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On certain nonstandard Calderón–Zygmund operators

by

STEVE HOFMANN (Dayton, Ohio)

Abstract. We formulate a version of the $T1$ theorem which enables us to treat singular integrals whose kernels need not satisfy the usual smoothness conditions. We also prove a weighted version. As an application of the general theory, we consider a class of multilinear singular integrals in $\mathbb{R}^n$ related to the first Calderón commutator, but with a kernel which is far less regular.

1. Introduction. The L^p mapping properties of non-convolution type Calderón–Zygmund singular integral operators with standard (i.e. "smooth") kernels are now rather well understood, thanks in no small part to the remarkable "$T1$" theorem of David and Journé [DJ]. Suppose that T is a singular integral operator with kernel $K(x,y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{x=y\}$ (this means that T is a bounded linear functional from D to D', where $\langle Tf, g\rangle \equiv \iint K(x,y)f(y)g(x)\,dx\,dy$, whenever $f,\, g \in C_0^\infty$ have disjoint supports), with

$$|K(x,y)| \le \frac{C}{|x-y|^n} \tag{1.1}$$

and, for all $|x-x'| < |x-y|/2$,

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \le C\frac{|x-x'|^\delta}{|x-y|^{n+\delta}}, \tag{1.2}$$

for some $0 < \delta \le 1$. We define the transpose T^* by $\langle Tf,\, g\rangle \equiv \langle f,\, T^*g\rangle$, so that T^* has kernel $K^*(x,y) = K(y,x)$. David and Journé's "$T1$" theorem states that for T as above with standard kernel, T extends to a bounded operator on L^2 if and only if $T1 \in \mathrm{BMO}$, $T^*1 \in \mathrm{BMO}$ and T satisfies the following "weak boundedness property" (WBP):

$$|\langle \varphi_1,\, T\varphi_2\rangle| \le Cr^n(\|\varphi_1\|_\infty + r\|\nabla\varphi_1\|_\infty)(\|\varphi_2\|_\infty + r\|\nabla\varphi_2\|_\infty), \tag{1.3}$$

for all $\varphi_1, \varphi_2 \in C_0^\infty$ with support in a ball of radius r.

The kernel conditions were weakened by Y. Meyer [M], who replaced the size condition (1.1) by

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$$(1.4)\qquad \sup_{R>0} \int_{R\le|x-y|\le 2R} [|K(x,y)|+|K(y,x)|]\,dx \le C<\infty\,,$$

and the smoothness condition (1.2) by

$$(1.5)\qquad \sup_{\substack{R>0\\|u|+|v|\le R}} \Big[\int_{R/s\le|x-y|\le 2R/s} |K(x+u,y+v)-K(x,y)|\,dx + \int_{R/s\le|x-y|\le 2R/s} |K(x+u,y+v)-K(x,y)|dy\Big] \le \widetilde{\omega}(s), \qquad 0<s<1\,,$$

where

$$\int_0^1 \widetilde{\omega}(s)\log\frac{1}{s}\,\frac{ds}{s}<\infty\,.$$

This last condition was relaxed slightly in recent work of Y. S. Han and the author [HH] to

$$(1.6)\qquad \sup_{\substack{R>0\\|u|+|v|\le R}} \Big[\int_{|x-y|\ge R/s} |K(x+u,y+v)-K(x,y)|\,dx + \int_{|x-y|\ge R/s} |K(x+u,y+v)-K(x,y)|\,dy\Big] \le \omega(s)\,,$$

where $\int_0^1 \omega(s)ds/s<\infty$. A result of Carbery and Seeger [CS, Prop. 2.3] could also be used to consider (1.6).

There are, however, large classes of singular integrals which are of interest in harmonic analysis, whose kernels may fail to satisfy even the relatively weak condition (1.6). In the present paper, we formulate a version of the $T1$ theorem which includes the above results, but which also permits one to treat at least some of these rougher kernels; we then consider a class of multilinear singular integrals in $\mathbb{R}^n$ related to the Calderón commutators, but rougher, and which falls outside of the scope of the earlier results. In a future paper [H2] we will apply this "$T1$ theorem for rough singular integrals" to extend some 1-dimensional results of Murai on singular integrals of "Calderón-type" to $\mathbb{R}^n$. It turns out that rough singular integrals arise in a natural way when treating such operators, even when the original "Calderón-type" kernel is smooth. The class of operators which we consider in the present paper are those of the form

$$(1.7)\qquad T^A f(x) \equiv \text{p.v.}\int \frac{\Omega(x-y)}{|x-y|^{n+1}}[A(x)-A(y)-\nabla A(y)\cdot(x-y)]f(y)\,dy\,,$$

where $\Omega\in L^r(S^{n-1})$, $r>1$, is positively homogeneous of degree zero and

satisfies the moment condition

$$(1.8)\qquad \int_{S^{n-1}} \theta\,\Omega(\theta)\,d\theta=0$$

and where $\nabla A\in \mathrm{BMO}$. We do not impose any smoothness condition on Ω. We recall that BMO is the Banach space of functions (modulo constants) with norm

$$\|b\|_* \equiv \sup_I \frac{1}{|I|}\int_I |b-m_I(b)|\,,$$

where $m_I(b)\equiv(1/|I|)\int_I b$, and I denotes a cube with sides parallel to the coordinate axes. A well known result of John–Nirenberg states that

$$(1.9)\qquad \|b\|_* \approx \sup_I\Big(\frac{1}{|I|}\int_I |b-m_I(b)|^q\Big)^{1/q}, \qquad 1\le q<\infty\,.$$

Since L^∞ is properly contained in BMO, we see that in general the kernel of the operator T^A fails to satisfy the "standard" kernel estimates even for "smooth" Ω. For $\nabla A\in L^\infty$, the L^p boundedness of T^A is a classical result, obtained by the method of rotations in work of Calderón [Ca], and Bajsanski and Coifman [BC]. For $\nabla A\in \mathrm{BMO}$, the method of rotations no longer applies, and in this case results for smooth Ω were obtained by means of a "good-λ" inequality by J. Cohen [Co] and Y. Hu [Hu].

We shall prove the following:

THEOREM 1.1. *Let T^A be defined as in* (1.7). *If $\Omega\in L^r$, $r>1$, then T^A is bounded on L^2, and*

$$(1.10)\qquad \|T^A f\|_2 \le C_n\|\Omega\|_r\|\nabla A\|_*\|f\|_2\,.$$

If $\Omega\in L^\infty$, then for all $w\in A_p$ and $1<p<\infty$ we have the weighted norm inequality

$$(1.11)\qquad \|T^A f\|_{p,w} \le C_{n,p,A_p}\|\Omega\|_\infty\|\nabla A\|_*\|f\|_{p,w}\,.$$

(*Here $\|f\|_{p,w}\equiv(\int|f(x)|^p w(x)\,dx)^{1/p}$, and the constants C_n and C_{n,p,A_p} depend only on dimension, and, in the latter case, p, and the A_p constant of w.*)

We remark that by generalizing (in a routine but technically messy way) the ideas of Duoandikoetxea and Rubio de Francia [DR] and Carbery and Seeger [CS] one can extend the unweighted bound (1.10) to all L^p, $1<p<\infty$, but this will not be done here. In order to avoid tiring the reader unduly, we shall instead content ourselves with the technically simpler case $\Omega\in L^\infty$, which permits one to obtain L^p bounds via the extrapolation theorem of Rubio de Francia [GR, Chapter IV].

This paper is a continuation of some work which began while the author held a post-doctoral fellowship at McMaster University. I would like to thank Eric Sawyer, who proposed the problem of formulating a version of the $T1$ theorem which would include "non-standard" kernels, and I am grateful to Y. S. Han for helpful discussions concerning the use of Calderón's reproducing formula. I also thank Anthony Carbery for bringing to my attention his joint work with A. Seeger [CS], in which some of the ideas used in the present paper had previously appeared. I also thank the referee for numerous helpful suggestions. In addition, it should be pointed out that the results in this paper were originally part of a much longer manuscript. I am also grateful to the referee of that paper for several helpful comments, including suggestions which have permitted a much shorter proof of Lemma 2.1 than I had given previously.

The paper is organized as follows. In Section 2 we discuss the weak smoothness conditions which replace the "standard" estimates. Section 3 contains the generalized $T1$ theorem and its proof, and in Section 4 we prove Theorem 1.1.

2. Conditions for nonstandard kernels. David, Journé and Semmes [DJS] have given a proof of the $T1$ theorem based on a continuous parameter Littlewood–Paley decomposition known as the "Calderón reproducing formula". Let $\psi \in C_0^\infty(|x| \le 1)$ be radial, with $\int \psi = 0$. Set $\psi_s(x) \equiv s^{-n}\psi(|x|/s)$, and $Q_s f \equiv \psi_s * f$. With the normalization $\int_0^\infty (\widehat{\psi}(s))^2\, ds/s = 1$, we then have $\int_0^\infty Q_s^2\, ds/s = I$, where the operator-valued integral converges in the strong operator topology on L^2. Thus, in analogy with the expansion of a function in terms of an orthonormal basis, one has $f = \int_0^\infty Q_s^2 f\, ds/s$, and formally,

$$\langle Tf, g\rangle = \int_0^\infty \int_0^\infty \langle (Q_s T Q_t) Q_t f, Q_s g\rangle \frac{ds}{s}\frac{dt}{t}. \tag{2.1}$$

David, Journé and Semmes observed that, under the assumptions of the $T1$ theorem (plus $T1 = 0 = T^*1$, a restriction which may be readily removed), the operator T is "almost diagonalized" by this expansion, and they were thus able to give an elegant proof of the $T1$ theorem. In fact, they were able to show rather easily that

$$L_{s,t}(x,y) \equiv \langle \psi_s(x - \cdot), T\psi_t(\cdot - y)\rangle$$

satisfies estimates which in particular imply that

$$\int |L_{s,t}(x,y)|\, dx + \int |L_{s,t}(x,y)|\, dy \le C\omega(\min(s/t, t/s)), \tag{2.2}$$

where $\int_0^1 \omega(s)\, ds/s < \infty$. It is not hard to show that even the weak smoothness condition (1.6) implies (2.2). It then follows easily that

$$\|Q_s T Q_t\|_{\mathrm{op}} \le C\omega(\min(s/t, t/s)), \tag{2.3}$$

where $\|\cdot\|_{\mathrm{op}}$ denotes the $L^2 \to L^2$ operator norm. Schwarz's inequality and Littlewood–Paley theory conclude the proof. Thus, it is not the kernel conditions per se that are crucial to this approach, but rather the fact that they yield the diagonalization (2.3), which in turn can be obtained in some cases for kernels which fail to satisfy the standard estimates, or even (1.6). The explicit use of (2.3) as the starting point of an L^2 theory has appeared previously in work of M. Christ [C2], in [CS, Prop. 2.3] and in somewhat simpler form in the convolution case, in [DR].

We now set some notation. Let $\varphi \in C_0^\infty(1/4, 1)$ define a partition of unity so that $\sum_{j=-\infty}^\infty \varphi(2^{-j}r) \equiv 1$, $r > 0$; set

$$K_j(x,y) \equiv K(x,y)\varphi(2^{-j}|x-y|), \quad \text{and} \quad K^N \equiv \sum_{j=N}^\infty K_j,$$

and let T_j and T^N be the corresponding pieces of the operator. Recall that $K^*(x,y) \equiv K(y,x)$, and T^* is the corresponding operator.

Let us introduce the replacements for the "standard" conditions. In lieu of the size condition (1.1) (or (1.4)) we have

$$\int_{B_{\lambda R}} \int_{R \le |x-y| \le 2R} [|K^*(x,y)| + |K(x,y)|]\, dy\, dx \le C_\lambda R^n, \tag{2.4}$$

where $B_{\lambda R}$ is any ball of radius λR, $\lambda \in (0,\infty)$. It is clear that (1.4) implies (2.4) with $C_\lambda = C\lambda^n$. However, for $\nabla A \in \mathrm{BMO}$, the operator T^A defined in (1.7) need not satisfy (1.4).

As a substitute for the smoothness condition (1.2) (or (1.5) or (1.6)) we will assume (for all Q_s defined as above)

$$\|Q_s T^N 1\|_\infty + \|Q_s T^{*N} 1\|_\infty \le C\omega(2^{-N}s), \quad s \le 2^N, \tag{2.5}$$

and a certain smoothness in the L^2 operator norm:

$$\|Q_s T^N\|_{\mathrm{op}} + \|Q_s T^{*N}\|_{\mathrm{op}} \le C\omega(2^{-N}s), \quad s \le 2^N, \tag{2.6}$$

where in each case $\int_0^1 \omega(s)\, ds/s < \infty$. (Here C may depend on the particular choice of ψ defining Q_s.) It is easy to see that standard kernels satisfy (2.5) and (2.6): since $Q_s 1 = 0$, this is a straightforward consequence of the mean value theorem. Moreover, it is not hard to show that even (1.6) implies both (2.5) and (2.6). We also point out that conditions similar to (2.6), as well as (2.8) below, have been considered in [CS, Props. 2.5 and 2.6]. See also the previous work of the author [H], where much of Section 3 had been implicit.

In addition, we impose a certain (qualitative) technical condition, which holds in particular for any kernel satisfying (1.6) (or even Hörmander's condition). It also holds trivially for doubly truncated principal value kernels. With ψ_s the kernel of Q_s, we assume

$$\int_{|x-u|>2s} \left| \int \psi_s(x-z)K^*(z,u)\,dz \right| du < \infty\,, \tag{2.7}$$
$$\int_{|x-u|>2s} \left| \int \psi_s(x-z)K(z,u)\,dz \right| du < \infty\,.$$

The precise bound in (2.7) is unimportant, as the condition will not be used quantitatively. It is only imposed so that we may define $T1$ and T^*1, and also to make the proof of the theorem rigorous. In fact, we can now define $T1$ by a standard device, in the sense of distributions modulo constants, as follows: for $\psi \in C_0^\infty$ having mean value zero and support in a ball of radius s with center x_0, we write $1 \equiv h + (1-h)$, where $h \in C_0^\infty$ and $h(x) \equiv 1$ if $|x - x_0| \le 2s$. Then we define $\langle \psi, T1\rangle \equiv \langle \psi, Th\rangle + \langle T^*\psi, 1-h\rangle$, where the last expression exists by (2.7). It is straightforward to verify that the definition is independent of h. Similarly, we can define $Q_sT^N1(x)$ as

$$\langle 1, T^{*N}\psi_s(x-\cdot)\rangle = \iint K^N(z,u)\psi_s(x-z)\,dz\,du\,,$$

which makes sense by (2.4) and (2.7).

Before stating and proving the nonstandard $T1$ theorem in the next section, we wish to make some comments about the weak smoothness condition (2.6). In our application (Theorem 1.1), we shall have a stronger version:

$$\|Q_sT_j\|_{\mathrm{op}} \le C(2^{-j}s)^\varepsilon, \quad s \le 2^j, \quad \text{for some } \varepsilon > 0\,. \tag{2.8}$$

We now proceed to show that (2.8) is satisfied in several cases of interest. The first (and easiest) case is that of the convolution operators satisfying Fourier transform estimates of Duoandikoetxea and Rubio de Francia [DR]. For example, if $K(x,y) = K(x-y) \equiv \Omega(x-y)/|x-y|^n$, where for some $r > 1$, $\Omega \in L^r(S^{n-1})$ has mean value zero, then (see [DR, Section 4])

$$|\widehat{K}_0(\xi)| \le C\min(|\xi|^\alpha, |\xi|^{-\alpha})\,,$$

for some $\alpha > 0$ (it is enough to take $j = 0$ by dilation invariance). Thus, by Plancherel,

$$\|Q_sT_0f\|_2^2 \le c\int |\widehat{\psi}(s|\xi|)|^2\min(1,|\xi|^{-\alpha})^2|\widehat{f}(\xi)|^2\,d\xi\,.$$

But $\widehat{\psi}(0) = 0$ and $\widehat{\psi} \in S$, so $\widehat{\psi}(s|\xi|) \le \min(s|\xi|, 1)$, and therefore, by an elementary computation, we have

$$\|Q_sT_0f\|_2 \le Cs^\varepsilon\|f\|_2\,. \tag{2.9}$$

The non-convolution case requires more work. In [H], the n-dimensional Calderón commutators with kernel $\Omega(x-y)|x-y|^{-n-k}[A(x)-A(y)]^k$ ($\nabla A \in L^\infty$ and $\Omega \in L^r$, $r > 1$) were shown to satisfy (2.8). Consider the kernel of T^A:

$$K^A(x,y) \equiv \frac{\Omega(x-y)}{|x-y|^{n+1}}[A(x) - A(y) - \nabla A(y)\cdot(x-y)]\,. \tag{2.10}$$

We will prove:

LEMMA 2.1. *Suppose that Ω is positively homogeneous of degree zero. Then, if $\Omega \in L^r(S^{n-1})$, $r > 1$, and $\nabla A \in \mathrm{BMO}$, we have, for $s \le 2^j$ and some $\varepsilon > 0$,*

$$\|Q_sT_j^A\|_{\mathrm{op}} \le C\|\Omega\|_r\|\nabla A\|_*(2^{-j}s)^\varepsilon\,.$$

Remark. We will see in Section 4 that T^{A*} nearly satisfies (2.8), except for an error term which is controlled by Carleson measure estimates.

Proof of Lemma 2.1. By scale invariance, we may take $j = 0$, and without loss of generality, we may take $\|\Omega\|_r = 1 = \|\nabla A\|_*$. We decompose $\mathbb{R}^n \equiv \bigcup I_i$, where each I_i is a cube with side length 1, and the cubes have disjoint interiors. Set $f_i \equiv \chi_{I_i}f$ so that $f \equiv \sum f_i$ a.e. Now, if $s < 1$, the support of $Q_sT_0f_i$ is contained in a fixed multiple of I_i, so that the supports of the various terms $Q_sT_0f_i$ have bounded overlaps. Thus we have the "almost orthogonality" property

$$\Big\|\sum_i Q_sT_0f_i\Big\|_2^2 \le C\sum_i\|Q_sT_0f_i\|_2^2\,,$$

and it is therefore enough to show

$$\|Q_sT_0f_i\|_2^2 \le Cs^\varepsilon\int|f_i|^2\,. \tag{2.11}$$

For fixed i, let $\widetilde{I} \equiv 1000nI_i$. Now, and frequently in the sequel, we use an observation from [Co] that $K^A \equiv K^{A_{\widetilde{I}}}$ (K^A defined as in (2.10)), where

$$A_{\widetilde{I}}(x) \equiv A(x) - m_{\widetilde{I}}(\nabla A)\cdot x\,.$$

(Here m_I denotes the mean value $m_I(f) \equiv \frac{1}{|I|}\int_I f$.) We write $Q_sT_0 = U + V + W$, where

$$Uf(x) \equiv \iint \psi_s(x-y)k_0(y-u) \times [A_{\widetilde{I}}(y) - A_{\widetilde{I}}(u)]|y-u|^{-1}f(u)\,du\,dy,$$
$$Vf(x) \equiv \iint \psi_s(x-y)k_0(y-u) \times [\nabla A(x) - \nabla A(u)]\cdot\frac{y-u}{|y-u|}f(u)\,du\,dy\,,$$
$$Wf(x) \equiv \iint \psi_s(x-y)k_0(y-u) \times [m_{\widetilde{I}}(\nabla A) - \nabla A(x)]\cdot\frac{y-u}{|y-u|}f(u)\,du\,dy\,,$$

with

$$k_0(y-u) \equiv \frac{\Omega(y-u)}{|y-u|^n}\varphi(|y-u|)\,.$$

We consider V and W first, as they are easier to handle than U. Now,

$$\int |Wf_i|^2 \le \int |\nabla A(x) - m_{\tilde I}(\nabla A)|^2 |Q_s \vec S_0 f_i(x)|^2\,dx\,, \tag{2.12}$$

where $\vec S_0$ has kernel $\vec K_0(x) \equiv k_0(x)x/|x|$. By Schwarz, the right hand side of (2.12) is no larger than

$$\Big(\int |\nabla A(x) - m_{\tilde I}(\nabla A)|^4 |Q_s\vec S_0 f_i(x)|^2\,dx\Big)^{1/2} \|Q_s \vec S_0 f_i\|_2\,. \tag{2.13}$$

By the result for convolution operators (2.9), the second factor in (2.13) is no larger than $Cs^\varepsilon\|f_i\|_2$. Thus, to obtain the required estimate (2.11) for W, it is enough to show that the first factor in (2.13) is bounded by $C\|\nabla A\|_*^2\|f_i\|_2$. By Schwarz, this factor is no larger than

$$\Big(\int |\nabla A(x) - m_{\tilde I}(\nabla A)|^4 \|L\|_1 \int |L(x-u)||f_i(u)|^2\,du\,dx\Big)^{1/2}, \tag{2.14}$$

where $L(x-u) \equiv \int |\psi_s(x-y)||\vec k_0(y-u)|\,dy$, so that $\|L\|_1 \le C\|\Omega\|_1 \le C$. Now, for all $u \in I_i$, $\operatorname{supp} L(\cdot - u) \subseteq \tilde I$, and $|\tilde I| = C_n$. Thus (2.14) is dominated by

$$C\Big(\int |f_i(u)|^2\Big[\frac{1}{|\tilde I|}\int_{\tilde I} |\nabla A(x) - m_{\tilde I}(\nabla A)|^4 |L(x-u)|\,dx\Big]\,du\Big)^{1/2},$$

and the required estimate then follows by applying Hölder's inequality to the inner integral with exponents r' and r, since $\|L\|_r \le C\|\psi\|_1\|\Omega\|_r$.

We now consider V, which is just the commutator $[\nabla A, Q_s\vec S_0]$. We have already observed that

$$\|Q_s\vec S_0 f\|_2 \le Cs^\varepsilon\|f\|_2\,.$$

Furthermore, trivially

$$\|Q_s\vec S_0 f\|_1 \le C\|f\|_1\,,$$

so by interpolation and duality, we have

$$\|Q_s\vec S_0 f\|_q \le Cs^{\varepsilon'}\|f\|_q, \quad 1 < q < \infty\,, \tag{2.15}$$

where $\varepsilon' > 0$ depends on q. Also, $Q_s\vec S_0$ maps L^p into $L^{p+\delta}$ for $p > r'$ since

$$\int |Q_s\vec S_0 f|^{p+\delta} \le C \int_{10\sqrt{n}I_i} |\vec S_0 f|^{p+\delta}$$

$$\le C\int\Big(\int \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}|f(y)|\,dy\Big)^{p+\delta} dx\,,$$

where we have used the fact that the kernel of $\vec S_0$ is supported in an annulus $|x-y| \approx 1$. But for $0 < \alpha < n/p$, and $(p+\delta)^{-1} = p^{-1} - \alpha/n$, the last expression is no larger than

$$C\|\Omega\|_r^{p+\delta}\|f\|_p^{p+\delta}\,,$$

by the Hardy–Littlewood–Sobolev theorem if $r = \infty$, or by a similar argument (see, e.g., [S, pp. 119–120]) if $p > r'$. Thus by interpolation with (2.15), we obtain (with smaller values of ε and δ),

$$\|Q_s\vec S_0 f\|_{q+\delta} \le Cs^\varepsilon\|f\|_q, \quad 1 < q < \infty\,. \tag{2.16}$$

Then $[\nabla A, Q_s\vec S_0] = [\nabla A_{\tilde I}, Q_s\vec S_0]$, and the desired estimate follows from (2.16) and Hölder's inequality applied to each of $\nabla A_{\tilde I}\cdot Q_s\vec S_0 f_i$ and $Q_s\vec S_0(\nabla A_{\tilde I} f_i)$. It remains to prove

$$\|Uf_i\|_2 \le Cs^\varepsilon\|f_i\|_2\,. \tag{2.17}$$

Now, $U \equiv Q_s T_{\tilde I}$, where $T_{\tilde I}$ has kernel

$$K_{\tilde I}(y,u) \equiv \frac{\Omega(y-u)}{|y-u|^{n+1}}\varphi(|y-u|)[A_{\tilde I}(y) - A_{\tilde I}(u)]\,. \tag{2.18}$$

We shall need the following unpublished inequality of M. Weiss, whose proof can be found in a paper of C. Calderón [CC, Lemma 1.4].

Lemma 2.2. *Suppose ∇A belongs to L^q_{loc}, $q > n$. Then for $\gamma > 1$ we have*

$$\frac{|A(x)-A(y)|}{|x-y|} \le C_{q,\gamma}\Big(\frac{1}{|x-y|^n}\int_{|x-z|\le\gamma|x-y|} |\nabla A(z)|^q\,dz\Big)^{1/q}.$$

We recall that I_i is a unit cube, and $\tilde I \equiv 1000nI_i$. Thus for $u \in 100nI_i$, by Lemma 2.2 and the definition of $A_{\tilde I}$, we have

$$\frac{|A_{\tilde I}(y) - A_{\tilde I}(u)|}{|y-u|}\varphi(|y-u|) \le C_n\|\nabla A\|_*\,. \tag{2.19}$$

We rewrite $Q_s T_{\tilde I} f_i(x)$ as

$$Q_s R_0(-A_{\tilde I} f_i)(x) + A_{\tilde I}(x) Q_s R_0 f_i(x) + [Q_s, A_{\tilde I}]R_0 f_i(x) \equiv I + II + III\,,$$

where R_0 denotes convolution with $(\Omega(y)/|y|^{n+1})\varphi(|y|)$. The kernel of $[Q_s, A_{\tilde I}]$ is bounded in absolute value by

$$\int |Q_s(x-y)||A_{\tilde I}(y) - A_{\tilde I}(x)|\,dy \le Cs^\varepsilon\|\psi\|_1\|\nabla A\|_*\,,$$

where in the last inequality we have used Lemma 2.2 to write for $x, y \in 20\sqrt{n}I_i$,

$$|A_{\tilde I}(y) - A_{\tilde I}(x)| \le |x-y|\Big(\frac{|\tilde I|}{|x-y|^n}\frac{1}{|\tilde I|}\int_{\tilde I} |\nabla A_{\tilde I}|^q\Big)^{1/q}.$$

The desired estimate for III follows easily.

In $I+II$, we can use an idea of Cohen [Co] to replace $A_{\tilde{I}}$ by

$$A_\eta(x) \equiv \eta(x)[A_{\tilde{I}}(x) - A_{\tilde{I}}(x_0)]$$

where x_0 is a point on the boundary of $500\sqrt{n}I_i$, and where we choose $\eta \in C_0^\infty$ such that $0 \le \eta \le 1$, η is identically one on $100\sqrt{n}I_i$ and vanishes outside of $200\sqrt{n}I_i$, and $\|\nabla\eta\|_\infty \le C_n$.

The kernel of R_0, call it k_0, satisfies $\|\widehat{k}_0(\xi)\|_\infty \le C\min(1, |\xi|^{-\alpha})$, so as above Q_sR_0 satisfies (2.9) and also (2.16). Thus we can treat $I+II$ in the same way that we handled $[\nabla A, Q_s\vec{S}_0]$, once we show that for all large, finite q_0, $\|A_\eta\|_{L^{q_0}} \le C\|\nabla A\|_*$. To do this, we use the identity $I = I_1\vec{R}\cdot\nabla$, where I_1 is the usual fractional integral of order 1, and $\vec{R} = (R_1,\ldots,R_n)$, and the R_j's are the Riesz transforms. By Sobolev's theorem and the L^p boundedness of $\vec{R}$, we have

$$\|A_\eta\|_{L^{q_0}} \le C\|\nabla A_\eta\|_{L^p}, \quad 1<p<n,$$

with $1/q_0 = 1/p - 1/n$. We now follow [Co, p. 698] to estimate ∇A_η. In fact, for $y \in \operatorname{supp}\eta \subseteq 200\sqrt{n}I_i$, by Lemma 2.2 we have

$$|\nabla A_\eta(y)| \le C\biggl\{|\nabla\eta(y)||x_0-y|\biggl\{\frac{1}{|x_0-y|}\int_{\tilde{I}}|\nabla A_{\tilde{I}}|^q\biggr\}^{1/q} + \nabla A_{\tilde{I}}(y)\biggr\}.$$

But $|x_0-y|\approx 1$, and $|\tilde{I}| = C_n$, so $\|\nabla A_\eta\|_{L^p} \le C_{\eta,p}\|\nabla A\|_*$, and this concludes the proof of Lemma 2.1.

3. $T1$ **theorem for nonstandard kernels.** Let T be a singular integral operator with associated kernel $K(x,y)$. We have the following:

THEOREM 3.1. *Suppose that T satisfies the weak size condition* (2.4), *the weak smoothness conditions* (2.5) *and* (2.6) *and the technical condition* (2.7). *Suppose also that $T1 \in \mathrm{BMO}$, $T^*1 \in \mathrm{BMO}$, and that T satisfies WBP* (1.3). *Then T extends to a bounded operator on L^2.*

3.1. Proof of Theorem 3.1. In the original proof of the $T1$ theorem by David and Journé [DJ], they first considered the special case $T1 = 0 = T^*1$. For if $T1 \equiv b$, $T^*1 \equiv b^* \in \mathrm{BMO}$, and $P_tf \equiv t^{-n}p(|\cdot|/t) * f$ with $p \in C_0^\infty$, $\int p = 1$, then

$$\Pi_b \equiv \int_0^\infty Q_t((Q_tb)P_t)\,\frac{dt}{t} \tag{3.1}$$

defines a bounded operator on L^2 (and therefore satisfying WBP) with standard kernel (as in (1.1) and (1.2)). Furthermore, $\Pi_b1 = b$, since (formally) $\Pi_b1 = \int_0^\infty Q_t^2b\,dt/t$ where $\int_0^\infty Q_t^2\,dt/t = I$. We refer the reader to [DJ] for

details. They then removed the restriction $T1 = 0 = T^*1$ by replacing T with $\widetilde{T} \equiv T - \Pi_b - \Pi^*_{b^*}$. Observe that $\widetilde{T}$ will also satisfy (2.4)–(2.8), since these easily hold for standard kernels, so we could also use this approach. We shall, however, instead treat T directly, which gives rise to error terms essentially like (3.1). This approach will be convenient for us, because additional such error terms will arise in our applications, and will be disposed of by the same method.

We now use the Calderón reproducing formula $\int_0^\infty Q_t^2\,dt/t = I$, and, by the same argument as in [DJS, Sections 1 and 2], it is enough to show that

$$\biggl|\int_{\varepsilon_1}^{1/\varepsilon_1}\int_{\varepsilon_2}^{1/\varepsilon_2}\langle Q_sTQ_t^2f, Q_sg\rangle\,\frac{ds}{s}\,\frac{dt}{t}\biggr| \le C\|f\|_2\|g\|_2, \tag{3.2}$$

with bound independent of ε_1 and ε_2. Since our hypotheses are symmetric with respect to T and T^*, by duality it is enough to consider the case $s \le t$. For each pair (s,t), let $N_0 \equiv N_0(s,t)$ be the least N such that $2^{N_0} \ge s^\theta t^{1-\theta}$, where $0<\theta<1$ will be chosen later. We write

$$Q_sTQ_t \equiv Q_sT^{N_0}Q_t + (Q_sTQ_t - Q_sT^{N_0}Q_t) \equiv R_{s,t} + S_{s,t}, \tag{3.3}$$

where we recall that $T^N \equiv \sum_{j\ge N}T_j$, and define corresponding operators

$$R \equiv R(\varepsilon_1,\varepsilon_2) \equiv \int_{\varepsilon_1}^{1/\varepsilon_1}\int_{\varepsilon_2}^{1/\varepsilon_2}\chi\{s\le t\}Q_sR_{s,t}Q_t\,\frac{ds}{s}\,\frac{dt}{t}, \tag{3.4}$$

$$S \equiv S(\varepsilon_1,\varepsilon_2) \equiv \int_{\varepsilon_1}^{1/\varepsilon_1}\int_{\varepsilon_2}^{1/\varepsilon_2}\chi\{s\le t\}Q_sS_{s,t}Q_t\,\frac{ds}{s}\,\frac{dt}{t}. \tag{3.5}$$

Here, with a slight abuse of notation, we indicate by $\chi\{s\le t\}$ the function which is identically one if $t \ge s$, and zero otherwise. Now, R can be handled trivially: by (2.6) we have

$$\|R_{st}Q_tf\|_2 \le C\omega((s/t)^{1-\theta})\|Q_tf\|_2,$$

and therefore the estimate

$$\begin{aligned}\int_0^\infty\int_0^t|\langle R_{s,t}Q_tf, Q_sg\rangle|\,\frac{ds}{s}\,\frac{dt}{t} &\le C\biggl(\int_0^\infty\int_0^t\omega\biggl(\Bigl(\frac{s}{t}\Bigr)^{1-\theta}\biggr)\|Q_tf\|_2^2\,\frac{ds}{s}\,\frac{dt}{t}\biggr)^{1/2}\\ &\quad\times\biggl(\int_0^\infty\int_s^\infty\omega\biggl(\Bigl(\frac{s}{t}\Bigr)^{1-\theta}\biggr)\|Q_sg\|_2^2\,\frac{dt}{t}\,\frac{ds}{s}\biggr)^{1/2}\\ &\le C\|f\|_2\|g\|_2\end{aligned} \tag{3.6}$$

follows immediately by Schwarz's inequality, Littlewood–Paley theory, and the fact that ω satisfies the Dini condition $\int_0^1 \omega(s)\,ds/s < \infty$.

Next, we rewrite $S_{s,t}$ as

$$[Q_sTQ_t - (Q_sT^{N_0}Q_t - (Q_sT^{N_0}1)Q_t)] - (Q_sT^{N_0}1)Q_t\,. \tag{3.7}$$

By (2.5), and the definition of N_0, we have the pointwise estimate

$$|(Q_sT^{N_0}1)Q_t^2 f| \le C\omega((s/t)^{1-\theta})M(Q_t f),$$

so for this term we have the bound

$$C\int_0^\infty\int_0^t \omega((s/t)^{1-\theta})\langle M(Q_tf), |Q_sg|\rangle\,\frac{ds}{s}\,\frac{dt}{t}\,, \tag{3.8}$$

and we can proceed as for (3.6).

We turn to the expression in square brackets in (3.7). This term will be treated by a technical modification of an argument in [CDMS, Lemma 2.3, the case $|x-y|\le 4t$]. By definition, for $\eta \in C_0^\infty(-11,11)$, and $\eta \equiv 1$ on $(-10,10)$, we have

$$\begin{aligned} Q_sT1(x) \equiv {} & \Big\langle \psi_s(x-\cdot), T\Big(\eta\Big(\frac{|x-\cdot|}{s^\theta t^{1-\theta}}\Big)\Big)\Big\rangle \\ & + \Big\langle T^*(\psi_s(x-\cdot)), 1-\eta\Big(\frac{|x-\cdot|}{s^\theta t^{1-\theta}}\Big)\Big\rangle, \end{aligned}$$

where the last expression is well defined by (2.7). We write $Q_sTQ_t = Q_s(T1)Q_t + Q_s(T-T1)Q_t$. Now, $Q_s(T-T1)Q_t$, for fixed $s\le t$, has kernel $H(x,y)$ given by

$$\begin{aligned} H(x,y) & \\ \equiv {} & \Big\langle \psi_s(x-\cdot), T\Big[\eta\Big(\frac{|x-\cdot|}{s^\theta t^{1-\theta}}\Big)(\psi_t(\cdot-y)-\psi_t(x-y))\Big]\Big\rangle \\ & + \Big\langle T^*(\psi_s(x-\cdot)), \Big[1-\eta\Big(\frac{|x-\cdot|}{s^\theta t^{1-\theta}}\Big)\Big][\psi_t(\cdot-y)-\psi_t(x-y)]\Big\rangle \\ \equiv {} & H^{(1)}(x,y) + H^{(2)}(x,y)\,. \end{aligned} \tag{3.9}$$

Since $s\le t$, $H^{(1)}(x,y)$ is supported in $\{(x,y) : |x-y|\le ct\}$, and a grubby computation involving WBP and the mean value theorem shows that

$$|H^{(1)}(x,y)| \le \Big(\frac{s}{t}\Big)^\varepsilon t^{-n},$$

if we take $\theta \equiv (n+1+\varepsilon)/(n+2) < 1$ for $0<\varepsilon<1$. Thus, the operator corresponding to $H^{(1)}(x,y)$ is controlled by $(s/t)^\varepsilon$ times the Hardy–Littlewood maximal function, and therefore this part of S can be handled exactly like (3.8).

Now, the only remaining parts of the expression in square brackets in (3.7) are $Q_s(T1)Q_t$, whose treatment we defer for the moment, and

$$H^{(2)} - (Q_sT^{N_0}Q_t - (Q_sT^{N_0}1)Q_t)\,, \tag{3.10}$$

where we have, by abuse of notation, denoted by $H^{(2)}$ the operator corresponding to $H^{(2)}(x,y)$. By (2.7), $H^{(2)}(x,y)$ is well defined as

$$\iint \psi_s(x-z)K(z,u)\,dz\Big[1-\eta\Big(\frac{|x-u|}{s^\theta t^{1-\theta}}\Big)\Big][\psi_t(u-y)-\psi_t(x-y)]\,du,$$

and we can replace K by K^{N_0} without changing the value of the integral. But this implies that the kernel of the operator in (3.10) is given by

$$\iint \psi_s(x-z)K^{N_0}(z,u)\eta\Big(\frac{|x-u|}{s^\theta t^{1-\theta}}\Big)[\psi_t(u-y)-\psi_t(x-y)]\,dz\,du\,,$$

which is bounded by

$$\begin{aligned} & \int_{|x-u|\le 11 s^\theta t^{1-\theta}} |\psi_s(x-z)||K(z,u)| \\ & \qquad \times \int_{s^\theta t^{1-\theta}/4\le|z-u|\le 12 s^\theta t^{1-\theta}} |\psi_t(u-y)-\psi_t(x-y)|\,dz\,du \\ & \le C(s/t)^{\theta-n(1-\theta)}t^{-n}\chi\{|x-y|\le Ct\}\,, \end{aligned}$$

where in the last inequality we have used the size condition (2.4) and the fact that $\|\psi_s\|_\infty \le Cs^{-n}$, as well as the mean value theorem. Now, $\theta - n(1-\theta) > 0$ for θ close enough to 1, so we can handle this part of the operator exactly like the corresponding estimate for $H^{(1)}$.

It now remains only to consider the part of S corresponding to $Q_s(T1)Q_t$. Since $T1\in \mathrm{BMO}$, it is enough to treat

$$\int_{\varepsilon_1}^{1/\varepsilon_1}\int_{\varepsilon_2}^{1/\varepsilon_2} \chi\{s\le t\}Q_s[Q_s(b)Q_t^2]\,\frac{ds}{s}\,\frac{dt}{t}\,, \tag{3.11}$$

with $b\in\mathrm{BMO}$. To simplify the exposition, we argue formally with $\varepsilon_1 = \varepsilon_2 = 0$, and we shall add a brief comment at the end of the proof explaining how to make the argument rigorous. We shall see that (3.11) is essentially Π_b of (3.1). We shall show that (3.11) is in fact bounded on L^2_w, $w\in A_2$, since we shall need this result in the sequel. To handle (3.11), we first recall an observation of Han and Sawyer [HS] that the radial kernel $p_s(|x|)$ of the operator $P_s \equiv \int_s^\infty Q_t^2\,dt/t$ satisfies

$$(3.12)\quad \begin{aligned}&\text{(i) } \|p_s(|x|)\|_\infty \le Cs^{-n}\,,\\ &\text{(ii) } \operatorname{supp} p_s \subseteq \{|x| \le 2s\}\,.\end{aligned}$$

The first of these is trivial, since

$$|p_s(|x|)| = \Big|\int_s^\infty t^{-n}\theta\Big(\frac{|x|}{t}\Big)\frac{dt}{t}\Big| \le C\|\theta\|_\infty s^{-n}\,,$$

where $\theta \equiv \psi * \psi$. The second follows by changing variables:

$$p_s(|x|) = |x|^{-n}\int_{s|x|^{-1}}^\infty t^{-n}\theta\Big(\frac{1}{t}\Big)\frac{dt}{t} = |x|^{-n}\int_0^{|x|/s}\theta(r)r^{n-1}\,dr\,.$$

But θ is radial, has mean value zero, and is supported in $\{|x|<2\}$, so the last integral is zero if $|x|>2s$. Thus (3.11) equals

$$\int_0^\infty Q_s[(Q_sb)P_s]\,\frac{ds}{s}\,.$$

But this is essentially Π_b, so by an observation of Journé [J, pp. 85–87], it is bounded on L^2_w, $w\in A_2$, with an operator norm no larger than $C\|b\|_*$.

To make the proof rigorous, one would split (3.11) into:

$$\int_{\varepsilon_2}^{1/\varepsilon_2}\chi\{s<\varepsilon_1\}\int_{\varepsilon_1}^{1/\varepsilon_1}\ldots\frac{dt}{t}\frac{ds}{s} + \int_{\varepsilon_2}^{1/\varepsilon_2}\chi\{\varepsilon_1\le s\}\int_s^{1/\varepsilon_1}\ldots\frac{dt}{t}\frac{ds}{s} \equiv \Pi_b^1 + \Pi_b^2\,.$$

Then $\langle \Pi_b^1 f, g\rangle$ splits into a difference of two terms of the form

$$\int_{\varepsilon_2}^{1/\varepsilon_2}\chi\{s<\varepsilon_1\}\langle (Q_sb)P_\beta f, Q_sg\rangle\,\frac{ds}{s}\,,$$

where $\beta=\varepsilon_1$ or $1/\varepsilon_1$. By Schwarz and weighted Littlewood–Paley theory, and the observation of [J, pp. 85–87], the square of the last expression is then bounded by

$$C\|g\|^2_{2,1/w}\cdot\|b\|_*^2\int_{\mathbb{R}^n}(N(\chi\{s<\varepsilon_1\}P_\beta f)(x))^2 w(x)\,dx\,,$$

where N is the nontangential maximal function

$$N(F_s(\cdot))(x) \equiv \sup_{(y,s):|x-y|<s}|F_s(y)|\,.$$

But for $\alpha\le\beta$, the reader may readily verify that $N(\chi\{s<\alpha\}P_\beta f)(x)$ is controlled by the maximal function.

To handle Π_b^2, we write

$$\chi\{\varepsilon_1\le s\le t\le 1/\varepsilon_1\} = \chi\{\varepsilon_1\le s\le 1/\varepsilon_1\}[\chi\{s\le t\}-\chi\{t>1/\varepsilon_1\}]\,.$$

Thus,

$$(3.13)\quad \begin{aligned}\langle \Pi_b^2 f, g\rangle = &\int_{\varepsilon_2}^{1/\varepsilon_2}\chi\{\varepsilon_1\le s\le 1/\varepsilon_1\}\langle (Q_sb)P_sf, Q_sg\rangle\,\frac{ds}{s}\\ &-\int_{\varepsilon_2}^{1/\varepsilon_2}\chi\{\varepsilon_1\le s\le 1/\varepsilon_1\}\langle (Q_sb)P_{1/\varepsilon_1}f, Q_sg\rangle\,\frac{ds}{s}\,.\end{aligned}$$

The second term in (3.13) can be handled exactly like Π_b^1. The first term in (3.13) is even easier, being bounded in absolute value by

$$\int_0^\infty |\langle (Q_sb)P_sf, Q_sg\rangle|\,\frac{ds}{s}\,,$$

and we repeat our previous argument.

3.2. *A short remark on weighted norm inequalities.* As above, to prove that T is bounded on L^2_w, it will be enough to show that, for $f, g\in C_0^\infty$, we have

$$(3.14)\quad \Big|\int_{\varepsilon_1}^{1/\varepsilon_1}\int_{\varepsilon_2}^{1/\varepsilon_2}\chi\{s\le t\}\langle Q_sTQ_t^2f, Q_sg\rangle\,\frac{ds}{s}\frac{dt}{t}\Big| \le C\|f\|_{2,w}\|g\|_{2,1/w}$$

for all $w\in A_2$, where C depends on the A_2 constant of w (but not on the weight w itself). The case $p\ne 2$ follows by the extrapolation theorem for A_p weights (see, e.g., García-Cuerva and Rubio de Francia [GR, Chapter IV]). Here we are using the notation

$$\|f\|_{p,w} \equiv \Big(\int |f|^pw\Big)^{1/p},\quad\text{and}\quad L^p_w \equiv \{f : \|f\|_{p,w}<\infty\}\,.$$

We have seen that $S_{s,t}$ (from the splitting of Q_sTQ_t, see (3.3)) is controlled by $C[\omega((s/t)^{1-\theta})+(s/t)^\varepsilon]\equiv\omega_0(s/t)$ times the maximal function, plus $Q_s(T1)Q_t$, which gave rise to the error term (3.11). The latter was shown to be bounded on L^2_w, $w\in A_2$. Thus, with S defined as in (3.5), we have (modulo the bounded error term)

$$\begin{aligned}|\langle Sf, g\rangle| &\le \int_0^\infty\int_0^t \omega_0(s/t)\langle M(Q_tf), |Q_sg|\rangle\,\frac{ds}{s}\frac{dt}{t}\\ &\le \int_0^\infty\int_0^t \omega_0(s/t)\|Q_tf\|_{2,w}\|Q_sg\|_{2,1/w}\,\frac{ds}{s}\frac{dt}{t}\,,\end{aligned}$$

for all $w\in A_2$, where in the last inequality we have used Schwarz and the weighted norm inequality for M. By weighted Littlewood–Paley theory and our previous argument, S is bounded on L^2_w, for all $w\in A_2$, and therefore on L^p_w, $w\in A_p$, $1<p<\infty$, by extrapolation. The only obstacle, then, to weighted norm inequalities is R (see (3.4)).

It will now be easy to prove a weighted version of Theorem 3.1.

THEOREM 3.2. *Suppose that T and K are as in Theorem* 3.1, *except that we impose the stronger smoothness condition* (2.8) *in place of* (2.6). *Suppose also that*

$$\|T_jf\|_{2,w}+\|T_j^*f\|_{2,w}\le C\|f\|_{2,w},\qquad w\in A_2\,. \tag{3.15}$$

Then T is bounded on L^p_w, $w\in A_p$, $1<p<\infty$.

We remark that (3.15) holds in particular in the case that $K(x,y)\le C|x-y|^{-n}$, in which case T_j is controlled by the maximal function.

Proof of Theorem 3.2. Duoandikoetxea and Rubio de Francia [DR] have proved weighted norm inequalities by interpolating between crude weighted estimates and sharper unweighted estimates, and Theorem 3.2 is based on their idea. In the present situation, by (2.8) we have

$$\|Q_sT_jf\|_2\le C(2^{-j}s)^{\varepsilon}\|f\|_2\,, \tag{3.16}$$

and the same for T_j^*. Now, $w\in A_2$ implies that $w^{1+\delta}\in A_2$ for some positive δ sufficiently small. Thus, since Q_s is controlled by the maximal function, by (3.15) we have

$$\|Q_sT_jf\|_{2,w^{1+\delta}}\le C\|f\|_{2,w^{1+\delta}}\,, \tag{3.17}$$

and the same for T_j^*. Thus, interpolation with change of measure between (3.16) and (3.17) yields a weighted version of (3.16) (with a smaller ε):

$$\|Q_sT_jf\|_{2,w}\le C(2^{-j}s)^{\varepsilon}\|f\|_{2,w}\,, \tag{3.18}$$

and the same for T_j^*.

The proof of Theorem 3.2 will now be easy. By extrapolation, we may assume $p=2$, and by the comments above, it is enough to consider the operator R defined in (3.4). By definition, and the self-adjointness of Q_s, we have

$$\langle Rf,g\rangle=\int\limits_{\varepsilon_1}^{1/\varepsilon_1}\int\limits_{\varepsilon_2}^{1/\varepsilon_2}\chi\{s\le t\}\sum_{2^j\ge s^\theta t^{1-\theta}}\langle Q_sT_jQ_t^2f,Q_sg\rangle\,\frac{ds}{s}\frac{dt}{t}\,.$$

By Schwarz and (3.18), this last expression is bounded in absolute value by a constant times

$$\int\limits_0^{\infty}\int\limits_0^{t}(s/t)^{(1-\theta)\varepsilon}\|Q_tf\|_{2,w}\|Q_sg\|_{2,1/w}\,\frac{ds}{s}\frac{dt}{t}\,,$$

and the desired estimate follows by another application of Schwarz's inequality and weighted Littlewood–Paley theory.

4. Proof of Theorem 1.1: L^2 and L^2_w bounds for T^A. The proof will be essentially a direct application of Theorems 3.1 and 3.2, except for an error term which can be controlled by Carleson measure estimates. In the sequel, we shall assume that the operator has been truncated; that is, we take $T^A\equiv T^A(N,M)\equiv\sum_{j=N}^M T_j^A$, and obtain bounds independent of the truncation. We remark that for such a truncation, the technical condition (2.7) holds trivially. We shall leave implicit the truncation of the sum, except for those parts of the argument where it plays a role.

We shall need to establish some facts. A crucial one is, of course, the following

LEMMA 4.1. $T^A1, T^{A*}1\in \mathrm{BMO}$.

Proof. We consider T^A1 first. We have

$$\begin{aligned}&T^A1(x)\\&=\int\sum_{j=N}^{M}\varphi(2^{-j}|x-y|)\frac{\Omega(x-y)}{|x-y|^{n+1}}[A(x)-A(y)-\nabla A(y)\cdot(x-y)]\,dy\,.\end{aligned}\tag{4.1}$$

In polar coordinates, $T^A1(x)$ equals

$$\begin{aligned}&\int\limits_{S^{n-1}}\Omega(\theta)\int\limits_0^{\infty}\sum_{j=N}^{M}\varphi(2^{-j}\varrho)[A(x)-A(x-\varrho\theta)]\,\frac{d\varrho}{\varrho^2}\,d\theta\\&\quad-\int\limits_{S^{n-1}}\theta\Omega(\theta)\cdot\int\limits_0^{\infty}\sum_{j=N}^{M}\varphi(2^{-j}\varrho)\nabla A(x-\varrho\theta)\frac{d\varrho}{\varrho}\,d\theta\,.\end{aligned}\tag{4.2}$$

By an integration by parts in the $d\varrho$ integral, the first term in (4.2) will exactly cancel the second term except for "boundary" terms

$$\int\limits_{S^{n-1}}\Omega(\theta)\int\limits_0^{\infty}\frac{A(x)-A(x-\varrho\theta)}{\varrho}[2^{-M}\varphi'(2^{-M}\varrho)+2^{-N}\varphi'(2^{-N}\varrho)]\,d\varrho\,d\theta\,.$$

Now, by the moment condition (1.8), this last expression equals

$$\int\limits_{S^{n-1}}\Omega(\theta)\int\limits_0^{\infty}\frac{A(x)-A(x-\varrho\theta)-m_I(\nabla A)\cdot\varrho\theta}{\varrho}(2^{-N}\varphi'(2^{-N}\varrho))\,d\varrho\,d\theta\,, \tag{4.3}$$

plus an analogous term corresponding to $2^{-M}\varphi'(2^{-M}\varrho)$, where $I=I(x)$ has center x and side length $10\sqrt{n}2^N$ (or $10\sqrt{n}2^M$ for the other term). Since the expression in the numerator in (4.3) equals $A_I(x)-A_I(x-\varrho\theta)$, where $A_I(x)\equiv A(x)-m_I(\nabla A)\cdot x$, a familiar argument involving Lemma 2.2 shows that (4.3) is bounded by $C\|\Omega\|_1\|\nabla A\|_*$.

We now consider $T^{A*}1$. This is (4.1), except with $\widetilde{\Omega}(x)\equiv-\Omega(-x)$ in place of Ω, and $\nabla A(x)$ in place of $\nabla A(y)$. Furthermore, by the moment condition (1.8),

$$\nabla A(x)\cdot\int_{S^{n-1}}\widetilde{\Omega}(\theta)\theta\,d\theta=0\,,$$

so in polar coordinates $T^{A*}1(x)$ equals the first term in (4.2), except with $\widetilde{\Omega}$ in place of Ω.

To show $T^{A*}1\in\mathrm{BMO}$, it is enough to prove that, for all H^1 atoms ψ, we have

$$|\langle\psi,T^{A*}1\rangle|\le C\|\Omega\|_r\|\nabla A\|_*\,. \tag{4.4}$$

Let ψ be supported in the cube with center x_0 and side length s, with $\|\psi\|_\infty\le s^{-n}$ and $\int\psi=0$. Now let $\eta\in C_0^\infty[-11\sqrt{n},11\sqrt{n}]$, and suppose $\eta\equiv1$ on $[-10\sqrt{n},10\sqrt{n}]$. Set $\Phi_N^M(\varrho)\equiv\sum_{j=N}^M\varphi(2^{-j}\varrho)$. We can write

$$\begin{aligned}T^{A*}1(x)&=\int_{S^{n-1}}\widetilde{\Omega}(\theta)\int_0^\infty\Phi_N^M(\varrho)[A(x)-A(x-\varrho\theta)][1-\eta(\varrho/s)]\,\frac{d\varrho}{\varrho^2}\,d\theta\\&\quad+\int_{S^{n-1}}\widetilde{\Omega}(\theta)\int_0^\infty\Phi_N^M(\varrho)[A(x)-A(x-\varrho\theta)]\eta(\varrho/s)\,\frac{d\varrho}{\varrho^2}\,d\theta\\&\equiv F(x)+G(x)\,.\end{aligned}$$

By the moment condition (1.8),

$$F(x)=\int_{S^{n-1}}\widetilde{\Omega}(\theta)\int_0^\infty\sum\varphi(2^{-j}\varrho)[A_j(x)-A_j(x-\varrho\theta)][1-\eta(\varrho/s)]\,\frac{d\varrho}{\varrho^2}\,d\theta\,, \tag{4.5}$$

where the sum runs over j such that $2^M\ge2^j\ge\max(2^N,\,10\sqrt{n}s)$, and where

$$A_j(x)\equiv A(x)-m_{I_j}(\nabla A)\cdot x\,,$$

and I_j is a cube with center x_0 and side length $10\cdot2^j$. Since ψ has mean value zero, we have

$$\langle\psi,F\rangle=\int\psi(x)[F(x)-F(x_0)]\,dx\,. \tag{4.6}$$

By Lemma 2.2, we have, for $\varrho\approx2^j$ and $|x-x_0|\le\sqrt{n}s<2^j/10$,

$$\begin{aligned}&|A_j(x)-A_j(x_0)|+|A_j(x-\varrho\theta)-A_j(x_0-\varrho\theta)|\\&\quad\le C|x-x_0|\bigg(\frac{1}{|x-x_0|^n}\int_{I_j(x_0)}|\nabla A_j|^q\bigg)^{1/q}\\&\quad\le C|x-x_0|^{1-n/q}2^{jn/q}\|\nabla A\|_*\le Cs^{1-n/q}2^{jn/q}\|\nabla A\|_*\,.\end{aligned}$$

A routine computation now yields the desired estimate (4.4) for $\langle\psi,F\rangle$.

We now turn to $G(x)$, which, after integrating by parts in the $d\varrho$ integral, equals

$$\begin{aligned}&\int_{S^{n-1}}\widetilde{\Omega}(\theta)\theta\cdot\int_0^\infty\frac{\nabla A(x-\varrho\theta)}{\varrho}\Phi_N^M(\varrho)\eta\Big(\frac{\varrho}{s}\Big)\,d\varrho\,d\theta\\&\quad+\int_{S^{n-1}}\widetilde{\Omega}(\theta)\int_0^\infty\frac{A(x)-A(x-\varrho\theta)}{\varrho}\Phi_N^M(\varrho)\frac{1}{s}\eta'\Big(\frac{\varrho}{s}\Big)\,d\varrho\,d\theta\equiv G_1(x)+G_2(x)\,,\end{aligned}$$

plus "boundary" terms which arise from differentiating Φ_N^M, and which can be handled exactly like the corresponding terms for T^A1 (see (4.3)).

We claim that, for $x\in\operatorname{supp}\psi$, we have

$$|G_2(x)|\le C\|\Omega\|_1\|\nabla A\|_*\,, \tag{4.7}$$

which yields the desired estimate (4.4) for $|\langle\psi,\,G_2\rangle|$. In fact, by definition, $\eta'(\varrho/s)$ is supported on the set $\{10\sqrt{n}s\le\varrho\le11\sqrt{n}s\}$. Furthermore, by the moment condition (1.8), we can replace $A(x)-A(x-\varrho\theta)$ by

$$A(x)-A(x-\varrho\theta)-m_{I(x_0)}(\nabla A)\cdot\varrho\theta\equiv A_I(x)-A_I(x-\varrho\theta), \tag{4.8}$$

where $I=I(x_0)$ has side length $20\sqrt{n}s$ and center x_0, and $A_I(x)\equiv A(x)-m_I(\nabla A)\cdot x$. Now x_0 is the center of the cube of side length s which supports ψ, so for $x\in\operatorname{supp}\psi$, (4.7) follows by a routine application of Lemma 2.2.

To handle $G_1(x)$, we return to rectangular coordinates, and write $G_1(x)$ equals

$$\begin{aligned}\int\frac{\Omega(x-y)}{|x-y|^n}\,\frac{x-y}{|x-y|}\cdot[\nabla A(y)-m_I(\nabla A)]&\\\times\Phi_N^M(|x-y|)\eta\Big(\frac{|x-y|}{s}\Big)\,dy\,,&\end{aligned} \tag{4.9}$$

for the same I as in (4.8), where the presence of $m_I(\nabla A)$ is permitted by the moment condition (1.8). For $x\in\operatorname{supp}\psi$, we can multiply the integrand in (4.9) by $\chi_I(y)$, so the estimate (4.4) follows from Schwarz and the well known L^2 boundedness of the (truncated) convolution singular integral with kernel

$$\frac{x\widetilde{\Omega}(x)}{|x|^{n+1}}\Phi_N^M(x)\eta\Big(\frac{|x|}{s}\Big)\,.$$

Next we prove a stronger version of (2.5).

Lemma 4.2. (i) $\|Q_sT_j^A1\|_\infty\le C\|\Omega\|_1\|\nabla A\|_*(2^{-j}s)^\varepsilon$, $2^j\ge s$.
(ii) $\|Q_sT_j^{A*}1\|_\infty\le C\|\Omega\|_r\|\nabla A\|_*(2^{-j}s)^\varepsilon$, $2^j\ge s$.

Sketch of proof. To treat $Q_sT_j^A1(x_0)$ we write $T_j^A1(x)$ in polar coordinates and integrate by parts exactly as in the proof of Lemma 4.1.

The result is an expression exactly like (4.3), with $N = j$, and where I is a cube with center x_0 and side length $5\sqrt{n}2^j$. Since $\int \psi = 0$, we have

$$Q_s T_j^A 1(x_0) = \int \psi_s(x_0 - x)[T_j^A 1(x) - T_j^A 1(x_0)]\,dx,$$

and an application of Lemma 2.2 (see the argument following (4.6)) concludes the proof of (i). The details are left to the reader. To prove (ii), we consider what is essentially a single j term in (4.5). The details are again left to the reader (see also (4.6) and the subsequent argument).

Next, we prove:

LEMMA 4.3. *T^A satisfies WBP.*

The proof follows a procedure with which the reader will by now be familiar. Suppose $\eta_1, \eta_2 \in C_0^\infty$ have support in the ball $\{x : |x - x_0| < r\}$. As usual, we may replace A by $A_I(x) \equiv A(x) - m_I(\nabla A)\cdot x$, where I has center x_0 and side length $20\sqrt{n}r$. As in the proof of Lemma 4.1, we change to polar coordinates and integrate by parts, so that $T^A\eta_2(x)$ equals

$$(4.10)\qquad -\int_{S^{n-1}} \Omega(\theta)\theta\cdot \int \frac{A_I(x) - A_I(x-\varrho\theta)}{\varrho}\Phi_N^M(\varrho)\nabla\eta_2(x-\varrho\theta)\,d\varrho\,d\theta$$

plus "boundary" terms

$$(4.11)\qquad \int_{S^{n-1}} \Omega(\theta)\theta \int \frac{A_I(x) - A_I(x-\varrho\theta)}{\varrho}\eta_2(x-\varrho\theta) \times [2^{-M}\varphi'(2^{-M}\varrho) + 2^{-N}\varphi'(2^{-N}\varrho)]\,d\varrho\,d\theta\,.$$

For $|x - x_0| \le r$, by Lemma 2.2 we see that (4.10) is bounded by

$$C\|\Omega\|_1\|\nabla A\|_*\|\nabla\eta_2\|_\infty \int_0^{Cr} (r/\varrho)^{n/q}\,d\varrho \le C\|\Omega\|_1\|\nabla A\|_*\|\nabla\eta\|_\infty r$$

since $q > n$, and integration against η_1 yields the desired estimate for this term.

Next, (4.11) equals (for $|x - x_0| \le r$)

$$(4.12)\qquad \int_{S^{n-1}} \Omega(\theta) \int_0^{Cr} \frac{A_I(x) - A_I(x-\varrho\theta)}{\varrho} \times [\eta_2(x-\varrho\theta) - \eta_2(x)](2^{-N}\varphi'(2^{-N}\varrho))\,d\varrho\,d\theta$$

$$(4.13)\qquad +\int_{S^{n-1}} \Omega(\theta) \int_0^{Cr} \frac{A_I(x) - A_I(x-\varrho\theta)}{\varrho}\eta_2(x)(2^{-N}\varphi'(2^{-N}\varrho))\,d\varrho\,d\theta\,,$$

plus analogous terms with M in place of N, which may be handled by the

same arguments. By Lemma 2.2,

$$|A_I(x) - A_I(x-\varrho\theta)| \le C(r/\varrho)^{n/q}\|\nabla A\|_*\,,$$

so by the mean value theorem, (4.12) is no larger than

$$(4.14)\qquad C\|\Omega\|_1\|\nabla A\|_*\|\nabla\eta_2\|_\infty \int_0^{Cr} (r/\varrho)^{n/q}\varrho 2^{-N}\varphi'(2^{-N}\varrho)\,d\varrho\,.$$

But $q > n$, and $2^N \le Cr$ (or the integral is zero), so the integral in (4.14) is bounded by

$$C2^{N(1-n/q)}r^{n/q}\|\varphi'\|_1 \le Cr\,,$$

and the desired estimate follows. Finally, in (4.13) we may use the moment condition (1.8) to replace A_I by A_J, where J has center x and side length $10\sqrt{n}2^N$. By Lemma 2.2, (4.13) is then no larger than $C\|\varphi'\|_1\|\Omega\|_1\|\eta_2\|_\infty \times \|\nabla A\|_*$, and WBP follows.

We now show that T^A satisfies the weighted estimates of Theorem 3.2 if $\Omega \in L^\infty$. We have

LEMMA 4.4. *If $\Omega \in L^\infty$, then for all $w \in A_2$,*

$$\|T_j^A f\|_{2,w} \le C\|\nabla A\|_*\|\Omega\|_\infty\|f\|_{2,w}\,.$$

Proof. The approach will be familiar. Let $I(x)$ be the cube with center x and side length $10\sqrt{n}2^j$. As usual we may replace A by $A_{I(x)}$, and by Lemma 2.2,

$$(4.15)\qquad \left|\int \frac{\Omega(x-y)}{|x-y|^{n+1}}[A_{I(x)}(x) - A_{I(x)}(y)]\varphi(2^{-j}|x-y|)f(y)\,dy\right| \le C\|\nabla A\|_*\|\Omega\|_\infty Mf(x)\,,$$

so the desired estimate follows for this part of the operator.

It remains to consider

$$\int \frac{\Omega(x-y)}{|x-y|^{n+1}}[\nabla A(y) - m_{I(x)}(\nabla A)]\cdot(x-y)\varphi(2^{-j}|x-y|)f(y)\,dy\,,$$

which is bounded in absolute value by

$$C\|\Omega\|_\infty (M(|f|^r))^{1/r}\left(\frac{1}{|I|}\int_I |\nabla A(y) - m_I(\nabla A)|^{r'}\,dy\right)^{1/r'}.$$

We may choose r so close to 1 that $w \in A_{2/r}$. This proves the lemma.

We are now in a position to finish the proof of Theorem 1.1. The only thing that prevents us from directly applying Theorems 3.1 and 3.2 is that T_j^A need not satisfy (2.8), but we shall see that the resulting error term can be controlled by Carleson measure estimates. We will show that (with $\|\Omega\|_r = 1 = \|\nabla A\|_*$)

$$\left| \int_{\varepsilon_1}^{1/\varepsilon_1} \int_{\varepsilon_2}^{1/\varepsilon_2} \langle Q_t^2 T^A Q_s^2 g, f\rangle \frac{ds}{s}\frac{dt}{t}\right| \le C\|f\|_2\|g\|_2 , \tag{4.16}$$

or that the left side of (4.16) is bounded by $C\|f\|_{2,w}\|g\|_{2,1/w}$, for all $w \in A_2$, when $\|\Omega\|_\infty = 1$. We split the left side of (4.16) so that we consider separately the cases $t \le s$ and $t \ge s$. We have shown in Lemma 2.1 and in Section 4 that T^A satisfies all the required estimates to carry out the proofs of the theorems in the case $t \le s$ (notice that in (4.16), we have reversed the order of the Q_t and Q_s operators). Furthermore, concerning the case $s \le t$, we have also verified in Section 4 that all the conditions hold which are used to treat (in the notation of (3.5))

$$\langle Sf, g\rangle \equiv \int_{\varepsilon_1}^{1/\varepsilon_1} \int_{\varepsilon_2}^{1/\varepsilon_2} \chi\{s \le t\} \langle S_{s,t} Q_t f, Q_s g\rangle \frac{ds}{s}\frac{dt}{t},$$

where

$$S_{s,t} \equiv Q_s T^{A*} Q_t - Q_s\Big(\sum_{2^j \ge s^\theta t^{1-\theta}} T_j^{A*}\Big) Q_t .$$

In the last expression, and in the sequel, we implicitly define $T_j^{A*} \equiv 0$ for $j > M$ or $j < N$. This is done to simplify the notation.

In fact, the only conditions used in Theorems 3.1 and 3.2 which we have not established are (2.8) for T_j^{A*}, and its weighted equivalent (3.18). Thus we need consider only that part of the operator whose treatment involved the use of those conditions, namely R of (3.4); i.e. we must show

$$\begin{aligned} &\langle Rf, g\rangle \\ &\equiv \int_{\varepsilon_1}^{1/\varepsilon_1} \int_{\varepsilon_2}^{1/\varepsilon_2} \chi\{s \le t\} \sum_{2^j \ge s^\theta t^{1-\theta}} \langle Q_s T_j^{A*} Q_t^2 f, Q_s g\rangle \frac{ds}{s}\frac{dt}{t} \\ &\le \begin{cases} C\|f\|_2\|g\|_2 & \text{if } \|\Omega\|_r = 1,\ 1 < r < \infty, \text{ or} \\ C\|f\|_{2,w}\|g\|_{2,1/w} & \text{if } \|\Omega\|_\infty = 1 . \end{cases} \end{aligned} \tag{4.17}$$

For fixed j, we again decompose $f \equiv \sum f_i$ a.e., where f_i is the restriction of f to a dyadic cube I_i with side length 2^j.

With $\widetilde{\Omega}(\theta) \equiv -\Omega(-\theta)$, and fixed, we write the kernel of T_j^{A*} as

$$\begin{aligned} K_j^{A*}(y, u) \equiv{} & \frac{\widetilde{\Omega}(y-u)}{|y-u|^{n+1}} \varphi(2^{-j}|y-u|)[A(y) - A(u) - m_{\widetilde{I}_i}(\nabla A)\cdot(y-u)] \\ &+ \frac{\widetilde{\Omega}(y-u)}{|y-u|^{n+1}} \varphi(2^{-j}|y-u|)[m_{\widetilde{I}_i}(\nabla A) - \nabla A(y)]\cdot(y-u) \\ \equiv{} & k_j^{(1)}(y,u) + k_j^{(2)}(y,u) , \end{aligned}$$

where $\widetilde{I}_i \equiv 1000nI_i$. Then $k_j^{(1)}$ is just the analogue of U in the proof of Lemma 2.1 but with $\widetilde{\Omega}$ in place of Ω, so it satisfies (2.8) and (for bounded Ω) condition (3.18). The corresponding part of (4.17) therefore satisfies the desired estimate.

We now turn to that part of (4.17) corresponding to $k_j^{(2)}$. We need to consider the action of Q_s on this kernel. We have

$$\begin{aligned} &\int \psi_s(x-y) k_j^{(2)}(y,u)\,dy \\ &\quad= \int \psi_s(x-y)[m_{\widetilde{I}_i}(\nabla A) - \nabla A(y)]\cdot \vec{k}_j(y-u)\,dy \equiv L(x,u) , \end{aligned} \tag{4.18}$$

where

$$\vec{k}_j(x) \equiv \frac{x\widetilde{\Omega}(x)}{|x|^{n+1}} \varphi\Big(\frac{|x|}{2^j}\Big) .$$

We let $\vec{S}_j$ denote the convolution operator $f \to \vec{k}_j * f$. We write

$$\begin{aligned} &L(x,u) \\ &\quad= \int \psi_s(x-y)[m_{\widetilde{I}_i}(\nabla A) - \nabla A(y)][\vec{k}_j(y-u) - \vec{k}_j(x-u)]\,dy \\ &\qquad+ Q_s(\nabla A)(x)\vec{k}_j(x-u) \equiv L_1(x,u) + L_2(x,u) , \end{aligned} \tag{4.19}$$

where in L_2 we have used the fact that $Q_s 1 = 0$. We will show that the usual program of Theorems 3.1 and 3.2 can be carried out for L_1, and that L_2 can be controlled by Carleson measure estimates. To treat L_1, it is enough to show that the operator

$$T_1 f(x) \equiv \int L_1(x,u) f(u)\,du$$

satisfies

$$\|T_1 f\|_2 \le C(2^{-j}s)^\varepsilon \|f\|_2, \qquad s \le 2^j , \tag{4.20}$$

and

$$\|T_1 f\|_{2,w} \le C\|f\|_{2,w}, \quad w \in A_2, \qquad \text{if } \|\Omega\|_\infty = 1 . \tag{4.21}$$

The proof of (4.21) is very simple. We have $L_1 = L - L_2$. But L_2 equals a bounded function times the kernel of the operator $\vec{S}_j$ which is controlled by the maximal function. To treat L, we apply Hölder's inequality in the dy integral, along with the John–Nirenberg inequality, to obtain an operator controlled by $(M(Mf)^{1+\varepsilon})^{1/(1+\varepsilon)}$.

To finish our treatment of L_1, we need to prove (4.20). By dilation invariance, we may take $j = 0$. As in the proof of Lemma 2.1, we may assume that f is supported in a unit cube I. The bound will hold more generally with ∇A replaced by an arbitrary BMO function b, and we write

$$m_{\widetilde{I}}(b) - b(y) = +m_{\widetilde{I}}(b) - m_{I_s}(b) + m_{I_s}(b) - b(y) ,$$

where I_s has center x and side length $\sqrt{n}s$, and $\widetilde{I}$ is concentric with I and has side length $1000n$. In place of L_1 we then consider

$$[m_{\widetilde{I}}(b) - m_{I_s}(b)] \int \psi_s(x-y)[\vec{k}_0(y-u) - \vec{k}_0(x-u)]\,dy$$
$$+ \int \psi_s(x-y)[m_{I_s}(b) - b(y)][\vec{k}_0(y-u) - \vec{k}_0(x-u)]\,dy$$
$$\equiv h_1(x,u) + h_2(x,u)\,.$$

Now, in h_1, the $\vec{k}_0(x-u)$ term is not there, since $\int \psi = 0$. Thus,

$$\int h_1(x,u) f(u)\,du \equiv [m_{\widetilde{I}}(b) - m_{I_s}(b)] Q_s \vec{S}_0 f(x)\,.$$

But $\|Q_s \vec{S}_0 f\|_2 \le C s^{\varepsilon} \|\Omega\|_r \|f\|_2$ (by (2.9)) and

$$|m_{\widetilde{I}}(b) - m_{I_s}(b)| \le C\|b\|_* \log \frac{1}{s}\,.$$

The latter is a well known property of BMO. Then (4.20) follows for h_1.

To finish the proof of (4.20), we must show

$$\int \left| \int h_2(x,u) f(u)\,du \right|^2 dx \le C\|b\|_*^2 s^{\varepsilon} \|f\|_2\,.$$

By the change of variables $y \to y + x$, we have

$$-\int h_2(x,u) f(u)\,du = \int \psi_s(y)[b(y+x) - m_{I_s}(b)][\vec{k}_0 * f(x+y) - \vec{k}_0 * f(x)]\,dy\,.$$

Now, by Schwarz, the square of the absolute value of this last expression is no larger than

(4.22)
$$\left[\int |\psi_s(y)||b(y+x) - m_{I_s}(b)|^2\,dy\right]\left[\int |\psi_s(y)||\vec{k}_0 * f(x+y) - \vec{k}_0 * f(x)|^2\,dy\right].$$

The first factor in (4.22) is no larger than $\|b\|_*^2$. Thus, by Fubini's theorem, it is enough to show that

$$\text{(4.23)}\qquad \sup_{|y|<s\le 1} \int |\vec{k}_0 * f(x+y) - \vec{k}_0 * f(x)|^2\,dx \le C s^{\varepsilon} \|f\|_2\,.$$

By Plancherel, the left hand side of (4.23) equals

$$\text{(4.24)}\qquad \sup_{|y|<s\le 1} \int |e^{i\xi\cdot y} - 1|^2 |\vec{k}_0{}^{\wedge}(\xi)|^2 |\widehat{f}(\xi)|^2\,d\xi\,.$$

But a result of [DR, Section 4] shows that

$$\text{(4.25)}\qquad |\vec{k}_0{}^{\wedge}(\xi)| \le |\xi|^{-\alpha}, \quad \text{for some } \alpha,\ 0 < \alpha < 1\,.$$

Thus (4.24) is dominated by

$$\sup_{|y|\le s} C s^{2\alpha} \int \frac{|e^{i\xi\cdot y} - 1|^{2\alpha}}{(|\xi||y|)^{2\alpha}} |\widehat{f}(\xi)|^2\,d\xi\,,$$

and the desired estimate follows.

To finish the proof of Theorem 1.1, we need only consider the part of (4.17) corresponding to $L_2(x,u)$ in (4.19). We need to prove that

$$\text{(4.26)}\qquad \left| \int_{\varepsilon_1}^{1/\varepsilon_1} \int_{\varepsilon_2}^{1/\varepsilon_2} \chi\{s \le t\} \sum_{2^j \ge s^{\theta} t^{1-\theta}} \langle Q_s(\nabla A) \cdot \vec{S}_j Q_t^2 f, Q_s g\rangle \frac{ds}{s}\frac{dt}{t} \right|$$
$$\le \begin{cases} C\|\nabla A\|_* \|\Omega\|_r \|f\|_2 \|g\|_2, & 1 < r < \infty, \\ C\|\nabla A\|_* \|\Omega\|_\infty \|f\|_{2,w} \|g\|_{2,1/w}, & r = \infty,\ w \in A_2\,. \end{cases}$$

The left hand side of (4.26) equals

$$\text{(4.27)}\qquad \left| \int_{\varepsilon_1}^{1/\varepsilon_1} \int_{\varepsilon_2}^{1/\varepsilon_2} \chi\{s \le t\} \langle Q_s(\nabla A) \cdot \vec{S} Q_t^2 f, Q_s g\rangle \frac{ds}{s}\frac{dt}{t} \right.$$
$$\left. - \int_{\varepsilon_1}^{1/\varepsilon_1} \int_{\varepsilon_2}^{1/\varepsilon_2} \chi\{s \le t\} \sum_{2^j \le s^{\theta} t^{1-\theta}} \langle Q_s(\nabla A) \cdot \vec{S}_j Q_t^2 f, Q_s g\rangle \frac{ds}{s}\frac{dt}{t} \right|$$
$$\equiv |\langle Uf, g\rangle - \langle Vf, g\rangle|\,,$$

where $\vec{S} \equiv \sum_{j=-\infty}^{\infty} \vec{S}_j$ (here we continue to use the convention that $\vec{S}_j = 0$ for $j < N$ or $j > M$). We recall that $\vec{S}_j$ is a convolution operator which annihilates constants, so the kernel of $\sum_{2^j \le s^{\theta} t^{1-\theta}} \vec{S}_j Q_t$ is no larger than

$$\sum_{2^j \le s^{\theta} t^{1-\theta}} \int |\vec{k}_j(x-y)||\psi_t(y-u) - \psi_t(x-u)|\,dy$$
$$\le C\|\nabla\psi\|_\infty \|\Omega\| \sum_{2^j \le s^{\theta} t^{1-\theta}} \frac{2^j}{t} \frac{\chi\{|x-u| < 2t\}}{t^n}$$
$$= C\|\Omega\|_1 (s/t)^{\theta} t^{-n} \chi\{|x-u| \le 2t\}\,,$$

where we have used the mean value theorem, that $\vec{k}_j(x-y)$ is supported in $\{2^{j-2} \le |x-y| \le 2^j\}$, and that $s \le t$. Thus, since $\|Q_s(\nabla A)\|_\infty \le C\|\nabla A\|_*$,

$$|\langle Vf, g\rangle| \le C\|\Omega\|_1 \|\nabla A\|_* \int_0^{\infty} \int_0^{t} (s/t)^{\theta} \langle M(Q_t f), |Q_s g|\rangle \frac{ds}{s}\frac{dt}{t}$$
$$\le C\|\Omega\|_1 \|\nabla A\|_* \|f\|_{2,w} \|g\|_{2,1/w}, \quad w \in A_2\,,$$

where the last inequality follows from Schwarz, weighted Littlewood–Paley theory, and the weighted norm inequality for the maximal function. We turn to U in (4.27). Now $\vec{S}$ is bounded on L^p by a classical result of Calderón–Zygmund [CZ], and if Ω is bounded, then $\vec{S}$ is bounded on L^p_w, $w \in A_p$, by [DR, Corollary 4.2]. Since $\vec{S}$, being a convolution operator, commutes with

Q_t, it is enough that

$$\Bigl|\int_{\varepsilon_1}^{1/\varepsilon_1}\int_{\varepsilon_2}^{1/\varepsilon_2}\chi\{s\le t\}\langle Q_s(\nabla A)Q_t^2 f, Q_s g\rangle\,\frac{ds}{s}\,\frac{dt}{t}\Bigr| \le C\|\nabla A\|_*\|f\|_{2,w}\|g\|_{2,1/w},\qquad w\in A_2\,.$$

But we have already proved this fact in Section 3—see (3.11) with $b\equiv\nabla A\in\mathrm{BMO}$.

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DEPARTMENT OF MATHEMATICS AND STATISTICS
WRIGHT STATE UNIVERSITY
DAYTON, OHIO 45435
U.S.A.
E-mail: SHOFMANN@DESIRE.WRIGHT.EDU

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