s \in \mathbb{R}$ and $0 < p \le \infty$. Then a function $b(x)$ is called an $(s,p)_{K,L}$-*atom* if

(39) $$\operatorname{supp} b \subset 5Q_{\nu m} \quad \text{for some } \nu \in \mathbb{N}_0 \text{ and } k \in \mathbb{Z}^n\,,$$

(40) $$|D^\alpha b(x)| \le |Q_{\nu k}|^{-1/p+s/n-|\alpha|/n} = 2^{-\nu(s-n/p)+\nu|\alpha|} \quad \text{if } |\alpha| \le K$$

and

(41) $$\int x^\beta b(x)\,dx = 0 \quad \text{if } |\beta| \le L$$

(again $L = -1$ means that there are no moment conditions).

Atomic representations for B^s_{pq}. Let $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$, $K \ge ([s]+1)_+$ and $L \ge \max(-1, [\sigma_p - s])$. Then $f \in B^s_{pq}$ if and only if f can be represented as

(42) $$f = \sum_{k\in\mathbb{Z}^n} \Big(\lambda_k b_k(x) + \sum_{\nu=0}^\infty \lambda_{\nu k} b_{\nu k}(x)\Big)$$

where $b_k(x)$ and $b_{\nu k}(x)$ are 1_K-atoms located in Q_{0k} and $b_{\nu k}(x)$ are $(s,p)_{K,L}$-atoms located in $Q_{\nu k}$, respectively, and

(43) $$\Big(\sum_k |\lambda_k|^p\Big)^{1/p} + \|\lambda \mid b_{pq}\| < \infty\,.$$

The infimum over all quasi-norms (43) with respect to all possible representations (42) is an equivalent quasi-norm in B^s_{pq}.

Atomic representations for F^s_{pq}. Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$, $\sigma_{pq} = \max(0, -n + n/\min(p,q))$,

(44) $$K \ge ([s]+1)_+\,, \qquad L \ge \max(-1, [\sigma_{pq} - s])\,.$$

Then $f \in F^s_{pq}$ if and only if f can be represented by (42) and

(45) $$\Big(\sum_k |\lambda_k|^p\Big)^{1/p} + \|\lambda \mid f_{pq}\| < \infty\,.$$

The infimum over all quasi-norms (45) with respect to all possible representations (42) is an equivalent quasi-norm in F^s_{pq}.

The theory of these atomic decompositions has been developed essentially in [3], [4] (see also [5; Section 5], [7] and [9; 1.9]).

3.5. *The left-hand inequality of* (3): *the case* B^s_{pq} *with* $s \le \sigma_p$. Since f is supported near the origin we may assume that an optimal atomic decomposition of f in the sense of (42) has the form

(46) $$f = \lambda_0 b_0(x) + \sum_{k\in\mathbb{Z}^n} \sum_{\nu=0}^\infty \lambda_{\nu k} b_{\nu k}(x)\,,$$

where $b_0(x)$ is the only 1_K-atom needed, located near the origin. The moment conditions in (2) and those in (46) for the $(s,p)_{K,L}$-atoms may be assumed to be the same. Then $b_0(x)$ also satisfies these moment conditions and may be incorporated in the sum in (46). Hence we may assume $\lambda_0 = 0$

and

$$\|f \mid B^s_{pq}\| \sim \|\lambda \mid b_{pq}\|. \tag{47}$$

By (1) and (46) with $\lambda_0 = 0$ we have

$$f^j(x) = \sum_{l \in \mathbb{Z}^n} a_l \sum_{k,\nu} \lambda_{\nu k} 2^{(j+1)(s-n/p)} b_{\nu k}(2^{j+1}(x - x^{l,j})) 2^{-j(s-n/p)}. \tag{48}$$

Since $\operatorname{supp} f \subset Q_d$ with $d > 0$ small we may assume that all non-vanishing terms in (46) are located near the origin, say, within $Q_{1/2}$. Hence the atoms $2^{-(j+1)(s-n/p)} b_{\nu k}(2^{j+1}(x - x^{l,j}))$ belonging to different l-terms have disjoint supports. In other words, (48) is an atomic representation of $f^j(x)$ and we have

$$\|f^j \mid B^s_{pq}\| \le c 2^{j(s-n/p)} \Big(\sum_{\nu=0}^{\infty} \Big(\sum_{l,k} |a_l|^p |\lambda_{\nu k}|^p \Big)^{q/p} \Big)^{1/q} \tag{49}$$
$$= c 2^{j(s-n/p)} \Big(\sum_l |a_l|^p \Big)^{1/p} \|\lambda \mid b_{pq}\|.$$

Now the left-hand inequality of (3) follows from (47) and (49).

3.6. *The left-hand inequality of* (3): *the case* F^s_{pq} *with* $s \le \sigma_p$, *I.* We proceed as in 3.5. But there is an additional difficulty since the assumed moment conditions in (2) and those needed in (41) and (44) are different if $q < p$. In that case we assume temporarily that (2) holds with $L = [\sigma_{pq} - s]$ (and $s \le \sigma_p$). Then we have (46) again with $\lambda_0 = 0$, the atomic representation (48) and

$$\|f \mid F^s_{pq}\| \sim \|\lambda \mid f_{pq}\|. \tag{50}$$

Now again, since different l-terms in (48) have disjoint supports, (37) yields the counterpart of (49),

$$\|f^j \mid F^s_{pq}\| \le c 2^{j(s-n/p)} \Big(\sum_l |a_l|^p \Big)^{1/p} \|\lambda \mid f_{pq}\|. \tag{51}$$

Now the left-hand inequality of (3) follows from (50) and (51).

3.7. *The left-hand inequality of* (3): *the case* F^s_{pq} *with* $s \le \sigma_p$, *II.* Now let $L = [\sigma_p - s]$ as assumed in (2). Let temporarily $\widetilde{L} = [\sigma_{pq} - s] \ge L$ be the number from 3.6. (Of course, besides $s \le \sigma_p$ we may assume $q < p$.) We have (46) where $b_{\nu k}(x)$ are $(s,p)_{K,\widetilde{L}}$-atoms. Then $b_0(x)$ is an $(s,p)_{K,L}$-atom located in $Q_{0,0}$. We assume that we have an optimal atomic decomposition, hence

$$|\lambda_0| + \|\lambda \mid f_{pq}\| \sim \|f \mid F^s_{pq}\|. \tag{52}$$

We write

$$f = f_1 + f_2 \quad \text{with } f_1(x) = \lambda_0 b_0(x). \tag{53}$$

We apply 3.6 to f_2. Then we obtain, by (51) and (52),

$$\|f_2^j \mid F^s_{pq}\| \le c 2^{j(s-n/p)} \Big(\sum_l |a_l|^p \Big)^{1/p} \|f \mid F^s_{pq}\|. \tag{54}$$

For f_1^j we may use 3.5 to obtain

$$\|f_1^j \mid B^s_{pq}\| \le c|\lambda_0| 2^{j(s-n/p)} \Big(\sum_k |a_k|^p \Big)^{1/p} \|b_0 \mid B^s_{pq}\|. \tag{55}$$

However, we have $\|b_0 \mid B^s_{pq}\| \le c$ uniformly for all admissible atoms b_0. Hence by (52) and $B^s_{pq} \subset F^s_{pq}$ it follows that

$$\|f_1^j \mid F^s_{pq}\| \le c 2^{j(s-n/p)} \Big(\sum_k |a_k|^p \Big)^{1/p} \|f \mid F^s_{pq}\|. \tag{56}$$

Now the left-hand inequality of (3) with $A^s_{pq} = F^s_{pq}$ follows from (53), (54) and (56). The proof of part (i) of the Theorem is complete.

3.8. *Parts* (ii) *and* (iii) *of the Theorem: the case* $1 \le p \le \infty$. Let $s < 0$, $1 \le p \le \infty$ ($p < \infty$ in the F-case) and $0 < q \le \infty$. Let $n = 1$ and let $\chi(x)$ be the characteristic function of the interval $[0,1]$. Then $\chi \in A^s_{pq}$ since $L_p \subset A^s_{pq}$. Let $m \in \mathbb{N}_0$. Then $f(x) = (d^m/dx^m)\chi(x) \in A^s_{pq}$ if $s < -m$. Let $x^{k,j} = 2^{-j}k$. Then we have, in accordance with (1),

$$f^j(x) = \sum_{k=0}^{2^j-1} f(2^j(x - x^{k,j})) \tag{57}$$
$$= 2^{-jm} \frac{d^m}{dx^m} \Big(\sum_{k=0}^{2^j-1} \chi(2^j(x - x^{k,j})) \Big) = 2^{-jm} f(x).$$

Assume that (3) holds. Then we have

$$2^{-jm}\|f \mid A^s_{pq}\| = \|f^j \mid A^s_{pq}\| \tag{58}$$
$$\le c 2^{j(s-1/p)} \Big(\sum_{k=0}^{2^j-1} 1 \Big)^{1/p} \|f \mid A^s_{pq}\| \le c' 2^{js} \|f \mid A^s_{pq}\|.$$

We obtain a contradiction as $j \to \infty$ since $s < -m$. Of course,

$$\int x^\beta f(x)\, dx = 0 \quad \text{if } |\beta| \le m - 1. \tag{59}$$

We have $\sigma_p = 0$ since $1 \le p \le \infty$ and hence $L = [-s] = m$ if $-m - 1 < s < -m$ in accordance with (ii) of the Theorem. Thus (59) coincides with (4) and we have the desired counterexample in that special case. If $s = -m - 1$ then

$L = m+1$ and (59) coincides with (5). If the number d used in (1) is such that the above arguments cannot be applied immediately, then we decompose $\chi(x)$ in a finite sum (in dependence on d), apply (3) to each component and sum up. We arrive again at (58) and the above contradiction. Upon replacing the interval $[0,1]$ by the unit cube in $\mathbb{R}^n$, there are no problems in extending these arguments from $n=1$ to $n>1$, which completes the proof of (ii) and (iii) if $1 \le p \le \infty$.

3.9. *The parts* (ii) *and* (iii) *of the Theorem: the case* $0<p<1$. Let $0<p<1$, $0<q\le\infty$ and $s<\sigma_p = n(1/p-1)$. Then we have $L = [n(1/p-1)-s] \in \mathbb{N}_0$ in (4) and $L \in \mathbb{N}$ in (5). Let $f \in A^s_{pq}$ with a compact support such that (4) resp. (5) holds, but

(60) $$(D^\alpha \hat{f})(0) \neq 0 \quad \text{for some } \alpha \text{ with } |\alpha| = L\,, \text{ resp. } |\alpha| = L-1$$

(see also Remark 1.1 for technical explanations). Then $\hat{f}(x)$ is an entire analytic function with the Taylor expansion

(61) $$\hat{f}(x) = \sum_{|\alpha|\ge k} a_\alpha x^\alpha \quad \text{with } k = L \text{ resp. } k = L-1\,,$$

convergent in $\mathbb{R}^n$. We assume that (3) holds with only one summand, hence

(62) $$\|f(2^j\cdot) \mid A^s_{pq}\| \le c 2^{j(s-n/p)} \|f \mid A^s_{pq}\|\,.$$

Then we have, by (8) and (9) with $\varphi = \varphi_0$,

(63) $$\|(\varphi f(2^j\cdot)^\wedge)^\vee \mid L_p\| \le c 2^{j(s-n/p)} \|f \mid A^s_{pq}\|\,.$$

Furthermore, we obtain

(64) $$\begin{aligned}\varphi(x) f(2^j\cdot)^\wedge(x) &= 2^{-jn}\varphi(x)\hat{f}(2^{-j}x)\\ &= \varphi(x) 2^{-jn-jk} \sum_{|\alpha|\ge k} a_\alpha 2^{-(|\alpha|-k)j} x^\alpha\,,\end{aligned}$$

(65) $$\varphi(x) \sum_{|\alpha|\ge k} a_\alpha 2^{-j(|\alpha|-k)} x^\alpha \to \varphi(x) \sum_{|\alpha|=k} a_\alpha x^\alpha \quad \text{in } S$$

as $j\to\infty$ and hence

(66) $$\Big(\varphi(x) \sum_{|\alpha|\ge k} a_\alpha 2^{-j(|\alpha|-k)} x^\alpha\Big)^\vee(\xi) \to (-i)^k \sum_{|\alpha|=k} a_\alpha D^\alpha \check{\varphi}(\xi) \quad \text{in } S$$

as $j\to\infty$. We assume that the function on the right-hand side of (66) does not vanish identically. Then (63), (65) and (66) yield

(67) $$\Big\| \sum_{|\alpha|=k} a_\alpha D^\alpha \check{\varphi} \;\Big|\; L_p \Big\| \le \lim_{j\to\infty} 2^{j(s-n/p)+j(n+k)} \|f \mid A^s_{pq}\|\,.$$

But this is a contradiction since $k < -s+\sigma_p = -s+n(1/p-1)$.

References

[1] I. Daubechies, *Ten Lectures on Wavelets*, CBMS-NSF Regional Conf. Ser. in Appl. Math. 61, SIAM, Philadelphia, 1992.

[2] D. E. Edmunds and H. Triebel, *Eigenvalue distributions of some degenerate elliptic operators: an approach via entropy numbers*, Math. Ann., to appear.

[3] M. Frazier and B. Jawerth, *Decomposition of Besov spaces*, Indiana Univ. Math. J. 34 (1985), 777–799.

[4] —, —, *A discrete transform and decomposition of distribution spaces*, J. Funct. Anal. 93 (1990), 34–170.

[5] M. Frazier, B. Jawerth and G. Weiss, *Littlewood–Paley Theory and the Study of Function Spaces*, CBMS Regional Conf. Ser. in Math. 79, Amer. Math. Soc., Providence, 1991.

[6] W. Sickel and H. Triebel, *Hölder inequalities and sharp embeddings in function spaces of B^s_{pq} and F^s_{pq} type*, submitted.

[7] R. H. Torres, *Boundedness results for operators with singular kernels on distribution spaces*, Mem. Amer. Math. Soc. 442 (1991).

[8] H. Triebel, *Theory of Function Spaces*, Birkhäuser, Basel, 1983.

[9] —, *Theory of Function Spaces, II*, Birkhäuser, Basel, 1992.

[10] —, *Approximation numbers and entropy numbers of embeddings of fractional Besov–Sobolev spaces in Orlicz spaces*, Proc. London Math. Soc. 66 (1993), 589–618.

MATHEMATISCHES INSTITUT
FAKULTÄT MATHEMATIK-INFORMATIK
UNIVERSITÄT JENA
D-07740 JENA, GERMANY

Received May 31, 1993 (3109)