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The converse of the Hölder inequality and its generalizations

by

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Abstract. Let (Ω,Σ,μ) be a measure space with two sets $A,B\in\Sigma$ such that $0<\mu(A)<1<\mu(B)<\infty$ and suppose that ϕ and ψ are arbitrary bijections of $[0,\infty)$ such that $\phi(0)=\psi(0)=0$. The main result says that if

$$\int_\Omega xy\,d\mu\le\phi^{-1}\Big(\int_\Omega\phi\circ x\,d\mu\Big)\psi^{-1}\Big(\int_\Omega\psi\circ x\,d\mu\Big)$$

for all μ-integrable nonnegative step functions x,y then ϕ and ψ must be conjugate power functions.

If the measure space (Ω,Σ,μ) has one of the following properties:

(a) $\mu(A)\le1$ for every $A\in\Sigma$ of finite measure;
(b) $\mu(A)\ge1$ for every $A\in\Sigma$ of positive measure,

then there exist some broad classes of nonpower bijections ϕ and ψ such that the above inequality holds true.

A general inequality which contains integral Hölder and Minkowski inequalities as very special cases is also given.

Introduction. Let (Ω,Σ,μ) be a measure space. Denote by $\mathbf{S}=\mathbf{S}(\Omega,\Sigma,\mu)$ the linear space of all μ-integrable step functions $x:\Omega\to\mathbb{R}$ and by $\mathbf{S}_+$ the set of all $x\in\mathbf{S}$ such that $x:\Omega\to\mathbb{R}_+$ where $\mathbb{R}_+=[0,\infty)$. One can easily verify that for every bijective function $\phi:\mathbb{R}_+\to\mathbb{R}_+$ such that $\phi(0)=0$ the functional $\mathbf{p}_\phi:\mathbf{S}_+\to\mathbb{R}_+$ given by the formula

$$\mathbf{p}_\phi(x)=\phi^{-1}\Big(\int_\Omega\phi\circ x\,d\mu\Big)\quad(x\in\mathbf{S}_+)\tag{1}$$

is well defined. In a recent paper [8] the author proved the following converse of Minkowski's inequality.

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Suppose that there are two sets $A, B \in \Sigma$ such that

$$0 < \mu(A) < 1 < \mu(B) < \infty . \tag{2}$$

If ϕ^{-1} is continuous at 0 and

$$\mathbf{p}_\phi(x+y) \le \mathbf{p}_\phi(x) + \mathbf{p}_\phi(y) \quad (x, y \in \mathbf{S}_+)$$

then there is a $p \ge 1$ such that $\phi(t) = \phi(1)t^p$ for all $t \ge 0$. Moreover, if the last inequality is reversed, then $\phi(t) = \phi(1)t^p$, $t \ge 0$, for some $p, 0 < p \le 1$.

It was also shown that assumption (2) is essential.

It seems to be quite natural to ask whether a similar result can be proved for Hölder's inequality. More precisely, suppose that ϕ and ψ are bijections of $\mathbb{R}_+$ such that $\phi(0) = \psi(0) = 0$ and condition (2) is satisfied. Does then the inequality

$$\int_\Omega xy\, d\mu \le \mathbf{p}_\phi(x)\mathbf{p}_\psi(y) \quad (x, y \in \mathbf{S}_+) \tag{3}$$

imply that ϕ and ψ are necessarily conjugate power functions?

The main purpose of this paper is to show that the answer is affirmative. This seems to be a little unexpected because the bijections ϕ and ψ are assumed to be unrelated at all. It turns out that assumption (2) can be replaced by the following: the functions ϕ and ψ are *multiplicatively conjugate*, i.e. there exists a constant $c > 0$ such that

$$\phi^{-1}(t)\psi^{-1}(t) = ct \quad (t \ge 0) .$$

Actually, our Theorem 1 extends the relevant result from the book by G. H. Hardy, J. E. Littlewood and G. Pólya [3], p. 82 (cf. also R. Cooper [1] where rather strong regularity conditions are assumed).

We also prove that the existence of two sets $A, B \in \Sigma$ satisfying (2) is essential. Namely, we indicate some broad classes of nonpower functions ϕ and ψ satisfying (3) if this assumption fails to hold.

A suitable theory for the inequality (3) reversed is also given.

At the end of this paper we present a general integral inequality containing Hölder's and Minkowski's inequalities as very special cases and supplying us with "one-line" proofs of them.

1. Auxiliary results.

A crucial part in the proof of the main result is played by

LEMMA 1. *Let a and b be real numbers such that*

$$0 < \min\{a, b\} < 1 < a + b .$$

If a function $f : (0, \infty) \to \mathbb{R}_+$ satisfies the inequality

$$af(s) + bf(t) \le f(as + bt) \quad (s, t > 0) ,$$

then $f(t) = f(1)t$ $(t > 0)$.

Remark 1. Lemma 1, recently proved in [10], gives an affirmative answer to a problem posed in [7] where a weaker result has been proved.

Concerning the reversed inequality we quote the following

LEMMA 2 ([7]). *Let a, b be real numbers such that*

$$0 < \min\{a, b\} < 1 < a + b .$$

If a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is bounded in a neighbourhood of 0, $f(0) = 0$ and

$$f(as + bt) \le af(s) + bf(t) \quad (s, t \ge 0) ,$$

then $f(t) = f(1)t$ $(t \ge 0)$.

We also need the following result on a system of two functional equations.

LEMMA 3 ([6]). *Let a, b, α, β be positive real and suppose that $\gamma : (0, \infty) \to (0, \infty)$ is continuous at least at one point and satisfies the system of functional equations*

$$\gamma(at) = \alpha\gamma(t), \quad \gamma(bt) = \beta\gamma(t) \quad (t > 0) .$$

If $a \ne 1$ and $\frac{\log b}{\log a}$ is irrational then there exists an $r \in \mathbb{R}$ such that $\gamma(t) = \gamma(1)t^r$ for all $t > 0$.

The next lemma is a consequence of a result of Kuhn [5].

LEMMA 4. *Let D be a convex subset of a linear space, $a \in (0, 1)$ and $F : D \to \mathbb{R}$. If*

$$F(ax + (1-a)y) \le aF(x) + (1-a)F(y) \quad (x, y \in D) ,$$

then F is Jensen convex, i.e.

$$F\left(\frac{x+y}{2}\right) \le \frac{F(x) + F(y)}{2} \quad (x, y \in D) .$$

Remark 2. The proof of Kuhn's result, based on an abstract Hahn–Banach theorem, is rather complicated. Therefore it is worth mentioning that Z. Daróczy and Z. Páles [2] found a very simple proof of Lemma 4. Namely, they observed that it immediately follows from the identity

$$\frac{x+y}{2} = a\left(a\frac{x+y}{2} + (1-a)y\right) + (1-a)\left(ax + (1-a)\frac{x+y}{2}\right) .$$

For a measure space (Ω, Σ, μ) denote by $\mathbf{L}^1(\Omega, \Sigma, \mu)$ the linear space of all μ-integrable functions $x : \Omega \to \mathbb{R}$ and by $\mathbf{L}^1_+(\Omega, \Sigma, \mu)$ the set of all positive $x \in \mathbf{L}^1(\Omega, \Sigma, \mu)$.

2. The converse of Hölder's inequality for multiplicatively conjugate functions. We begin this section with the following

THEOREM 1. *Let (Ω,Σ,μ) be a measure space with two disjoint sets $A,B\in\Sigma$ of finite and positive measure. If $\phi,\psi:\mathbb{R}_+\to\mathbb{R}_+$ are bijections such that for some $c>0$,*

$$\phi^{-1}(t)\psi^{-1}(t)=ct \quad (t>0), \tag{4}$$

and

$$\int_\Omega xy\,d\mu\le \mathbf{p}_\phi(x)\mathbf{p}_\psi(y) \quad (x,y\in \mathbf{S}_+), \tag{5}$$

then ϕ and ψ are conjugate power functions, i.e. there are $p,q>1$ such that $p^{-1}+q^{-1}=1$, $\phi(t)=\phi(1)t^p$ and $\psi(t)=\psi(1)t^q$ $(t\ge 0)$.

Proof. Relation (4) implies that $\phi(0)=\psi(0)=0$ and, consequently, the functionals $\mathbf{p}_\phi$ and $\mathbf{p}_\psi$ are well defined. We denote by χ_A the characteristic function of a set A. Put $a=\mu(A)$, $b=\mu(B)$. Setting in (5)

$$x=x_1\chi_A+x_2\chi_B,\quad y=y_1\chi_A+y_2\chi_B \quad (x_i,y_i\ge 0),$$

we get

$$ax_1y_1+bx_2y_2\le \phi^{-1}(a\phi(x_1)+b\phi(x_2))\psi^{-1}(a\psi(y_1)+b\psi(y_2)) \quad (x_i,y_i\ge 0).$$

Replacing x_i by $\phi^{-1}(x_i)$ and y_i by $\psi^{-1}(y_i)$ $(i=1,2)$, we obtain

$$\begin{aligned}a\phi^{-1}(x_1)\psi^{-1}(y_1)+b\phi^{-1}(x_2)\psi^{-1}(y_2)\\ \le \phi^{-1}(ax_1+bx_2)\psi^{-1}(ay_1+by_2)\end{aligned} \tag{6}$$

for all nonnegative x_1,x_2,y_1,y_2. Setting $x_2=y_2=0$ we have

$$a\phi^{-1}(x_1)\psi^{-1}(y_1)\le \phi^{-1}(ax_1)\psi^{-1}(ay_1) \quad (x_1,y_1\ge 0).$$

Since, in view of (4), $\psi^{-1}(t)=ct[\phi^{-1}(t)]^{-1}$ for $t>0$, we get

$$\frac{\phi^{-1}(ay_1)}{\phi^{-1}(y_1)}\le\frac{\phi^{-1}(ax_1)}{\phi^{-1}(x_1)} \quad (x_1,y_1>0).$$

This implies that the function $t\to\phi^{-1}(t)/\phi^{-1}(a^{-1}t)$ is constant in $(0,\infty)$, and, consequently,

$$\frac{\phi^{-1}(a^{-1}x_1)}{\phi^{-1}(a^{-1}y_1)}=\frac{\phi^{-1}(x_1)}{\phi^{-1}(y_1)} \quad (x_1,y_1>0). \tag{7}$$

In the same way we show that

$$\frac{\phi^{-1}(b^{-1}x_2)}{\phi^{-1}(b^{-1}y_2)}=\frac{\phi^{-1}(x_2)}{\phi^{-1}(y_2)} \quad (x_2,y_2>0). \tag{8}$$

From (6) and (4) we obtain

$$ay_1\frac{\phi^{-1}(x_1)}{\phi^{-1}(y_1)}+by_2\frac{\phi^{-1}(x_2)}{\phi^{-1}(y_2)}\le(ay_1+by_2)\frac{\phi^{-1}(ax_1+bx_2)}{\phi^{-1}(ay_1+by_2)}.$$

Replacing x_1,x_2,y_1,y_2 by $a^{-1}x_1,b^{-1}x_2,a^{-1}y_1,b^{-1}y_2$ resp. we get

$$y_1\frac{\phi^{-1}(a^{-1}x_1)}{\phi^{-1}(a^{-1}y_1)}+y_2\frac{\phi^{-1}(b^{-1}x_2)}{\phi^{-1}(b^{-1}y_2)}\le(y_1+y_2)\frac{\phi^{-1}(x_1+x_2)}{\phi^{-1}(y_1+y_2)}.$$

Now from (7) and (8) we obtain the inequality

$$y_1\frac{\phi^{-1}(x_1)}{\phi^{-1}(y_1)}+y_2\frac{\phi^{-1}(x_2)}{\phi^{-1}(y_2)}\le(y_1+y_2)\frac{\phi^{-1}(x_1+x_2)}{\phi^{-1}(y_1+y_2)}, \tag{9}$$

valid for all $x_1,x_2,y_1,y_2>0$. Making again use of (4) we can write this inequality in the following symmetric form:

$$\phi^{-1}(x_1)\psi^{-1}(y_1)+\phi^{-1}(x_2)\psi^{-1}(y_2)\le\phi^{-1}(x_1+x_2)\psi^{-1}(y_1+y_2),$$

and, consequently, we have

$$\phi^{-1}(x_1)\psi^{-1}(y_1)\le\phi^{-1}(x_1+x_2)\psi^{-1}(y_1+y_2) \quad (x_1,x_2,y_1,y_2>0).$$

Now we can prove that ϕ and ψ are homeomorphic in $(0,\infty)$. In view of (4) it is sufficient to show that either ϕ^{-1} or ψ^{-1} is increasing in $(0,\infty)$. Suppose for instance that ψ^{-1} is not increasing in $(0,\infty)$. Then we have $\psi^{-1}(y_1)>\psi^{-1}(y_1+y_2)$ for some positive y_1,y_2 and the last inequality implies that $\phi^{-1}(x_1)<\phi^{-1}(x_1+x_2)$ $(x_1,x_2>0)$, i.e. ϕ^{-1} is increasing in $(0,\infty)$.

From (9), by induction, we obtain

$$y_1\frac{\phi^{-1}(x_1)}{\phi^{-1}(y_1)}+\ldots+y_k\frac{\phi^{-1}(x_k)}{\phi^{-1}(y_k)}\le(y_1+\ldots+y_k)\frac{\phi^{-1}(x_1+\ldots+x_k)}{\phi^{-1}(y_1+\ldots+y_k)}$$

for all positive $x_1,\ldots,x_k$, $y_1,\ldots,y_k$ and $k\in\mathbb{N}$. Setting $x_1=\ldots=x_k=s$, $y_1=\ldots=y_k=t$, we get

$$\frac{\phi^{-1}(kt)}{\phi^{-1}(t)}\le\frac{\phi^{-1}(ks)}{\phi^{-1}(s)} \quad (s,t>0;\ k\in\mathbb{N}).$$

It follows that for every $k\in\mathbb{N}$ the function $t\to\phi^{-1}(kt)/\phi^{-1}(t)$ $(t>0)$ is constant. Hence for every $k\in\mathbb{N}$ there is $\alpha_k>0$ such that

$$\phi^{-1}(kt)=\alpha_k\phi^{-1}(t) \quad (t>0).$$

Taking $k=2$ and $k=3$ we see that $\gamma=\phi^{-1}|_{(0,\infty)}$ satisfies

$$\gamma(2t)=\alpha\gamma(t),\quad \gamma(3t)=\beta\gamma(t) \quad (t>0),$$

where $\alpha=\alpha_2$, $\beta=\alpha_3$. Since γ is continuous and $\frac{\log 3}{\log 2}$ is irrational, Lemma 3 implies that there is a $p\in\mathbb{R}$ such that $\phi^{-1}(t)=\phi^{-1}(1)t^{1/p}$ $(t>0)$. By the monotonicity of ϕ^{-1} we have $p>0$. In the same way one can show that $\psi^{-1}(t)=\psi^{-1}(1)t^{1/q}$ $(t>0)$ for some $q>0$. In view of (4) we have $p^{-1}+q^{-1}=1$. This completes the proof.

Modifying the above reasoning in an obvious way we can prove

THEOREM 2. *Let* (Ω, Σ, μ) *be a measure space with two disjoint sets of finite and positive measure. If* $\phi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ *are bijections of* $\mathbb{R}_+$ *such that* (4) *holds and*

$$\mathbf{p}_\phi(x)\mathbf{p}_\psi(y) \le \int_\Omega xy\,d\mu \qquad (x, y \in \mathbf{S}_+),$$

then there exist $p, q \in \mathbb{R}$ *such that* $\phi(t) = \phi(1)t^p$ *and* $\psi(t) = \psi(1)t^q$ $(t > 0)$, $p^{-1} + q^{-1} = 1$, $pq < 0$.

Remark 3. Suppose that $\phi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy the following conditions:

(i) the function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ given by $\Phi(t) = t^{-1}\phi(t)$ for $t > 0$ and $\Phi(0) = 0$ is increasing and continuous in $\mathbb{R}_+$ and twice continuously differentiable in $(0, \infty)$;

(ii) the function $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ given by $\Psi(t) = t^{-1}\psi(t)$ for $t > 0$ and $\Psi(0) = 0$ is the inverse of Φ;

(iii) for every $n \in \mathbb{N}$ and for all positive $q_1, \ldots, q_n$ such that $q_1 + \ldots + q_n = 1$ we have

$$q_1x_1y_1 + \ldots + q_nx_ny_n \le \phi^{-1}(q_1\phi(x_1) + \ldots + q_n\phi(x_n))\psi^{-1}(q_1\psi(x_1) + \ldots + q_n\psi(x_n))$$

for all nonnegative $x_1, \ldots, x_n$, $y_1, \ldots, y_n$.

Under these assumptions G. H. Hardy, J. E. Littlewood and G. Pólya (cf. [3], p. 82, Theorem 101(a)) proved that ϕ and ψ are conjugate power functions. Because condition (ii) implies that $\phi^{-1}(t)\psi^{-1}(t) = t$ $(t \ge 0)$, this is a special case of Theorem 1. Moreover, the condition of "multiplicative conjugacy" seems to be more convenient than (ii).

3. The main theorem. The main result of this paper reads as follows.

THEOREM 3. *Suppose that* (Ω, Σ, μ) *is a measure space with two sets* $A, B \in \Sigma$ *such that*

$$0 < \mu(A) < 1 < \mu(B) < \infty.$$

If $\phi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ *are bijections such that* $\phi(0) = \psi(0) = 0$ *and*

$$\int_\Omega xy\,d\mu \le \mathbf{p}_\phi(x)\mathbf{p}_\psi(y) \qquad (x, y \in \mathbf{S}_+),$$

then $\phi(t) = \phi(1)t^p$ *and* $\psi(t) = \psi(1)t^q$ $(t \ge 0)$ *for some* $p, q > 1$ *such that* $p^{-1} + q^{-1} = 1$.

Proof. Put $a = \mu(A)$ and $b = \mu(B \setminus A)$. Setting

$$x = x_1\chi_A + x_2\chi_{B\setminus A}, \quad y = y_1\chi_A + y_2\chi_{B\setminus A} \in \mathbf{S}_+$$

we obtain, as in the proof of Theorem 1,

$$a\phi^{-1}(x_1)\psi^{-1}(y_1) + b\phi^{-1}(x_2)\psi^{-1}(y_2) \le \phi^{-1}(ax_1 + bx_2)\psi^{-1}(ay_1 + by_2)$$

for all nonnegative x_1, x_2, y_1, y_2. Define $f : \mathbb{R}_+ \to \mathbb{R}_+$ by $f(t) = \phi^{-1}(t)\psi^{-1}(t)$ $(t \ge 0)$. For $x_1 = y_1 = s$ and $x_2 = y_2 = t$ we hence get $af(s) + bf(t) \le f(as + bt)$ $(s, t > 0)$. Since $0 < a < 1 < a + b$ it follows by Lemma 1 that $f(t) = f(1)t$ $(t > 0)$. Thus ϕ and ψ are multiplicatively conjugate and our result is a consequence of Theorem 1.

Remark 4. It can be easily verified that if $\mu(\Omega) < \infty$ then the functionals $\mathbf{p}_\phi$ and $\mathbf{p}_\psi$ are well defined for all bijections ϕ and ψ of $\mathbb{R}_+$. Therefore in this case the assumption $\phi(0) = \psi(0) = 0$ can be dropped.

Analogously, applying Lemma 2 and Theorem 2, we can prove

THEOREM 4. *Suppose that* (Ω, Σ, μ) *is a measure space with two sets* $A, B \in \Sigma$ *such that* $0 < \mu(A) < 1 < \mu(B) < \infty$. *If* $\phi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ *are bijections such that* $\phi(0) = \psi(0) = 0$ *and the function* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *given by* $f(t) = \phi^{-1}(t)\psi^{-1}(t)$ $(t \ge 0)$ *is bounded in a neighbourhood of* 0 *and*

$$\int_\Omega xy\,d\mu \ge \mathbf{p}_\phi(x)\mathbf{p}_\psi(y) \qquad (x, y \in \mathbf{S}_+),$$

then $\phi(t) = \phi(1)t^p$ *and* $\psi(t) = \psi(1)t^q$ $(t > 0)$ *for some* $p, q \in \mathbb{R}$ *such that* $pq < 0$ *and* $p^{-1} + q^{-1} = 1$.

Remark 5. Note that in Theorems 4 and 2 one of the power functions is increasing in $\mathbb{R}_+$ and the other is decreasing in $(0, \infty)$.

4. Discussion of the assumptions and generalizations of Hölder's inequality. We begin this section with a generalization of Hölder's inequality which shows that the assumption of the existence of a set $B \in \Sigma$ with $\mu(B) > 1$ in Theorem 3 is essential.

THEOREM 5. *Suppose that* (Ω, Σ, μ) *is a normalized measure space* (*i.e.* $\mu(\Omega) = 1$) *with at least one set* $A \in \Sigma$ *such that* $0 < \mu(A) < 1$ *and let* $\phi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ *be arbitrary bijections. Under these conditions,*

$$\int_\Omega xy\,d\mu \le \mathbf{p}_\phi(x)\mathbf{p}_\psi(y) \qquad (x, y \in \mathbf{S}_+)$$

if and only if the function $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ *given by* $F(s,t) = \phi^{-1}(s)\psi^{-1}(t)$ *is concave.*

Proof. Assume that the inequality holds for all $x, y \in \mathbf{S}_+$ and put $B = \Omega \setminus A$, $a = \mu(A)$, $b = \mu(B)$. Substituting

$$x = x_1\chi_A + x_2\chi_B, \quad y = y_1\chi_A + y_2\chi_B \quad (x_i, y_i \geq 0),$$

and then replacing x_1, x_2, y_1, y_2 by $\phi^{-1}(x_1), \phi^{-1}(x_2), \psi^{-1}(y_1), \psi^{-1}(y_2)$ resp. we obtain (6). Since $a+b=1$, inequality (6) says that the function $F(s,t) = \phi^{-1}(s)\psi^{-1}(t)$ $(s,t \geq 0)$ satisfies the concavity condition for one value $a \in (0,1)$:

$$aF(x_1,y_1) + (1-a)F(x_2,y_2) \leq F(ax_1 + (1-a)x_2, ay_1 + (1-a)y_2) \qquad (x_i, y_i \geq 0).$$

In view of Lemma 4, F is Jensen concave. Since it is nonnegative, by the Bernstein–Doetsch Theorem (cf. M. Kuczma [4], p. 145), F is concave in $\mathbb{R}^2_+$. The converse implication is obvious. This completes the proof.

Remark 6. If the reverse inequality holds, then in the same way one can show that F satisfies the convexity condition for at least one value $a = \mu(A)$. To get an analogous result in this case we have to assume additionally that F is bounded above on an open subset of $\mathbb{R}^2_+$ (cf. [4], p. 145, Bernstein–Doetsch Theorem) or that it is measurable (cf. [4], p. 218, Sierpiński's Theorem).

Remark 7. Suppose that $\mu(\Omega) \leq 1$. Then it is easy to verify that concavity of F is a sufficient condition for the inequality of Theorem 5 to hold. With obvious modifications, the same concerns the reverse inequality.

The theorem below also generalizes Hölder's inequality and proves that the assumption of the existence of a set $A \in \Sigma$ such that $0 < \mu(A) < 1$ in Theorem 3 is essential.

THEOREM 6. *Let (Ω, Σ, μ) be a measure space such that for every $A \in \Sigma$, either*

$$(10) \qquad \mu(A) = 0 \quad \text{or} \quad \mu(A) \geq 1,$$

and suppose that ϕ and ψ are homeomorphisms of $\mathbb{R}_+$. If there are $p, q > 0$, $p^{-1} + q^{-1} = 1$, such that one of the following conditions is satisfied:

(i) *the functions $\Phi, \Psi : (0,\infty) \to (0,\infty)$ defined by the formulas*

$$\Phi(t) = t^{-p}\phi(t), \quad \Psi(t) = t^{-q}\psi(t) \quad (t > 0),$$

are nonincreasing;

(ii) *ϕ or ψ is subadditive and*

$$(11) \qquad \phi(a^{1/p}t) \leq a\phi(t), \quad \psi(a^{1/q}t) \leq a\psi(t) \quad (a \in \mu(\Sigma),\ t \geq 0),$$

then

$$\int_\Omega xy\, d\mu \leq \mathbf{p}_\phi(x)\mathbf{p}_\psi(y) \qquad (x, y \in \mathbf{S}_+).$$

Proof. Suppose that (i) holds. Then we have

$$(12) \qquad \phi(a^{1/p}t) \leq a\phi(t), \quad \psi(a^{1/q}t) \leq a\psi(t) \quad (a \geq 1,\ t \geq 0).$$

Since the functions $\Phi(t^{1/p}) = t^{-1}\phi(t^{1/p})$ and $\Psi(t^{1/q}) = t^{-1}\psi(t^{1/q})$ $(t > 0)$, are also nonincreasing, the functions $\phi(t^{1/p})$ and $\psi(t^{1/q})$ are subadditive (cf. for instance [3], p. 239, Theorem 7.2.4(i)). Consequently,

$$\phi\Big[\Big(\sum_{i=1}^n t_i\Big)^{1/p}\Big] \leq \sum_{i=1}^n \phi(t_i^{1/p}) \quad (t_i \geq 0;\ i = 1, \ldots, n).$$

Replacing t_i by t_i^p $(i = 1, \ldots, n)$ and making use of the monotonicity of ϕ we can write this inequality in the form

$$(13) \qquad \Big(\sum_{i=1}^n t_i^p\Big)^{1/p} \leq \phi^{-1}\Big(\sum_{i=1}^n \phi(t_i)\Big) \quad (t_i \geq 0;\ i = 1, \ldots, n).$$

In the same way we get

$$(14) \qquad \Big(\sum_{i=1}^n t_i^q\Big)^{1/q} \leq \psi^{-1}\Big(\sum_{i=1}^n \psi(t_i)\Big) \quad (t_i \geq 0;\ i = 1, \ldots, n).$$

Take now arbitrary $x, y \in \mathbf{S}_+$. Then, by (10), there exist $n \in \mathbb{N}$ and disjoint sets $A_i \in \Sigma$ with $a_i = \mu(A_i) \geq 1$ $(i = 1, \ldots, n)$ such that

$$x = \sum_{i=1}^n x_i\chi_{A_i}, \quad y = \sum_{i=1}^n y_i\chi_{A_i} \quad (x_i, y_i \geq 0;\ i = 1, \ldots, n).$$

The classical Hölder inequality and (12)–(14) imply

$$\begin{aligned}\int_\Omega xy\, d\mu &= \sum_{i=1}^n a_i x_i y_i = \sum_{i=1}^n (a_i^{1/p}x_i)(a_i^{1/q}y_i) \\ &\leq \Big(\sum_{i=1}^n a_i x_i^p\Big)^{1/p}\Big(\sum_{i=1}^n a_i y_i^q\Big)^{1/q} \\ &\leq \phi^{-1}\Big(\sum_{i=1}^n \phi(a_i^{1/p}x_i)\Big)\psi^{-1}\Big(\sum_{i=1}^n \psi(a_i^{1/q}y_i)\Big) \\ &\leq \phi^{-1}\Big(\sum_{i=1}^n a_i\phi(x_i)\Big)\psi^{-1}\Big(\sum_{i=1}^n a_i\psi(y_i)\Big) \\ &= \mathbf{p}_\phi(x)\mathbf{p}_\psi(y),\end{aligned}$$

which completes the proof in this case.

Suppose now that (ii) is satisfied and let, for instance, ψ be subadditive. Hence, since ψ is increasing, ψ^{-1} is superadditive. Writing $x, y \in \mathbf{S}_+$ as above and making use in turn of (11), the monotonicity of ϕ^{-1} and superadditivity of ψ^{-1} we have

$$\begin{aligned}\int_{\Omega} xy\,d\mu &= \sum_{i=1}^{n} a_i x_i y_i = \sum_{i=1}^{n}(a_i^{1/p}x_i)(a_i^{1/q}y_i) \le \sum_{i=1}^{n}\phi^{-1}(a_i\phi(x_i))\psi^{-1}(a_i\psi(y_i))\\ &\le \sum_{i=1}^{n}\phi^{-1}\Big(\sum_{j=1}^{n}a_j\phi(x_j)\Big)\psi^{-1}(a_i\psi(y_i))\\ &= \mathbf{p}_\phi(x)\sum_{i=1}^{n}\psi^{-1}(a_i\psi(y_i)) \le \mathbf{p}_\phi(x)\psi^{-1}\Big(\sum_{i=1}^{n}a_i\psi(y_i)\Big) = \mathbf{p}_\phi(x)\mathbf{p}_\psi(y),\end{aligned}$$

which completes the proof.

Remark 8. One can easily check that the assumptions of Theorem 6(i) are satisfied if the functions $\phi(t^{1/p})$ and $\psi(t^{1/q})$ are concave. Taking in particular $\phi(t) = t^p$ and $\psi(t) = t^q$, with $p^{-1}+q^{-1} = 1$, $p, q > 0$, we hence get Hölder's inequality. Thus, on the one hand, Theorem 6(i) generalizes Hölder's inequality, and on the other hand, it shows that the assumption of the existence of a set $A \in \Sigma$ such that $0 < \mu(A) < 1$ in Theorem 3 is indispensable. Note also that if ϕ and ψ are subadditive and $\mu(\Sigma)$, the range of the measure μ, is contained in $\mathbb{N}\cup\{0,\infty\}$ then conditions (11) are satisfied.

Remark 9. Modifying the assumptions of the above Theorem 6 in an obvious way one can obtain the corresponding result for the reverse inequality.

5. A generalized integral Hölder–Minkowski inequality. In [7], using a weaker version of Lemma 1, we determined the form of all the functions $f : \mathbb{R}_+^k \to \mathbb{R}_+$ satisfying the inequality

$$af(x) + bf(y) \le f(ax+by) \qquad (x, y \in \mathbb{R}_+^k)$$

for some a, b such that $0 < a < 1 < a+b$. This result for $k = 2$ leads to the following obvious observation: *a function* $h : (0,\infty) \to \mathbb{R}$ *is convex iff*

$$h\Big(\frac{x_1+x_2}{y_1+y_2}\Big)(y_1+y_2) \le h\Big(\frac{x_1}{y_1}\Big)y_1 + h\Big(\frac{x_2}{y_2}\Big)y_2 \qquad (x_1, x_2, y_1, y_2 > 0). \tag{15}$$

It seems to be of interest that this is a generalization of Hölder's and Minkowski's inequalities in the discrete case. To get for instance a one-line proof of Minkowski's inequality it is enough to apply (15) to the convex function $h(t) = (t^p+1)^{1/p}, p \ge 1$. It should be emphasized that we do not need Hölder's inequality in this argument. Taking $p > 1$ and applying (15) for $h(t) = -t^{1/p}$ we obtain Hölder's inequality.

In this section we prove the following integral counterpart of inequality (15).

THEOREM 7 (Generalized Integral Hölder–Minkowski Inequality). *Let* (Ω, Σ, μ) *be a measure space such that* $\mu(\Omega) > 0$. *If a function* $h : (0,\infty) \to \mathbb{R}$ *is convex* (*resp. concave*), *then*

$$h\Big(\frac{\int_\Omega x\,d\mu}{\int_\Omega y\,d\mu}\Big)\int_\Omega y\,d\mu \le \int_\Omega\Big[h\circ\Big(\frac{x}{y}\Big)\Big]y\,d\mu \qquad (x, y \in \mathbf{L}_+^1(\Omega, \Sigma, \mu)), \tag{16}$$

(*resp. the reverse inequality holds*).

Proof. Take $x, y \in \mathbf{L}_+^1(\Omega, \Sigma, \mu)$ and define the measure $\nu : \Sigma \to [0,\infty]$ by

$$\nu(A) = \frac{\int_A y\,d\mu}{\int_\Omega y\,d\mu} \qquad (A \in \Sigma).$$

Clearly (Ω, Σ, ν) is a normalized measure space and $x/y \in \mathbf{L}_+^1(\Omega, \Sigma, \nu)$. By the integral Jensen inequality for convex functions (cf. M. Kuczma [4], p. 181) we obtain

$$h\Big(\int_\Omega \frac{x}{y}\,d\nu\Big) \le \int_\Omega h\circ\Big(\frac{x}{y}\Big)\,d\nu.$$

Using the formula for $d\nu$ we get (16). This completes the proof.

Now we apply this result to get the integral Hölder and Minkowski inequalities.

Hölder's Inequality. Take $p, q \in \mathbb{R}\setminus\{0,1\}$ such that $p^{-1}+q^{-1} = 1$ and nonnegative functions x, y such that $x^p, y^q \in \mathbf{L}_+^1(\Omega, \Sigma, \mu)$. The function

$$h(t) = t^{1/p} \qquad (t > 0)$$

is concave for $p > 1$ and convex for $p < 1$. Writing inequality (16) for this function and replacing x by x^p and y by y^q we obtain the integral Hölder inequality for $p > 1$ and the "companion" inequality for $p < 1$.

Minkowski's Inequality. Take $p \in \mathbb{R}$, $p \ne 0$. The function

$$h(t) = (t^{1/p}+1)^p \qquad (t > 0)$$

is concave for $p \ge 1$ and convex for $p \le 1$. Substituting this function into (16) and replacing x and y by $x^p, y^p \in \mathbf{L}_+^1(\Omega, \Sigma, \mu)$, resp., we get the integral Minkowski inequality for $p \ge 1$ and the "companion" inequality for $p \le 1$.

Remark 10. A weaker version of Theorem 7 in which x and y are assumed to be elements of $\mathbf{S}_+$ has been proved in [9] (cf. also [7]).

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A localization property for B^s_{pq} and F^s_{pq} spaces

by

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Abstract. Let $f^j=\sum_k a_k f(2^{j+1}x-2k)$, where the sum is taken over the lattice of all points k in $\mathbb{R}^n$ having integer-valued components, $j\in\mathbb{N}$ and $a_k\in\mathbb{C}$. Let A^s_{pq} be either B^s_{pq} or F^s_{pq} ($s\in\mathbb{R}$, $0<p<\infty$, $0<q\le\infty$) on $\mathbb{R}^n$. The aim of the paper is to clarify under what conditions $\|f^j\mid A^s_{pq}\|$ is equivalent to $2^{j(s-n/p)}(\sum_k|a_k|^p)^{1/p}\|f\mid A^s_{pq}\|$.

1. Introduction and theorem. The spaces B^s_{pq} and F^s_{pq} with $s\in\mathbb{R}$, $0<p\le\infty$ ($p<\infty$ for the F-scale), $0<q\le\infty$, on $\mathbb{R}^n$ cover many well-known classical function spaces, such as the Sobolev spaces $W^k_p=F^k_{p,2}$ (with $k\in\mathbb{N}_0$, $1<p<\infty$), the fractional Sobolev spaces $H^s_p=F^s_{p,2}$ (with $s\in\mathbb{R}$, $1<p<\infty$), the Hölder–Zygmund spaces $\mathcal{C}^s=B^s_{\infty,\infty}$ (with $s>0$), the (inhomogeneous) Hardy spaces $h_p=F^0_{p,2}$ (with $0<p<\infty$) and the classical Besov spaces B^s_{pq} (with $s>0$, $1<p<\infty$, $1\le q\le\infty$). The theory of these spaces has been developed in [8, 9]. The aim of this paper is to prove a localization property for all these spaces which in this generality and in its almost final form is unexpected and rather surprising.

Let $\mathbb{Z}^n$ be the lattice of all points in $\mathbb{R}^n$ having integer-valued components. Let $x^{k,j}=2^{-j}k$ with $k\in\mathbb{Z}^n$ and $j\in\mathbb{N}$. Let $f\in S'$ with $\operatorname{supp} f\subset Q_d=\{x\in\mathbb{R}^n:|x_l|<d \text{ if } l=1,\ldots,n\}$, where $d>0$ is assumed to be small, at least $d\le 1/2$, and let

$$f^j(x)=\sum_{k\in\mathbb{Z}^n}a_k f(2^{j+1}(x-x^{k,j}))\,,\qquad a_k\in\mathbb{C}\,. \tag{1}$$

Of course, the terms in (1) have mutually disjoint supports. Let $\sigma_p=\max(0,n(1/p-1))$ and let $[a]$ be the largest integer less than or equal to $a\in\mathbb{R}$.

THEOREM. *Let $s\in\mathbb{R}$, $0<p\le\infty$ ($p<\infty$ for the F-scale), $0<q\le\infty$. Let A^s_{pq} be either B^s_{pq} or F^s_{pq} and let $0<d\le 1/4$.*

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