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A weighted vector-valued weak type (1, 1) inequality and spherical summation

by

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Abstract. We prove a weighted vector-valued weak type (1, 1) inequality for the Bochner–Riesz means of the critical order. In fact, we prove a slightly more general result.

1. Introduction. For a nonnegative function w on $\mathbb{R}^n$ $(n \ge 2)$, let $L^p_w(\mathbb{R}^n) = \{f : \|fw^{1/p}\|_p = \|f\|_{p,w} < \infty\}$ be the weighted L^p space and let $L^{1,\infty}_w$ be the weighted weak L^1 space. We write for $f \in L^{1,\infty}_w$,

$$\|f\|^*_w = \sup_{\lambda>0} \lambda w(\{x : |f(x)| > \lambda\}),$$

where $w(E) = \int_E w$. Next for $R > 0$ let

$$S^\delta_R(f)(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi)(1 - |\xi|^2R^{-2})^\delta_+ e^{2\pi i x\xi}\, d\xi$$

be the Bochner–Riesz means of order δ. In this note we shall prove a weighted vector-valued version of Christ [1, Theorem 1].

THEOREM 1. *Let $w(x) = |x|^\beta$, $-n < \beta \le 0$, and let $\alpha = (n-1)/2$ be the critical index. Then for a sequence $\{R_k\}$ of positive numbers, we have*

$$\Big\|\Big(\sum |S^\alpha_{R_k}(f_k)|^2\Big)^{1/2}\Big\|^*_w \le c\Big\|\Big(\sum |f_k|^2\Big)^{1/2}\Big\|_{1,w}.$$

See [2, 3, 4, 10] for related results. We shall prove a more general result. Following [3], we consider a sequence $\{T_k\}$ of bounded linear operators on L^2 such that there exists a sequence $\{K^k\}$ of kernels satisfying

$$\langle T_k(f), g\rangle = \iint g(x)f(y)K^k(x-y)\,dy\,dx$$

for $f, g \in C^\infty_0$ with disjoint supports. Furthermore, we assume the following.

(1.1) The operators T_k are bounded on L^2_w and $\sup_k \|T_k\|_{2,w} = c_1 < \infty$, where $\|\cdot\|_{2,w}$ denotes the operator norm.

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(1.2) The kernels K^k can be written in polar coordinates as

$$K^k(r,\theta) = r^{-n}\Omega^k(r,\theta),$$

where $\sup_{r,\theta,k}(|\Omega^k(r,\theta)| + |\partial_\theta \Omega^k(r,\theta)|) = c_2 < \infty$.

Then we can obtain a weighted vector-valued version of a special case of [3, Theorem 4].

THEOREM 2. *Let $w(x) = |x|^\beta$, $-n < \beta \le 0$, and $\{T_k\}$ be as above. Then there exists a constant c depending only on c_1, c_2, n and w such that*

$$\Big\|\Big(\sum |T_k(f_k)|^2\Big)^{1/2}\Big\|_w^* \le c\Big\|\Big(\sum |f_k|^2\Big)^{1/2}\Big\|_{1,w}.$$

Theorem 1 immediately follows from Theorem 2. In the rest of this note, we consider only a weight w as in Theorems 1 and 2. As a consequence of Theorem 1 for $R_k = 2^k$, by a standard argument we have the following.

COROLLARY 1. *Define*

$$\sigma(f)(x) = \Big(\sum_{k\in\mathbb{Z}} |S_{2^k}^{\alpha+1}(f)(x) - S_{2^k}^{\alpha}(f)(x)|^2\Big)^{1/2}.$$

Then $\|\sigma(f)\|_w^* \le c\|f\|_{H_w^1}$, where H_w^1 denotes the weighted Hardy space (see [14]).

Here we give a sketch of the proof. First we note that there are $\widehat{\varphi}, \widehat{\psi} \in C_0^\infty$ such that $\widehat{\varphi}(0) = \widehat{\psi}(0) = 0$ and

$$S_R^{\alpha+1}(f) - S_R^{\alpha}(f) = f * \varphi_R + S_R^{\alpha}(f * \psi_R),$$

where $g_R(x) = R^n g(Rx)$. Then we have

$$\sigma(f) \le \Big(\sum |f * \varphi_{2^k}|^2\Big)^{1/2} + \Big(\sum |S_{2^k}^{\alpha}(f * \psi_{2^k})|^2\Big)^{1/2}.$$

By Chebyshev's inequality, Theorem 1 and the Littlewood–Paley inequality for H_w^1, we obtain the assertion of Corollary 1.

By Corollary 1 we have the following.

COROLLARY 2. *Let $S_*^\delta(f)(x) = \sup_k |S_{2^k}^\delta(f)(x)|$. Then*

$$\|S_*^\alpha(f)\|_w^* \le c\|f\|_{H_w^1}.$$

The inequality $S_*^\alpha(f) \le S_*^{\alpha+1}(f) + \sigma(f)$ proves the corollary. From this we obtain almost everywhere convergence of the lacunary Bochner–Riesz means for H_w^1. See [13] and also [7], [8], [16, Chap. XV]. We can prove in the same way a continuous analogue of Theorem 1 where ℓ^2 is replaced by $L^2((0,\infty), dR/R)$. Using this, we obtain the following similarly to Corollary 1.

COROLLARY 3. *Let $f \in H_w^1$. Then $\|\widetilde{\sigma}(f)\|_w^* \le c\|f\|_{H_w^1}$, where*

$$\widetilde{\sigma}(f)(x) = \Big(\int_0^\infty |S_R^{\alpha+1}(f)(x) - S_R^{\alpha}(f)(x)|^2 \frac{dR}{R}\Big)^{1/2}.$$

See [6] for the pointwise equivalence between $\widetilde{\sigma}$ and other square functions.

The proof we shall give below is a combination of arguments of Christ–Rubio de Francia [3] and Hofmann [5]. Theorems 1, 2 and their corollaries for $w = 1$ can be found in [9].

2. Outline of proof of Theorem 2. Let $L_w^p(\ell^q)$ be the space of ℓ^q-valued functions $f = (f_k)$ such that $|f|_q \in L_w^p$, where $|\cdot|_q$ denotes the ℓ^q norm. We also write $\|f\|_{p,w} = (\int |f|_2^p w\,dx)^{1/p}$ for the norm of $f \in L_w^p(\ell^2)$ and when $w = 1$ this norm is denoted by $\|\cdot\|_p$ (this will not cause any confusion).

Let $f = (f_k) \in L_w^1(\ell^2) \cap L_w^2(\ell^2)$ and $\lambda > 0$. We use a Calderón–Zygmund decomposition, i.e. a collection $\{Q\}$ of nonoverlapping dyadic cubes and a decomposition $f = g + b$, $b = \sum b_Q$, with the following properties:

$$\|g\|_\infty \le c\lambda, \quad \|g\|_{1,w} \le c\|f\|_{1,w}, \tag{2.1}$$

$$w\Big(\bigcup Q\Big) \le c\|f\|_{1,w}/\lambda, \tag{2.2}$$

$$\|b_Q\|_1 \le c\lambda|Q|, \quad \int b_Q = 0, \quad b_Q \text{ is supported on } Q. \tag{2.3}$$

Define $S(f) = (T_k(f_k))$. Then by (1.1), S is bounded on $L_w^2(\ell^2)$. Thus by (2.1) we have

$$w(\{|S(g)|_2 > \lambda\}) \le \lambda^{-2}\|S(g)\|_{2,w}^2 \le c\lambda^{-2}\|g\|_{2,w}^2 \le c\lambda^{-1}\|f\|_{1,w},$$

so that, by (2.2), Theorem 2 follows from

$$w(\{x \in \mathbb{R}^n \setminus E^* : |S(b)(x)|_2 > \lambda\}) \le \frac{c}{\lambda}\|f\|_{1,w}, \tag{2.4}$$

where $E^* = \bigcup Q^*$ with Q^* denoting the cube with the same center as Q and with sidelength 2^{10+n} times that of Q.

Let $\eta \in C_0^\infty$ be radial ($\eta(x) = \eta_0(|x|)$), nonnegative and such that $\operatorname{supp}(\eta) \subset \{1/4 \le |x| \le 4\}$ and $\sum_{j\in\mathbb{Z}} \eta(2^{-j}x) = 1$ for $x \in \mathbb{R}^n \setminus \{0\}$. Define $K_j(x) = (\eta(2^{-j}x)K^k(x))$. Then to obtain (2.4) it is sufficient to prove that

$$\Big\|\sum_j K_j * B_{j-s}\Big\|_{2,w}^2 \le c2^{-\varepsilon s}\lambda\|f\|_{1,w} \tag{2.5}$$

for all $s > n + 4$ with some $\varepsilon > 0$, where $B_i = \sum_{|Q|=2^{in}} b_Q$, the convolution is defined by $f * g(x) = (f_k * g_k(x))$ for $f = (f_k)$, $g = (g_k)$ and by our construction of the exceptional set E^* we may assume that $s > n+4$. (See [3].)

Now using the Schwarz inequality, we see that

$$\Big\|\sum_j K_j * B_{j-s}\Big\|_{2,w}^2 \le c\sum_j \|K_j * B_{j-s}\|_{2,w}^2 + c\sum_j \sum_{i\le j-10} |\langle K_j * B_{j-s}, K_i * B_{i-s}\rangle_w|,$$

where $\langle\,,\rangle_w$ denotes the inner product of the Hilbert space $L^2_w(\ell^2)$. Let $K_j = (K_j^k)$, $B_i = (B_i^k)$. Then

$$\begin{aligned}\langle K_j * B_{j-s}, K_i * B_{i-s}\rangle_w &= \sum_k \int K_j^k * B_{j-s}^k(x)\overline{K}_i^k * \overline{B}_{i-s}^k(x)w(x)\,dx\\
&= \sum_k \iint K_j^k(x-y)B_{j-s}^k(y)\,dy \int \overline{K}_i^k(x-z)\overline{B}_{i-s}^k(z)\,dz\,w(x)\,dx\\
&= \sum_k \int B_{j-s}^k(y) \int \overline{B}_{i-s}^k(z) \int K_j^k(x-y)\overline{K}_i^k(x-z)w(x)\,dx\,dz\,dy\\
&= \sum_k \int B_{j-s}^k(y) \int \overline{B}_{i-s}^k(z)(K_j^k w_y) * \widetilde{K}_i^k(z-y)\,dz\,dy\\
&= \int (B_{j-s}(y), B_{i-s} * L_{ij}^y(y))_2\,dy,\end{aligned}$$

where $\widetilde{K}_i^k(x) = \overline{K}_i^k(-x)$, $w_y(x) = w(x+y)$, $L_{ij}^y(z) = (\widetilde{K}_j^k\widetilde{w}_y * K_i^k(z))$ and $(\,,)_2$ denotes the inner product in ℓ^2.

Next, let $B_{1,j-s} = \sum b_Q$, where b_Q ranges over the collection of those b_Q which satisfy $\mathrm{supp}(b_Q) \subset \{2^{j-3} \le |x| \le 2^{j+3}\}^c$ and $|Q| = 2^{n(j-s)}$. Then following Hofmann [5], we make a decomposition

$$B_{j-s} = B_{1,j-s} + B_{2,j-s}.$$

We note that since $s > n+4$, if $B_{2,j-s} = \sum b_Q$, then each Q is contained in $\{2^{j-4} \le |x| \le 2^{j+4}\}$. We shall prove (2.5) for $B_{1,j-s}$ and $B_{2,j-s}$ separately. By the above expression of $\langle K_j * B_{j-s}, K_i * B_{i-s}\rangle_w$ and the inequality $\sum_j \|B_{j-s}\|_{1,w} \le c\|f\|_{1,w}$, for this it is sufficient to prove the following results.

LEMMA 1. *Let $y \in \mathrm{supp}(B_{1,j-s})$. Then*

$$\sum_{i\le j-10} |B_{1,i-s} * L_{ij}^y(y)|_2 \le c\lambda 2^{-\varepsilon s} w(y).$$

LEMMA 2. *Let $y \in \mathrm{supp}(B_{2,j-s})$. Then*

$$\sum_{i\le j-10} |B_{2,i-s} * L_{ij}^y(y)|_2 \le c\lambda 2^{-\varepsilon s} w(y).$$

LEMMA 3. *Let $y \in \mathrm{supp}(B_{1,j-s})$. Then*

$$|B_{1,j-s} * L_{jj}^y(y)|_2 \le c\lambda 2^{-\varepsilon s} w(y).$$

LEMMA 4. *Let $y \in \mathrm{supp}(B_{2,j-s})$. Then*

$$|B_{2,j-s} * L_{jj}^y(y)|_2 \le c\lambda 2^{-\varepsilon s} w(y).$$

We observe that by dilation invariance, to prove these lemmas we may assume that $j=0$. Thus in the following sections, we shall give the proofs only for $j=0$, and then we shall use a (vector-valued) version of [3, Lemma 6.1]. Let $E = (E^k)$ and $F_i = (F_i^k)$ be kernels which can be written in polar coordinates as

$$E^k(r,\theta) = r^{-n}\Phi^k(r,\theta)\eta_0(r), \qquad F_i^k(r,\theta) = r^{-n}\Psi^k(r,\theta)\eta_0(2^{-i}r).$$

We assume that

$$\sup_{r,\theta}(|\Phi^k(r,\theta)| + |\partial_\theta\Phi^k(r,\theta)|) \le 1 \quad \text{uniformly in } k, \tag{2.6}$$

$$\sup_{r,\theta}(|\Psi^k(r,\theta)| + |\partial_\theta\Psi^k(r,\theta)|) \le 1 \quad \text{uniformly in } k. \tag{2.7}$$

Then we have the following (see [3, Lemma 6.1]).

LEMMA 5. *Let $x \in \mathbb{R}^n \setminus \{0\}$, $|h| < |x|/2$. Then*

(a) $$|E * F_i(x+h) - E * F_i(x)|_\infty \le c|2^{-i}h|^{1/2} \quad (i \le -10),$$

(b) $$|E * F_0(x+h) - E * F_0(x)|_\infty \le c|h|^{1/2}|x|^{-3/2}.$$

We shall give a sketch of the proof in §7 for completeness.

3. Proof of Lemma 1.

Let $\zeta \in C_0^\infty(\mathbb{R})$ be nonnegative and such that $\zeta(r) = 1$ if $1/4 \le r \le 4$ and $\mathrm{supp}(\zeta) \subset \{1/5 \le r \le 5\}$. We define

$$K^y(x) = (\widetilde{K}_0^k(x)\widetilde{w}_y(x)) = (r^{-n}\omega_y^k(r,\theta)\eta_0(r)),$$

where $\omega_y^k(r,\theta) = \overline{\Omega}^k(r,-\theta)|y - r\theta|^\beta\zeta(r)$. Then $L_{i0}^y(z) = K^y * K_i(z)$.

SUBLEMMA 1. *Let $y \in \mathrm{supp}(B_{1,-s})$. Then*

(a) $$\sup_{k,r,\theta}|\omega_y^k(r,\theta)| \le c|y|^\beta,$$

(b) $$\sup_{k,r,\theta}|\partial_\theta\omega_y^k(r,\theta)| \le c|y|^\beta.$$

Proof. If $y \in \mathrm{supp}(B_{1,-s})$, then $|y| \le 2^{-3}$ or $|y| \ge 2^3$. Thus for $r \in [1/5, 5]$, we have $|y - r\theta| \approx \max(|y|, 1)$, so that

$$|y - r\theta|^\beta \le c\max(|y|,1)^\beta \le c|y|^\beta.$$

Combined with (1.2), this proves (a). Similarly we have

$$|\partial_\theta \omega_y^k(r,\theta)| \le c(|y-r\theta|^\beta + |y-r\theta|^{\beta-1})\zeta(r)$$
$$\le c\max(|y|,1)^\beta + c\max(|y|,1)^{\beta-1} \le c\max(|y|,1)^\beta \le c|y|^\beta,$$

proving (b).

By Lemma 5 and Sublemma 1 we have the following.

SUBLEMMA 2. *Let $y \in \operatorname{supp}(B_{1,-s})$, $x \in \mathbb{R}^n \setminus \{0\}$ and $|h| < |x|/2$. Then*

(a) $$|L_{i0}^y(x+h) - L_{i0}^y(x)|_\infty \le cw(y)|2^{-i}h|^{1/2} \quad (i \le -10),$$

(b) $$|L_{00}^y(x+h) - L_{00}^y(x)|_\infty \le cw(y)|h|^{1/2}|x|^{-3/2}.$$

Now we prove Lemma 1. Denote by c_Q and $d(Q)$ the center and the diameter of a cube Q, respectively. Then for $s > n+4$ and $y \in \operatorname{supp}(B_{1,-s})$, we have

$$\sum_{i\le -10}\Big|\int B_{1,i-s}(z)L_{i0}^y(y-z)\,dz\Big|_2 = \sum_i \Big|\sum_{|c_Q-y|>d(Q)}\int b_Q(z)L_{i0}^y(y-z)\,dz\Big|_2,$$

where $\int f(z)g(z)\,dz = (\int f_k(z)g_k(z)\,dz)$ for $f=(f_k)$, $g=(g_k)$. By Sublemma 2(a), (2.3) and Minkowski's inequality, this is majorized by

$$\sum_i\sum_Q\Big|\int b_Q(z)(L_{i0}^y(y-z) - L_{i0}^y(y-c_Q))\,dz\Big|_2$$
$$\le c\sum_i\sum_Q\int |b_Q(z)|_2|z-c_Q|^{1/2}w(y)2^{-i/2}\,dz$$
$$\le c\lambda w(y)2^{-s/2}\sum|Q| \le c\lambda 2^{-s/2}w(y),$$

where in the last summation, Q ranges over a family of nonoverlapping dyadic cubes contained in $\{x : |x-y| < 100\}$. This completes the proof of Lemma 1.

4. Proof of Lemma 2.

Let $\mu,\nu \in C^\infty(\mathbb{R}^n)$ be radial, nonnegative and such that $\mu(x)+\nu(x) = 1$ for all $x\in\mathbb{R}^n$, $\operatorname{supp}(\mu)\subset\{|x|\le 1\}$ and $\mu(x) = 1$ if $|x|\le 1/2$. Let

$$w_y^0(x) = w(x+y)\mu(2^{\delta s}(x+y)) \quad\text{and}\quad w_y^1(x) = w(x+y)\nu(2^{\delta s}(x+y))$$

with $\delta > 0$ which will be specified later. We decompose L_{ij}^y as $L_{ij}^y(z) = M_{ij}^y(z) + N_{ij}^y(z)$, where

$$M_{ij}^y(z) = ((\widetilde{K}_j^k\widetilde{w}_y^0) * K_i^k(z)),\qquad N_{ij}^y(z) = ((\widetilde{K}_j^k\widetilde{w}_y^1) * K_i^k(z)).$$

Let $y\in\operatorname{supp}(B_{2,-s})$. We note that $|y|\approx 1$. Thus in order to prove Lemma 2 it is sufficient to prove

(4.1) $$\sum_{i\le -10}|B_{2,i-s} * M_{i0}^y(y)|_2 \le c\lambda 2^{-\varepsilon s}$$

and

(4.2) $$\sum_{i\le -10}|B_{2,i-s} * N_{i0}^y(y)|_2 \le c\lambda 2^{-\varepsilon s}.$$

First we prove (4.1). Since $|z|\le 2^{i+4}$ if $z\in\operatorname{supp}(B_{2,i-s})$, we have

$$|B_{2,i-s}^k * (\widetilde{K}_0^k\widetilde{w}_y^0) * K_i^k(y)| = \Big|\int B_{2,i-s}^k(z)\int_{|y-x|\le c2^i}\widetilde{K}_0^k(x)$$
$$\times w(x-y)\mu(2^{\delta s}(x-y))K_i^k(y-z-x)\,dx\,dz\Big|$$
$$\le c2^{-in}\int |B_{2,i-s}^k(z)|\,dz\int_{|x|\le c2^i} w(x)\mu(2^{\delta s}x)\,dx.$$

Thus by Minkowski's inequality we have

$$|B_{2,i-s} * M_{i0}^y(y)|_2 \le c2^{-in}\int_{|x|\le c2^i} w(x)\mu(2^{\delta s}x)\,dx\,\|B_{2,i-s}\|_1$$
$$\le c\lambda\int_{|x|\le c2^i} w(x)\mu(2^{\delta s}x)\,dx,$$

where we have used

$$\|B_{2,i-s}\|_1 \le c\lambda\sum|Q| \le c\lambda 2^{ni},$$

which holds since in the last summation Q ranges over a family of nonoverlapping dyadic cubes contained in $\{2^{i-4}\le|x|\le 2^{i+4}\}$. Thus

$$\sum_{i\le-10}|B_{2,i-s} * M_{i0}^y(y)|_2 \le c\lambda\sum_{i\le -10}\int_{|x|\le c2^i} w(x)\mu(2^{\delta s}x)\,dx$$
$$\le c\lambda\sum_{2^i\le 2^{-\delta s}}\int_{|x|\le c2^i}|x|^\beta\,dx + c\lambda\sum_{2^i>2^{-\delta s}}\int_{|x|\le c2^{-\delta s}}|x|^\beta\,dx$$
$$\le c\lambda\sum_{2^i\le 2^{-\delta s}}2^{i(n+\beta)} + c\lambda\sum_{2^i>2^{-\delta s}}2^{-\delta s(n+\beta)} \le c\lambda s2^{-\delta s(n+\beta)},$$

which proves (4.1).

Next we prove (4.2). Let

$$J^y(x) = (\widetilde{K}_0^k(x)w(x-y)\nu(2^{\delta s}(x-y))) = (r^{-n}\sigma_y^k(r,\theta)\eta_0(r)),$$

where $\sigma_y^k(r,\theta)=\overline{\Omega}^k(r,-\theta)|y-r\theta|^\beta\nu_0(2^{\delta s}|y-r\theta|)\zeta(r)$, $\nu_0(|x|)=\nu(x)$ and ζ is as in §3. Then $N_{i0}^y(z)=J^y*K_i(z)$. In order to apply Lemma 5 we use the following obvious estimates.

SUBLEMMA 3. *Let $y\in\operatorname{supp}(B_{2,-s})$. Then*

(a) $$\sup_{k,r,\theta}|\sigma_y^k(r,\theta)|\le c2^{-\delta s\beta},$$

(b) $$\sup_{k,r,\theta}|\partial_\theta\sigma_y^k(r,\theta)|\le c2^{(-\beta+1)\delta s}.$$

By Lemma 5 and Sublemma 3 we have the following.

SUBLEMMA 4. *Let $y\in\operatorname{supp}(B_{2,-s})$, $x\in\mathbb{R}^n\setminus\{0\}$ and $|h|<|x|/2$. Then*

(a) $$|N_{i0}^y(x+h)-N_{i0}^y(x)|_\infty\le c|2^{-i}h|^{1/2}2^{(-\beta+1)\delta s}\quad(i\le-10),$$

(b) $$|N_{00}^y(x+h)-N_{00}^y(x)|_\infty\le c|h|^{1/2}|x|^{-3/2}2^{(-\beta+1)\delta s}.$$

We first see that

$$\sum_{i\le-10}|B_{2,i-s}*N_{i0}^y(y)|_2\le\sum_i\Big|\sum_{|c_Q-y|<d(Q)}\int b_Q(z)N_{i0}^y(y-z)\,dz\Big|_2+\sum_i\Big|\sum_{|c_Q-y|\ge d(Q)}\int b_Q(z)N_{i0}^y(y-z)\,dz\Big|_2=I+II,\quad\text{say}.$$

By Sublemma 3(a) we have $\sup_z|N_{i0}^y(z)|_\infty\le c2^{-\delta s\beta}$. Thus by Minkowski's inequality and (2.3) we see that

$$I\le c2^{-\delta s\beta}\sum_i\sum_{|c_Q-y|<d(Q)}\int|b_Q(z)|_2\,dz\le c\lambda2^{-\delta s\beta}\sum_i\sum_{|c_Q-y|<d(Q)}|Q|\le c\lambda2^{-\delta s\beta}\sum_i2^{n(i-s)}\le c\lambda2^{-\delta s\beta}2^{-ns}.$$

Next, using Sublemma 4(a), (2.3) and Minkowski's inequality, we have

$$II\le\sum_i\sum_{|c_Q-y|\ge d(Q)}\Big|\int b_Q(z)(N_{i0}^y(y-z)-N_{i0}^y(y-c_Q))\,dz\Big|_2\le c\sum_i\sum_Q\int|b_Q(z)|_2|z-c_Q|^{1/2}2^{-i/2}2^{(-\beta+1)\delta s}\,dz\le c\lambda2^{-s/2}2^{(-\beta+1)\delta s}\sum|Q|\le c\lambda2^{-s/2}2^{(-\beta+1)\delta s},$$

where the last inequality follows as in the proof of Lemma 1. Combining the estimates for I, II and taking δ small enough, we obtain (4.2).

5. Proof of Lemma 3. Let $y\in B_{1,-s}$. Then

$$|B_{1,-s}*L_{00}^y(y)|_2\le\sum_{|c_Q-y|<d(Q)}\Big|\int b_Q(z)L_{00}^y(y-z)\,dz\Big|_2+\Big|\sum_{|c_Q-y|\ge d(Q)}\int b_Q(z)L_{00}^y(y-z)\,dz\Big|_2=I+II,\quad\text{say}.$$

By Sublemma 1(a) we have $\sup_z|L_{00}^y(z)|_\infty\le cw(y)$. Thus by Minkowski's inequality we see that

$$I\le cw(y)\sum_{|c_Q-y|<d(Q)}\|b_Q\|_1\le c\lambda w(y)\sum|Q|\le c\lambda w(y)2^{-sn}.$$

Next by Sublemma 2(b), (2.3) and Minkowski's inequality, we have

$$II\le\sum_{|c_Q-y|\ge d(Q)}\Big|\int b_Q(z)(L_{00}^y(y-z)-L_{00}^y(y-c_Q))\,dz\Big|_2\le c\sum_Q\int|b_Q(z)|_2w(y)|z-c_Q|^{1/2}|c_Q-y|^{-3/2}\,dz\le c\lambda w(y)2^{-s/2}\sum|Q|\,|c_Q-y|^{-3/2}.$$

If $|c_Q-y|\ge d(Q)$, we have $|c_Q-y|\approx|x-y|$ for $x\in Q$. Thus

$$II\le c\lambda w(y)2^{-s/2}\sum_Q\int_Q|x-y|^{-3/2}\,dx\le c\lambda w(y)2^{-s/2}\int_B|x|^{-3/2}\,dx,$$

where B is a fixed bounded set. Combining the estimates for I and II, we obtain the conclusion of Lemma 3.

6. Proof of Lemma 4. Let M_{ij}^y and N_{ij}^y be as in §4. Then to obtain Lemma 4, it is sufficient to prove the following estimates for $y\in B_{2,-s}$:

(6.1) $$|B_{2,-s}*M_{00}^y(y)|_2\le c\lambda2^{-\varepsilon s},$$

(6.2) $$|B_{2,-s}*N_{00}^y(y)|_2\le c\lambda2^{-\varepsilon s}.$$

We first prove (6.1). As in the proof of (4.1) we see that

$$|B_{2,-s}*M_{00}^y(y)|_2\le c\int w(x)\mu(2^{\delta s}x)\,dx\,\|B_{2,-s}\|_1\le c\lambda2^{-\delta s(n+\beta)}\sum|Q|\le c\lambda2^{-\delta s(n+\beta)},$$

since in the last summation Q ranges over a family of cubes contained in a fixed bounded set. This proves (6.1).

Next we prove (6.2). First we have

$$|B_{2,-s} * N_{00}^y(y)|_2 \le \Big| \sum_{|c_Q - y| < d(Q)} \int b_Q(z) N_{00}^y(y-z)\,dz \Big|_2 + \Big| \sum_{|c_Q - y| \ge d(Q)} \int b_Q(z) N_{00}^y(y-z)\,dz \Big|_2 = I + II\,, \quad \text{say}.$$

Since $\sup_z |N_{00}^y(z)|_\infty \le c2^{-\delta s\beta}$ by Sublemma 3(a), using Minkowski's inequality and (2.3), we see that

$$I \le c2^{-\delta s\beta} \sum_{|c_Q-y|<d(Q)} \|b_Q\|_1 \le c\lambda 2^{-\delta s\beta} \sum_{|c_Q-y|<d(Q)} |Q| \le c\lambda 2^{-\delta s\beta} 2^{-sn}.$$

Next by Sublemma 4(b), (2.3) and Minkowski's inequality, arguing as in §5 we have

$$\begin{aligned} II &= \Big| \sum_{|c_Q-y|\ge d(Q)} \int b_Q(z)(N_{00}^y(y-z) - N_{00}^y(y-c_Q))\,dz \Big|_2 \\ &\le c\sum_Q \int |b_Q(z)|_2 |z-c_Q|^{1/2} |y-c_Q|^{-3/2} 2^{(-\beta+1)\delta s}\,dz \\ &\le c\lambda 2^{-s/2} 2^{(-\beta+1)\delta s} \sum |Q|\,|y-c_Q|^{-3/2} \le c\lambda 2^{-s/2} 2^{(-\beta+1)\delta s}. \end{aligned}$$

Combining the estimates for I and II and taking δ small enough, we obtain (6.2).

7. Sketch of proof of Lemma 5. We fix k and write $E = E^k$, $F_i = F_i^k$, $\Phi = \Phi^k$, $\Psi = \Psi^k$. Then

$$(7.1) \qquad E * F_i(x) = c \int_0^\infty \int_0^\infty (\Phi_r d\sigma_r) * (\Psi_s d\sigma_s)(x) \eta_0(r) \eta_0(2^{-i}s) \frac{dr\,ds}{rs},$$

where $\Phi_r(\theta) = \Phi(r,\theta)$, $\Psi_s(\theta) = \Psi(s,\theta)$ and σ_r denotes the uniform surface probability measure of the sphere $\{x : |x| = r\}$. By (2.6) and (2.7) we have the following result of [3] (see [3, Lemma 6.2]).

SUBLEMMA 5. *Let $r \ge s$ and $r \in [1/4, 4]$. Then $(\Phi_r d\sigma_r) * (\Psi_s d\sigma_s)(x) = 0$ if $|x| \le r - s$ or $|x| \ge r + s$, and if $r - s < |x| < r + s$ we have*

$$|(\Phi_r d\sigma_r) * (\Psi_s d\sigma_s)(x)| \le c(|x|(r+s-|x|)(|x|-r+s))^{-1/2},$$
$$|\nabla((\Phi_r d\sigma_r) * (\Psi_s d\sigma_s))(x)| \le cs(|x|(r+s-|x|)(|x|-r+s))^{-3/2}.$$

When $r \ge s$, by a straightforward computation we see that

$$\sigma_r * \sigma_s(x) = c_n r^{-n+2} s^{-n+2} |x|^{-n+2} (((r+s)^2 - |x|^2)(|x|^2 - (r-s)^2))^{(n-3)/2}$$

if $r - s < |x| < r + s$, and $\sigma_r * \sigma_s(x) = 0$ otherwise. From this, Sublemma 5 follows when $\Phi_r = \Psi_s = 1$. The proof of the general case is similar. We omit the details.

We can prove (a) and (b) of Lemma 5 similarly by using Sublemma 5. Here we only give the proof of (b). First we may assume that $|x| < 100$ and $|h| < 10^{-10}|x|$ since $E * F_0$ is bounded and supported in $\{|x| \le 10\}$. Put $G(r,s,x,h) = |(\Phi_r d\sigma_r) * (\Psi_s d\sigma_s)(x+h) - (\Phi_r d\sigma_r) * (\Psi_s d\sigma_s)(x)|$. Then let

$$\iint G(r,s,x,h)\eta_0(r)\eta_0(s)\,dr\,ds = \iint_{r\ge s} G(r,s,x,h)\eta_0(r)\eta_0(s)\,dr\,ds + \iint_{r\le s} G(r,s,x,h)\eta_0(r)\eta_0(s)\,dr\,ds = I_1 + I_2\,, \quad \text{say}.$$

We can estimate I_1 and I_2 similarly. We consider I_1. For $s \in [1/4, 4]$, let $A = \{r : r \ge s,\ ||x| - s - r| < 2|h| \text{ or } ||x| + s - r| < 2|h|\}$, $B = \{r : r \ge s,\ |x| - s + 2|h| < r < |x| + s - 2|h|\}$ and put

$$J_1(s) = \int_A G(r,s,x,h)\eta_0(r)\,dr\,, \qquad J_2(s) = \int_B G(r,s,x,h)\eta_0(r)\,dr.$$

Then since $\operatorname{supp}((\Phi_r d\sigma_r) * (\Psi_s d\sigma_s)) \subset \{x : r - s \le |x| \le r + s\}$ $(r \ge s)$, we have $I_1 = \int (J_1(s) + J_2(s))\eta_0(s)\,ds$. By Sublemma 5, for $s \in [1/4, 4]$ we see that $J_1(s)$ is dominated by

$$c\int_A ((|x+h| \cdot |r+s-|x+h|| \cdot ||x+h| - r + s|)^{-1/2} + (|x| \cdot |r+s-|x|| \cdot ||x|-r+s|)^{-1/2})\,dr.$$

By a direct computation, this is bounded by

$$c|x|^{-1/2} \int_{|r|\le 5|h|} |r|^{-1/2}\,dr \le c|x|^{-1/2}|h|^{1/2}.$$

Next by Sublemma 5 and the mean value theorem, via a direct computation, for $s \in [1/4, 4]$ we see that $J_2(s)$ is bounded by

$$c|h| \int_{|x|-s+2|h|}^{|x|+s-2|h|} \sup_{0<\theta<1} (|x+\theta h| \cdot |r+s-|x+\theta h|| \cdot ||x+\theta h| - r + s|)^{-3/2}\,dr \le c|h| \cdot |x|^{-3/2} \int_{|h|}^{10} r^{-3/2}\,dr \le c|h|^{1/2}|x|^{-3/2}.$$

Collecting the results we have $I_1 \le c|h|^{1/2}|x|^{-3/2}$. We obtain the same estimate for I_2. Since these estimates are uniform in k, by (7.1) we obtain Lemma 5(b).

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The converse of the Hölder inequality and its generalizations

by

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Abstract. Let (Ω,Σ,μ) be a measure space with two sets $A,B\in\Sigma$ such that $0<\mu(A)<1<\mu(B)<\infty$ and suppose that ϕ and ψ are arbitrary bijections of $[0,\infty)$ such that $\phi(0)=\psi(0)=0$. The main result says that if

$$\int_\Omega xy\,d\mu\le\phi^{-1}\Big(\int_\Omega\phi\circ x\,d\mu\Big)\psi^{-1}\Big(\int_\Omega\psi\circ x\,d\mu\Big)$$

for all μ-integrable nonnegative step functions x,y then ϕ and ψ must be conjugate power functions.

If the measure space (Ω,Σ,μ) has one of the following properties:

(a) $\mu(A)\le1$ for every $A\in\Sigma$ of finite measure;
(b) $\mu(A)\ge1$ for every $A\in\Sigma$ of positive measure,

then there exist some broad classes of nonpower bijections ϕ and ψ such that the above inequality holds true.

A general inequality which contains integral Hölder and Minkowski inequalities as very special cases is also given.

Introduction. Let (Ω,Σ,μ) be a measure space. Denote by $\mathbf{S}=\mathbf{S}(\Omega,\Sigma,\mu)$ the linear space of all μ-integrable step functions $x:\Omega\to\mathbb{R}$ and by $\mathbf{S}_+$ the set of all $x\in\mathbf{S}$ such that $x:\Omega\to\mathbb{R}_+$ where $\mathbb{R}_+=[0,\infty)$. One can easily verify that for every bijective function $\phi:\mathbb{R}_+\to\mathbb{R}_+$ such that $\phi(0)=0$ the functional $\mathbf{p}_\phi:\mathbf{S}_+\to\mathbb{R}_+$ given by the formula

$$\mathbf{p}_\phi(x)=\phi^{-1}\Big(\int_\Omega\phi\circ x\,d\mu\Big)\quad(x\in\mathbf{S}_+)\tag{1}$$

is well defined. In a recent paper [8] the author proved the following converse of Minkowski's inequality.

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