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Real interpolation for families of Banach spaces

by

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Abstract. We develop a new method of real interpolation for infinite families of Banach spaces that covers the methods of Lions–Peetre, Sparr for N spaces, Fernández for 2^N spaces and the recent method of Cobos–Peetre.

1. Introduction. The notion of interpolation family was introduced by the St. Louis group in [CCRSW]. In that paper, the theory of complex interpolation for families is developed.

In the setting of real interpolation the Peetre K- and J-functionals are constructed, in the case of a compatible couple (A_0, A_1), by introducing a positive weight factor t in the norms of the sum space $A_0 + A_1$ and intersection space $A_0 \cap A_1$ respectively. More precisely, $K(t,a) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1\}$ and $J(t,a) = \max\{\|a\|_{A_0}, t\|a\|_{A_1}\}$.

The real method for a compatible couple developed by Lions and Peetre ([LP]) has had many generalizations in the last decades. Namely, the Sparr method for a finite collection of spaces (see [S]), the Fernández method for 2^N spaces (see [F]) and the recent method of Cobos–Peetre for N spaces associated with the vertices of a polygon in $\mathbb{R}^2$ (see [CP]).

In all of them, both the K- and J-functionals are defined introducing a positive weight factor $\overline{w}$ in the norms of the sum space $\sum_{j=1}^N A_j$ and intersection space $\bigcap_{j=1}^N A_j$. In general,

$$K_M(\overline{w}, a) = \inf\Big\{\sum_{j=1}^N w_j\|a_j\|_{A_j} : \sum_{j=1}^N a_j = a\Big\},$$

where $\overline{w} = (w_1, \ldots, w_N) \in \mathbb{R}^N$, $w_j \ge 0$ for every $j = 1, \ldots, N$ and, for each method M (Lions–Peetre, Sparr, Fernández, Cobos–Peetre), the weight $\overline{w}$ is chosen in a different way.

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To define the interpolation spaces with the K-functional, they make $\overline{w}$ vary in a certain set and impose the condition $K(\overline{w},a)/C(\overline{w}) \in E$ where E is a quasi-Banach space and $C(\overline{w})$ is a measurable function such that

$$\frac{\min_{j=1,\ldots,N}(w_j)}{C(\overline{w})} \in E. \tag{1}$$

In the above mentioned methods $E = L^p(d\mu(\overline{w}))$ and $C(\overline{w})$ satisfies

$$\log C(\overline{w}) = \sum_{j=1}^{N} \theta_j \log w_j, \qquad \theta_j \in \mathbb{R}.$$

However, the choice of a different $C(\overline{w})$ gives rise, for example, to the function parameter method due to Gustavsson and Kalugina (see [GU]) and the choice of different spaces E such as Orlicz spaces has been studied by Mastyło (see [M]).

In this paper, we want to study a method of real interpolation for families of Banach spaces which can be related to the one described by the St. Louis group; that is, given an interpolation family $\overline{A}$, in the sense of [CCRSW], we want to define, for each $z_0 \in D = \{|z| < 1\}$, two interpolation functors F and G such that

$$G(\overline{A}) \subset A[z_0] \subset F(\overline{A}),$$

with $A[z_0]$ the interpolation space defined with the complex method.

The paper will be developed as follows. In Section 2 we define the K- and J-functionals for families in the same context of weight factors and study their first properties. In Section 3 we introduce the interpolation spaces with the K-method, give some examples and prove some interpolation theorems. Section 4 is dedicated to the interpolation spaces constructed with the J-method, and in Section 5 we study the relationship between these spaces and the one constructed with the complex method (see [CCRSW]). Finally, we end up with an appendix related to some other interpolation spaces which can be defined by a slight modification of the ones we describe in the paper.

Throughout this paper, $\sum'$ will indicate a finite sum and the symbol $f \sim g$ will be used to indicate the existence of two positive constants a, b such that $af(\cdot) \le g(\cdot) \le bf(\cdot)$.

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I am also very grateful to Prof. J. Peetre who gave me useful advice about the way this paper should be written and informed me about example VI in 3.8.

2. K- and J-functionals for families. Let us first recall what an interpolation family is. Let D denote the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ and Γ its boundary. We say that $\overline{A} = \{A(\gamma) : \gamma \in \Gamma; \mathcal{A}, \mathcal{U}\}$ is a complex *interpolation family* (i.f.) on Γ with $\mathcal{U}$ as the containing Banach space and $\mathcal{A}$ as the log-intersection space, in the sense of [CCRSW], if:

(a) the complex Banach spaces $A(\gamma)$ are continuously embedded in $\mathcal{U}$ ($\|\cdot\|_\gamma$ will be the norm on $A(\gamma)$ and $\|\cdot\|_{\mathcal{U}}$ the norm on $\mathcal{U}$),

(b) for every $a \in \bigcap_{\gamma\in\Gamma} A(\gamma)$, $\Gamma \ni \gamma \to \|a\|_\gamma$ is a measurable function on Γ,

(c) $\mathcal{A} = \{a \in A(\gamma)$ for a.e. $\gamma \in \Gamma : \int_\Gamma \log^+ \|a\|_\gamma \, d\gamma < \infty\}$, and there exists a measurable function P on Γ such that

$$\int_\Gamma \log^+ P(\gamma)\, d\gamma < \infty \quad \text{and} \quad \|a\|_{\mathcal{U}} \le P(\gamma)\|a\|_\gamma \quad \text{for a.e. } \gamma \ (a \in \mathcal{A}).$$

Observe that in the definition of the log-intersection space $\mathcal{A}$ we require $a \in A(\gamma)$ for almost every $\gamma \in \Gamma$ and not for every $\gamma \in \Gamma$. This change of the usual definition (see [CCRSW]) will be specially useful in §4, Theorem 4.5.

Let

$$\mathcal{L} = \{\alpha : \Gamma \to \mathbb{R}^+ : \alpha \text{ measurable}, \ \log\alpha \in L^1(\Gamma)\},$$

and let

$$\mathcal{G} = \Big\{b = {\sum}' b_j \chi_{E_j} : b_j \in \mathcal{A} \text{ and } E_j \text{ pairwise disjoint measurable sets in } \Gamma\Big\},$$

where χ_E denotes the characteristic function of E. We shall write $a(\cdot) \in \overline{\mathcal{G}}$ whenever $a(\cdot)$ is a Bochner integrable function in $\mathcal{U}$ such that $a(\gamma) \in A(\gamma)$ for a.e. $\gamma \in \Gamma$ and such that $a(\cdot)$ can be a.e. approximated in the $A(\cdot)$-norm by functions $a_n(\cdot)$ belonging to $\mathcal{G}$.

DEFINITION 2.1. Let $\alpha \in \mathcal{L}$.

(a) For each $a \in \mathcal{U}$, we define the K-functional with respect to the i.f. $\overline{A}$ by

$$K(\alpha, a) = \inf\Big\{\int_\Gamma \alpha(\gamma)\|a(\gamma)\|_\gamma \, d\gamma\Big\},$$

where the infimum extends over all representations $a = \int_\Gamma a(\gamma)\, d\gamma$ (convergence in $\mathcal{U}$), with $a(\cdot) \in \overline{\mathcal{G}}$.

(b) For each $a \in \mathcal{A}$, we also define the J-functional by

$$J(\alpha, a) = \operatorname*{ess\,sup}_{\gamma\in\Gamma}(\alpha(\gamma)\|a\|_\gamma).$$

Remark 2.2. (1) Given $a \in \overline{\mathcal{G}}$, let

$$E(a) = \{\gamma \in \Gamma : \|a(\gamma)\|_\gamma \ne 0\}.$$

Then, since $a_n(\gamma)\stackrel{n\to\infty}{\to} a(\gamma)$ for a.e. $\gamma\in\Gamma$ (in the $A(\gamma)$-norm), we can always assume (considering, if necessary, $a'_n = a_n\chi_{\{\gamma\in E(a):\|a_n(\gamma)\|_\gamma\le C\|a(\gamma)\|_\gamma\}}$) that, for every constant $C>1$, $\|a_n(\gamma)\|_\gamma \le C\|a(\gamma)\|_\gamma$ for a.e. $\gamma\in E(a)$. And, if $\|a(\cdot)\|_\cdot \in L^1(\alpha)$ with $\alpha\in\mathcal{L}$, we can assume that there exists a constant $C'>0$ such that $\alpha(\gamma)\|a_n(\gamma)\|_\gamma \le C'$ for every $\gamma\in\Gamma\setminus E(a)$. This will allow us to assert that, if $a\in\overline{\mathcal{G}}$ is such that $\|a(\cdot)\|_\cdot\in L^1(\alpha)$ and $a_n\in\mathcal{G}$ converges to a, then the sequence αa_n is dominated up to a multiplicative constant by $\alpha a+1$. That is, we can apply dominated convergence to the sequence αa_n whenever needed.

(2) Obviously, $J(\alpha,a)$ can be $+\infty$.

Before stating the first elementary properties of these functionals we need the following definitions.

DEFINITION 2.3. Let $\overline{A}=\{A(\gamma):\gamma\in\Gamma;\mathcal{A},\mathcal{U}\}$ and $\overline{B}=\{B(\gamma):\gamma\in\Gamma;\mathcal{B},\mathcal{V}\}$ be two i.f. An operator $T:\overline{A}\to\overline{B}$ is said to be an *interpolation operator* if $T:\mathcal{U}\to\mathcal{V}$ is bounded, $T:\mathcal{A}\to\bigcap_{\gamma\in\Gamma}B(\gamma)$ and $\|T\|_{A(\gamma)\to B(\gamma)}\le M(\gamma)$ with $M\in\mathcal{L}$.

We shall also use the following theorem due to Szegö (see [G]):

THEOREM 2.4. *Let $w\in\mathcal{L}$. Then*

$$\inf\Big\{\int_\Gamma |\varphi(\gamma)w(\gamma)|P_z(\gamma)\,d\gamma : \varphi\in H^\infty(D),\ \varphi(z)=1\Big\} = \exp\int_\Gamma \log(w(\gamma))P_z(\gamma)\,d\gamma\,,$$

where P_z is the Poisson kernel at $z\in D$ and $H^\infty(D)$ is the space of bounded analytic functions on D.

When not specified, we will always be dealing with the interpolation family $\overline{A}=\{A(\gamma):\gamma\in\Gamma;\mathcal{A},\mathcal{U}\}$.

Given $a\in\mathcal{A}$, we define

$$\varphi_a(z)=\exp\int_\Gamma \log(\|a\|_\gamma)P_z(\gamma)\,d\gamma\,,$$

and, for $\alpha\in\mathcal{L}$,

$$\alpha(z)=\exp\int_\Gamma \log\alpha(\gamma)P_z(\gamma)\,d\gamma\,.$$

PROPOSITION 2.5. (1) *The K- and J-functionals are seminorms.*

(2) *For every $a\in\mathcal{U}$, $\|a\|_\mathcal{U}\le K(P,a)$ where P is the function associated with the i.f. in* (c). *Therefore, if $K(P,a)=0$, we get $a=0$* (we shall see that this is not true in general for $\alpha\ne P$).

(3) *For every $a\in\mathcal{A}$ and every $z\in D$, $K(\alpha,a)\le\varphi_a(z)\alpha(z)$.*

(4) *For every $a\in\mathcal{A}$, $K(\alpha,a)\le J(\alpha,a)$.*

(5) *If T is an interpolation operator and $\|T\|_\gamma\le M(\gamma)\in\mathcal{L}$ for a.e. $\gamma\in\Gamma$, then*

$$K(\alpha,Ta)\le K(\alpha M,a),\quad a\in\mathcal{U}\,,$$
$$J(\alpha,Ta)\le J(\alpha M,a),\quad a\in\mathcal{A}\,.$$

Proof. Properties (1) and (4) are trivial.

(2) If $a=\int_\Gamma a(\gamma)\,d\gamma$, then $\|a\|_\mathcal{U}\le\int_\Gamma\|a(\gamma)\|_\mathcal{U}\,d\gamma\le\int_\Gamma P(\gamma)\|a(\gamma)\|_\gamma\,d\gamma$. Therefore, $\|a\|_\mathcal{U}\le K(P,a)$.

(3) This is a consequence of Szegö's Theorem. We write $a=a\int_\Gamma\varphi(\gamma)\times P_z(\gamma)\,d\gamma$ with $\varphi\in H^\infty(D)$ such that $\varphi(z)=1$. Then

$$K(\alpha,a)\le\inf\Big\{\int_\Gamma\alpha(\gamma)\|a\|_\gamma|\varphi(\gamma)|P_z(\gamma)\,d\gamma:\varphi(z)=1,\ \varphi\in H^\infty\Big\}$$
$$\le\exp\int_\Gamma\log(\alpha(\gamma)\|a\|_\gamma)P_z(\gamma)\,d\gamma=\varphi_a(z)\alpha(z)\,.$$

(5) Let $a=\int_\Gamma a(\gamma)\,d\gamma$. Then $Ta=\int_\Gamma T(a(\gamma))\,d\gamma$. Hence,

$$K(\alpha,Ta)\le\int_\Gamma\alpha(\gamma)\|T(a(\gamma))\|_{B(\gamma)}\,d\gamma\le\int_\Gamma\alpha(\gamma)M(\gamma)\|a(\gamma)\|_{A(\gamma)}\,d\gamma\,.$$

Taking the infimum over all the representations of a we are done. The result for J is proved in the same way. ∎

The following definitions will be very useful in the sequel.

DEFINITION 2.6. Let N be a countable set. A sequence $(\alpha_n)_{n\in N}$ of measurable functions in Γ is said to be a *base sequence* (b.s.) if it satisfies

(i) there exists $n_0\in N$ such that $\alpha_{n_0}(\gamma)=1$ for a.e. $\gamma\in\Gamma$,

(ii) there exists a partition $\{\Gamma_1,\dots,\Gamma_m\}$ of Γ such that α_n is constant in each Γ_j, and

(iii) $\sum_{n\in N}(\operatorname{ess\,inf}(\alpha_n(\gamma))/\alpha_n(z))^p<\infty$ for every $z\in D$, and every $p>0$.

If we have the following condition, stronger than (iii):

(iv) for every $z\in D$, there exists a compact set $K\subset D$ so that

$$\sum_{n\in N}(\inf_{\xi\in K}\alpha_n(\xi)/\alpha_n(z))^p<\infty\quad\text{for every } p>0\,,$$

then we say that $(\alpha_n)_n$ is a *special base sequence* (s.b.s.).

EXAMPLE 2.7. (I) Let $\{\Gamma_1,\Gamma_2\}$ be a partition of Γ, and consider, for $n\in\mathbb{Z}$,

$$\alpha_n(\gamma)=\begin{cases}1 & \text{if } \gamma\in\Gamma_1\,,\\ 2^n & \text{if } \gamma\in\Gamma_2\,.\end{cases}$$

Then $(\alpha_n)_n$ is a s.b.s. Since $\alpha_n(z) = 2^{n|\Gamma_2|_z}$ (where $|E|_z$ denotes the harmonic measure of $E \subset \Gamma$ at $z \in D$), we can easily prove that property (iv) holds.

(II) Let $\{\Gamma_0, \Gamma_1, \ldots, \Gamma_m\}$ be a partition of Γ, consider $\overline{n} = (n_1, \ldots, n_m) \in \mathbb{Z}^m$ and

$$\alpha_{\bar{n}}(\gamma) = \begin{cases} 1 & \text{if } \gamma \in \Gamma_0, \\ 2^{n_j} & \text{if } \gamma \in \Gamma_j,\ j = 1, \ldots, m. \end{cases}$$

Then one can easily check that $(\alpha_{\bar{n}})_{\bar{n} \in \mathbb{Z}^m}$ is a b.s. for $m > 1$.

(III) Let $\{\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3\}$ be a partition of Γ, consider $\overline{n} = (n, k) \in \mathbb{Z}^2$ and

$$\alpha_{\bar{n}}(\gamma) = \begin{cases} 1 & \text{if } \gamma \in \Gamma_0, \\ 2^n & \text{if } \gamma \in \Gamma_1, \\ 2^k & \text{if } \gamma \in \Gamma_2, \\ 2^n 2^k & \text{if } \gamma \in \Gamma_3. \end{cases}$$

Then $(\alpha_{\bar{n}})_{\bar{n} \in \mathbb{Z}^2}$ is a s.b.s. This example can be generalized to a partition of Γ into 2^N subsets and $\overline{n} \in \mathbb{Z}^N$.

(IV) By choosing a_j and b_j properly, we can construct a b.s. defined for $n, k \in \mathbb{Z}$ by

$$\alpha_{n,k}(\gamma) = \{(a_j)^n (b_j)^k : \gamma \in \Gamma_j,\ j = 1, \ldots, m\}.$$

PROPOSITION 2.8. *Let $M : \Gamma \to \mathbb{R}^+$ be a measurable function in $\mathcal{L}$. Then, for every $a \in \mathcal{A}$,*

$$K(M, a) \le \inf(C_\Omega(a) \inf_{z \in \Omega} M(z)),$$

where the infimum is taken over all sets $\Omega \subset D$ and $C_\Omega(a) = \sup_{\sigma \in \Omega} \varphi_a(\sigma)$. If $\Omega = D$, then

$$K(M, a) \le (\operatorname*{ess\,inf}_{\gamma \in \Gamma} M(\gamma)) C_\Gamma(a),$$

with $C_\Gamma(a) = \operatorname{ess\,sup}_{\gamma \in \Gamma} \|a\|_\gamma = J(1, a)$.

Proof. By condition (3) of Proposition 2.5, $K(M, a) \le M(z)\varphi_a(z)$, for every $z \in D$. Therefore, for a fixed $\Omega \subset D$ we get

$$K(M, a) \le \inf_{z \in \Omega}(M(z)\varphi_a(z)) \le \sup_{z \in \Omega} \varphi_a(z) \inf_{z \in \Omega} M(z) = C_\Omega(a) \inf_{z \in \Omega} M(z). \ \blacksquare$$

Remark 2.9. By Proposition 2.8, if we set $(B(\gamma), \|\cdot\|_{B(\gamma)}) = (A(\gamma), \beta(\gamma)\|\cdot\|_{A(\gamma)})$ with $\beta \in \mathcal{L}$, we get

$$K(M\beta, a; \overline{A}) = K(M, a; \overline{B}) \le \operatorname{ess\,inf} M(\gamma) J(1, a; \overline{B}) = \operatorname{ess\,inf} M(\gamma) J(\beta, a; \overline{A});$$

that is, for every $\alpha, \beta \in \mathcal{L}$ and every $a \in \mathcal{A}$,

$$K(\alpha, a) \le \operatorname{ess\,inf} \frac{\alpha(\gamma)}{\beta(\gamma)} J(\beta, a).$$

In the case of a compatible pair (A_0, A_1), if the i.f. $\overline{A}$ is defined by $A(\gamma) = A_0$ for every $\gamma \in \Gamma_0$ and $A(\gamma) = A_1$ for every $\gamma \in \Gamma_1$, where $\{\Gamma_0, \Gamma_1\}$ is a partition of Γ, and taking, for $t, s \in \mathbb{R}$,

$$\alpha(\gamma) = \begin{cases} 1 & \text{if } \gamma \in \Gamma_0, \\ t & \text{if } \gamma \in \Gamma_1, \end{cases} \qquad \beta(\gamma) = \begin{cases} 1 & \text{if } \gamma \in \Gamma_0, \\ s & \text{if } \gamma \in \Gamma_1, \end{cases}$$

we get the well-known formula (see [BL])

$$K(t, a) \le \min\left(1, \frac{t}{s}\right) J(s, a),$$

for every $a \in A_0 \cap A_1$.

3. Interpolation spaces with the K-method

DEFINITION 3.1. Let $S \subset \mathcal{L}$ and $0 < p \le \infty$. Let $\overline{A} = \{A(\gamma) : \gamma \in \Gamma;\ \mathcal{A}, \mathcal{U}\}$ be an i.f. The space $[A]^S_{z_0,p}$ consists of all $a \in \mathcal{U}$ for which

$$\left(\frac{K(\alpha, a)}{\alpha(z_0)}\right)_{\alpha \in S} \in l^p(S),$$

endowed with the norm

$$\|a\|_{[A]^S_{z_0,p}} = \left(\sum_{\alpha \in S} \left(\frac{K(\alpha, a)}{\alpha(z_0)}\right)^p\right)^{1/p}.$$

Our next goal is to choose S in such a way that the spaces constructed are first intermediate, and then interpolation spaces (with or without convexity).

LEMMA 3.2. *If $P \in S$, then $[A]^S_{z_0,p}$ is continuously embedded in the containing space $\mathcal{U}$ with norm less than or equal to $P(z_0)$.*

Proof. By (2) of Proposition 2.5, $\|a\|_{\mathcal{U}} \le K(P, a) \le P(z_0)\|a\|_{[A]^S_{z_0,p}}$. $\blacksquare$

From now on, we will always assume that $P \in S$ or at least that there exist $K \in S$ and $C > 0$ such that $P(\gamma) \le CK(\gamma)$ for a.e. $\gamma \in \Gamma$.

PROPOSITION 3.3. *$\mathcal{A}$ is contained in $[A]^S_{z_0,\infty}$ for every S.*

Proof. We only have to use property (3) of Proposition 2.5 to obtain

$$\sup_{\alpha \in S} \frac{K(\alpha, a)}{\alpha(z_0)} \le \varphi_a(z_0) < \infty. \ \blacksquare$$

Now if $p \ne \infty$ we have the following result.

PROPOSITION 3.4. (a) *If, for every $a \in \mathcal{A}$, $J(1, a) < \infty$ and*

$$\sum_{\alpha \in S} \left(\frac{\operatorname{ess\,inf}_{\gamma \in \Gamma} \alpha(\gamma)}{\alpha(z_0)}\right)^p < \infty, \tag{2}$$

then $\mathcal{A}$ is contained in $[A]^S_{z_0,p}$.

(b) *If $J(1,a)=\infty$ for some $a\in\mathcal{A}$, then $\mathcal{A}$ is contained in $[A]^S_{z_0,p}$ under the stronger condition that there exists a compact set K in D such that*

$$\sum_{\alpha\in S}\left(\frac{\inf_{z\in K}\alpha(z)}{\alpha(z_0)}\right)^p<\infty. \tag{3}$$

Remark 3.5. (i) We want to point out that condition (2) is analogous to condition (1) in the introduction.

(ii) Condition (2) implies that $\operatorname{ess\,inf}\alpha(\gamma)\neq 0$ only for a countable subset of S and condition (3) implies that S is a countable set.

Proof of Proposition 3.4. (a) By 2.8, for every $a\in\mathcal{A}$, $K(\alpha,a)\le J(1,a)\operatorname{ess\,inf}\alpha(\gamma)$ and therefore

$$\|a\|_{[A]^S_{z_0,p}}=\left(\sum_{\alpha\in S}\left(\frac{K(\alpha,a)}{\alpha(z_0)}\right)^p\right)^{1/p}$$
$$\le J(1,a)\left(\sum_{\alpha\in S}\left(\frac{\operatorname{ess\,inf}_{\gamma\in\Gamma}\alpha(\gamma)}{\alpha(z_0)}\right)^p\right)^{1/p}<\infty.$$

(b) If $J(1,a)=\infty$ for some $a\in\mathcal{A}$, we have to use 2.8 with $\Omega=\mathring{K}$ (K the compact set given in the hypothesis) and condition (3). ■

THEOREM 3.6. *Let $\overline{A}$ and $\overline{B}$ be two i.f. and let $T:\overline{A}\to\overline{B}$ be an interpolation operator with norm $\|T\|_{A(\gamma)\to B(\gamma)}\le M(\gamma)\in\mathcal{L}$.*

(a) *If $\|M\|_\infty<\infty$, then for every $S\subset\mathcal{L}$,*

$$T:[A]^S_{z_0,p}\to[B]^S_{z_0,p}$$

is bounded with norm less than or equal to $\|M\|_\infty$.

(b) *If $MS=\{M\alpha:\alpha\in S\}\subset S$ then*

$$T:[A]^S_{z_0,p}\to[B]^S_{z_0,p}$$

is bounded with norm less than or equal to $M(z_0)$ (that is, with convexity).

Proof. The proof of both parts is obvious by property (5) of Proposition 2.5. ■

PROPOSITION 3.7. *Let $\overline{A}=(A_0,A_1)$ be the i.f. defined by $A(\gamma)=A_j$ for $\gamma\in\Gamma_j$, $j=0,1$, with $\{\Gamma_0,\Gamma_1\}$ a partition of Γ. Let $\alpha\in\mathcal{L}$ be such that $\alpha_j=\operatorname{ess\,inf}_{\gamma\in\Gamma_j}\alpha(\gamma)\neq0$ for $j=0,1$. Then, for every $a\in A_0\cap A_1$,*

$$K(\alpha,a)=\inf\{\alpha_0\|a_0\|_{A_0}+\alpha_1\|a_1\|_{A_1}:a=a_0+a_1\}=\alpha_0K(\alpha_1/\alpha_0,a),$$

where the last K-functional is the classical one.

Proof. Let $a=a_0+a_1\in A_0\cap A_1$ with $a_j\in A_j$ and consider $a(\gamma)=a_0\varphi_0(\gamma)+a_1\varphi_1(\gamma)$ where φ_j are arbitrary measurable functions such that

$$\operatorname{supp}\varphi_j\subset\Gamma_j\quad\text{and}\quad\int_{\Gamma_j}\varphi_j(\gamma)\,d\gamma=1. \tag{$*$}$$

Then $\int_\Gamma a(\gamma)\,d\gamma=a$ and

$$K(\alpha,a)\le\inf\Big\{\sum_{j=0}^1\int_{\Gamma_j}\|a_j\|_{A_j}\alpha(\gamma)|\varphi_j(\gamma)|\,d\gamma:\varphi_j\text{ satisfies }(*)\Big\}$$
$$\le\|a_0\|_{A_0}\alpha_0+\|a_1\|_{A_1}\alpha_1.$$

Conversely, given $\varepsilon>0$, if $a=\int_\Gamma a(\gamma)\,d\gamma$ with

$$\int_{\Gamma_0}\alpha(\gamma)\|a(\gamma)\|_0\,d\gamma+\int_{\Gamma_1}\alpha(\gamma)\|a(\gamma)\|_1\,d\gamma\le K(\alpha,a)+\varepsilon,$$

then defining $a_j=\int_{\Gamma_j}a(\gamma)\,d\gamma\in A_j$ we get $a=a_0+a_1$ and

$$\alpha_0\|a_0\|_{A_0}+\alpha_1\|a_1\|_{A_1}$$
$$\le\sum_{j=0}^1\inf\Big\{\int_{\Gamma_j}\alpha(\gamma)\varphi_j(\gamma)\,d\gamma:\int_{\Gamma_j}\varphi_j(\gamma)\,d\gamma=1\Big\}\int_{\Gamma_j}\|a(\gamma)\|_{A(\gamma)}\,d\gamma.$$

Taking $\varphi_j(\gamma)=\|a(\gamma)\|_{A(\gamma)}\chi_{\Gamma_j}(\gamma)(\int_{\Gamma_j}\|a(\gamma)\|_{A(\gamma)}\,d\gamma)^{-1}$ we see that the last term is less than or equal to

$$\sum_{j=0}^1\int_{\Gamma_j}\alpha(\gamma)\|a(\gamma)\|_{A(\gamma)}\,d\gamma=\int_\Gamma\alpha(\gamma)\|a(\gamma)\|_{A(\gamma)}\,d\gamma\le K(\alpha,a)+\varepsilon,$$

and, thus, letting ε go to zero and taking the appropriate infimum we get the result. ■

EXAMPLE 3.8. (I) If $\overline{A}=(A_0,A_1)$ is the i.f. corresponding to a partition $\{\Gamma_0,\Gamma_1\}$ of Γ, then $P(\gamma)\le1$ for a.e. $\gamma\in\Gamma$. In this case, we can take for S the b.s. of Example 2.7(I) and we get $[A]^S_{z_0,p}=(A_0,A_1)_{|\Gamma_1|_{z_0},p}$ since, by Proposition 3.7, $K(\alpha_n,a)=K(2^n,a)$ with $K(2^n,a)$ the classical K-functional and $a\in A_0\cap A_1$.

We shall assume in the next examples (II), (III) and (IV) that $\bigcap_jA_j$ is dense in every A_j. Then one can easily see that the following equivalences hold.

(II) If $\overline{A}=(A_0,A_1,\ldots,A_N)$, that is, $A(\gamma)=A_j$ for $\gamma\in\Gamma_j$, $j=0,1,\ldots,N$, $\{\Gamma_0,\Gamma_1,\ldots,\Gamma_N\}$ a partition of Γ, and we take for S the b.s. of Example 2.7(II), then $[A]^S_{z_0,p}=(A_0,A_1,\ldots,A_N)_{(|\Gamma_j|_{z_0},j=1,\ldots,N),p}$ (Sparr K-space, see [S]).

(III) If $\overline{A} = (A_0, A_1, A_2, A_3)$, that is, $A(\gamma) = A_j$ when $\gamma \in \Gamma_j$, $j = 0, \ldots, 3$, and we take for S the s.b.s. of Example 2.7(III), then $[A]^S_{z_0,p} = (A_0, A_1, A_2, A_3)_{(\theta_1,\theta_2),p;K}$ where $\theta_1 = |\Gamma_1 \cup \Gamma_3|_{z_0}$ and $\theta_2 = |\Gamma_2 \cup \Gamma_3|_{z_0}$ (Fernández K-space, see [F]).

(IV) If $\overline{A} = (A_1, \ldots, A_N)$ and we take S to be the b.s. of Example 2.7(IV) with $a_j = 2^{x_j}$ and $b_j = 2^{y_j}$ and (x_j, y_j) the vertices of a polygon in the affine plane $\mathbb{R}^2$, then $[A]^S_{z_0,p} = (A_1, \ldots, A_N)_{(\alpha,\beta),p;K}$ where $(\alpha, \beta) = \sum_{j=1}^N |\Gamma_j|_{z_0}(x_j, y_j)$ (Cobos–Peetre interpolation spaces, see [CP]).

(V) The following example will be very close to those defined by Cwikel–Janson in [CJ] and denoted by $U_M(A, Z)$. We shall use their notation in this example.

Assume that $A(\gamma) \le \mathcal{U}$ (bounded family on Γ), that is, $P(\gamma) = 1$ for a.e. $\gamma \in \Gamma$. Let $\mathcal{P}$ be a countable subset of the set of partitions of Γ and let $\overline{\Gamma} = \{\Gamma_0, \Gamma_1, \ldots, \Gamma_n\} \in \mathcal{P}$. We define (instead of the space $\sup_{\gamma\in\Gamma_j} A(\gamma)$ as defined in [CJ]) the space

$$A^{\overline{\Gamma}}(\delta) = \{a \in \mathcal{U} : K_j(a) < \infty\}, \qquad \delta \in \Gamma_j,\ j = 0, 1, \ldots, n,$$

where $K_j(a) = \inf\{\int_{\Gamma_j} \|a(\gamma)\|_\gamma \, d\gamma : a = \int_{\Gamma_j} a(\gamma)\, d\gamma\}$. Consider the finite family $\{A^{\overline{\Gamma}}(\delta_0), A^{\overline{\Gamma}}(\delta_1), \ldots, A^{\overline{\Gamma}}(\delta_n)\}$ with $\delta_j \in \Gamma_j$, and define

$$U_M(A, \overline{\Gamma}) = (A^{\overline{\Gamma}}(\delta_0), A^{\overline{\Gamma}}(\delta_1), \ldots, A^{\overline{\Gamma}}(\delta_n))_M,$$

where M denotes the Sparr method with index p and $\overline{\theta} = (|\Gamma_1|_{z_0}, \ldots, |\Gamma_n|_{z_0})$ for a fixed point $z_0 \in D$. Let $S = \{S_{\overline{\Gamma}} : \overline{\Gamma} \in \mathcal{P}\}$ where, if $\overline{\Gamma} = \{\Gamma_0, \Gamma_1, \ldots \ldots, \Gamma_n\}$, $S_{\overline{\Gamma}} = \{\alpha_k\}_{k\in\mathbb{Z}^n}$ is the b.s. of Example 2.7(II). Finally, set $U_M(A) = \bigcap_{\overline{\Gamma}\in\mathcal{P}} U_M(A, \overline{\Gamma})$. Then we have:

PROPOSITION. *The space $U_M(A)$ coincides with $[A]^S_{z_0,p}$.*

Proof. We shall prove the following two facts:

(i) $\bigcap_{\overline{\Gamma}\in\mathcal{P}} [A]^{S_{\overline{\Gamma}}}_{z_0,p} = [A]^S_{z_0,p}$ and

(ii) $[A]^{S_{\overline{\Gamma}}}_{z_0,p} = U_M(A, \overline{\Gamma})$.

Proof of (i). Since $S_{\overline{\Gamma}} \subset S$, we have (see Remark 3.11) $[A]^S_{z_0,p} \subset [A]^{S_{\overline{\Gamma}}}_{z_0,p}$ and, therefore, we can deduce one of the embeddings of part (i). Let now $a \in \bigcap_{\overline{\Gamma}\in\mathcal{P}} [A]^{S_{\overline{\Gamma}}}_{z_0,p}$. Then there exists a constant C such that for every $\overline{\Gamma} \in \mathcal{P}$, $\|a\|_{[A]^{S_{\overline{\Gamma}}}_{z_0,p}} \le C$. Let $S = \{\alpha_n\}_n$ and consider the set $\widetilde{\alpha}_N = \{\alpha_0, \alpha_1, \ldots, \alpha_N\}$. Assume that $\alpha_j \in S_{\overline{\Gamma}_j}$. Then, if $\overline{\Gamma} = \bigcup_{j=0}^N \overline{\Gamma}_j$, for every $j = 0, \ldots, N$ we have $\alpha_j \in S_{\overline{\Gamma}}$ and therefore

$$\left(\sum_{j=0}^N \left(\frac{K(\alpha_j, a)}{\alpha_j(z_0)}\right)^p\right)^{1/p} \le \|a\|_{[A]^{S_{\overline{\Gamma}}}_{z_0,p}} \le C.$$

Letting N go to infinity, we get $\|a\|_{[A]^S_{z_0,p}} \le C$.

Proof of (ii). In order to be precise, we shall write K_{S} for the Sparr K-functional. Let $a \in U_M(A, \overline{\Gamma})$. Then $a = \sum_{j=0}^n a_j$ with $a_j \in A^{\overline{\Gamma}}(\delta_j)$ and

$$\|a\|_{U_M} = \left(\sum_{\overline{n}\in\mathbb{Z}^n} \left(\frac{K_{\mathrm{S}}(\alpha_{\overline{n}}, a)}{\alpha_{\overline{n}}(z_0)}\right)^p\right)^{1/p},$$

where, if $\overline{n} = (n_1, \ldots, n_m)$,

$$K_{\mathrm{S}}(\alpha_{\overline{n}}, a) = \inf\Big\{\|a_0\|_{A^{\overline{\Gamma}}(\delta_0)} + \sum_{j=1}^m 2^{n_j}\|a_j\|_{A^{\overline{\Gamma}}(\delta_j)}\Big\}$$
$$= \inf\Big\{K_0(a_0) + \sum_{j=1}^m 2^{n_j} K_j(a_j)\Big\}.$$

But, as in Example 3.8(I), one can easily prove that $K_{\mathrm{S}}(\alpha_{\overline{n}}, a) \sim K(\alpha_{\overline{n}}, a)$ and we have the equivalence of both spaces. ■

Using the argument given in Theorem 2.21 of [CJ] we can prove the following:

THEOREM. *Let $\overline{A}$ and $\overline{B}$ be two i.f. Let T be an interpolation operator such that $\|T\|_{A(\gamma)\to B(\gamma)} \le N(\gamma)$ for a.e. $\gamma \in \Gamma$, with $N(\cdot) \in L^\infty(\Gamma)$ and Riemann integrable. Then*

$$T : [A]^S_{z_0} \to [B]^S_{z_0,p},$$

with norm less than or equal to $N(z_0)$.

Proof. We only need to observe that $T : A^{\overline{\Gamma}}(\delta_j) \to B^{\overline{\Gamma}}(\delta_j)$ with norm less than or equal to $N_j = \operatorname{ess\,sup}_{\gamma\in\Gamma_j} N(\gamma)$ and apply the same kind of proof as in [CJ]. ■

(VI) Let $\overline{A} = \{A(\gamma) : \gamma \in \Gamma\}$ be an i.f. with $A(\gamma) = L^1(w(\gamma, \cdot))$ and $w(\gamma, \cdot)$ a family of weights on a measure space $\mathcal{M}$. Then

$$K(\alpha, f) = \inf\Big\{\int_\Gamma \alpha(\gamma)\|F(\gamma)\|_{L^1(w(\gamma,\cdot))}\, d\gamma : \int_\Gamma F(\gamma) = f\Big\}$$
$$= \inf\Big\{\int_{\mathcal{M}}\int_\Gamma \alpha(\gamma)|F(\gamma, x)|w(\gamma, x)\, d\gamma\, dx : \int_\Gamma F(\gamma, x)\, d\gamma = f(x)\Big\}$$

$$= \int_{\mathcal{M}} |f(x)| \Big(\inf \Big\{ \int_\Gamma \alpha(\gamma)|\varphi(\gamma,x)|w(\gamma,x)\,d\gamma : \int_\Gamma \varphi(\gamma,x)\,d\gamma = 1 \Big\} \Big)\,dx$$

$$= \int_{\mathcal{M}} |f(x)| \operatorname*{ess\,inf}_{\gamma\in\Gamma} (\alpha(\gamma)w(\gamma,x))\,dx\,,$$

and hence, if we define

$$w(S,z_0) = \sum_{\alpha\in S} \frac{\operatorname{ess\,inf}_{\gamma\in\Gamma}(\alpha(\gamma)w(\gamma,x))}{\alpha(z_0)}\,,$$

we can easily see that, for every S and every $z_0 \in D$, $[A]^S_{z_0,1} = L^1(w(S,z_0))$.

Remark 3.9. For a general family $\overline{A}$ and a general S, we cannot assure that $[A]^S_{z_0,p}$ is a Banach space ($1 \le p < \infty$). However, if, for every $\alpha \in S$, there exists a constant $C_\alpha > 0$ such that $P(\gamma) \le C_\alpha \alpha(\gamma)$ for a.e. $\gamma \in \Gamma$, then standard arguments show the completeness of these spaces. This is the case for all the given examples. Let us show that:

THEOREM 3.10. *If for every $\alpha \in S$, there exists a constant $C_\alpha > 0$ such that $P(\gamma) \le C_\alpha \alpha(\gamma)$ for a.e. $\gamma \in \Gamma$ then, for $1 \le p < \infty$, the spaces $[A]^S_{z_0,p}$ are Banach spaces.*

Proof. Let $(a_n)_n$ be such that $\sum_n \|a_n\|_{[A]^S_{z_0,p}} < \infty$, and so the series $\sum_n a_n$ converges in $\mathcal{U}$ to an element $a \in \mathcal{U}$. For $S = \{\alpha_m\}_m$, we have

$$\sum_n \Big(\sum_m \Big(\frac{K(\alpha_m,a_n)}{\alpha_m(z_0)} \Big)^p \Big)^{1/p} < \infty\,. \tag{4}$$

Given $\varepsilon > 0$, let $a_{n,m}(\cdot)$ be such that $a_n = \int_\Gamma a_{n,m}(\gamma)\,d\gamma$ and

$$\int_\Gamma \alpha_m(\gamma)\|a_{n,m}(\gamma)\|_\gamma\,d\gamma \le (1+\varepsilon)K(\alpha_m,a_n)\,.$$

Then

$$\int_\Gamma \Big\|\sum_n a_{n,m}(\gamma)\Big\|_{\mathcal{U}}\,d\gamma \le \int_\Gamma \sum_n P(\gamma)\|a_{n,m}(\gamma)\|_\gamma\,d\gamma$$
$$\le C_{\alpha_m} \sum_n \int_\Gamma \alpha_m(\gamma)\|a_{n,m}(\gamma)\|_\gamma\,d\gamma < \infty\,,$$

and hence, there exists a function $a^m(\cdot) \in L^1(\mathcal{U})$ such that $\sum_{|n|\le N} a_{n,m} \to a^m$ in the $L^1(\mathcal{U})$-norm and

$$a = \sum_n a_n = \sum_n \int_\Gamma a_{n,m}(\gamma)\,d\gamma = \int_\Gamma a^m(\gamma)\,d\gamma\,.$$

Finally, by (4),

$$\|a\|_{[A]^S_{z_0,p}} \le \Big(\sum_m \Big(\frac{\int_\Gamma \alpha_m(\gamma)\|a^m(\gamma)\|_\gamma\,d\gamma}{\alpha_m(z_0)} \Big)^p \Big)^{1/p}$$
$$\le \sum_n \Big(\sum_m \Big(\frac{\int_\Gamma \alpha_m(\gamma)\|a_{n,m}(\gamma)\|_\gamma\,d\gamma}{\alpha_m(z_0)} \Big)^p \Big)^{1/p}$$
$$\le (1+\varepsilon)\sum_n \Big(\sum_m \Big(\frac{K(\alpha_m,a_n)}{\alpha_m(z_0)} \Big)^p \Big)^{1/p} < \infty\,,$$

and, hence, we get $a \in [A]^S_{z_0,p}$ and $\|a\|_{[A]^S_{z_0,p}} \le \sum_n \|a_n\|_{[A]^S_{z_0,p}} < \infty$. ■

Remark 3.11. Observe that:

(i) If $S \subset S'$ then $[A]^{S'}_{z_0,p}$ is embedded in $[A]^S_{z_0,p}$.

(ii) Moreover, if $S = S_1 \cup S_2$ then $[A]^S_{z_0,p}$ is equivalent to the intersection space $[A]^{S_1}_{z_0,p} \cap [A]^{S_2}_{z_0,p}$.

4. Interpolation spaces with the J-method

DEFINITION 4.1. We define $(A)^S_{z_0,p}$ to be the space of all elements $a \in \mathcal{U}$ such that there exists $\{u(\alpha)\}_{\alpha\in S}$ in $\mathcal{A}$ satisfying $a = \sum_{\alpha\in S} u(\alpha)$ (in the $\mathcal{U}$-norm) and

$$\Big(\sum_\alpha \Big(\frac{J(\alpha,u(\alpha))}{\alpha(z_0)} \Big)^p \Big)^{1/p} < \infty\,.$$

This space will be endowed with the quasi-seminorm

$$\|a\|_{(A)^S_{z_0,p}} = \inf\Big\{ \Big(\sum_\alpha \Big(\frac{J(\alpha,u(\alpha))}{\alpha(z_0)} \Big)^p \Big)^{1/p} \Big\}$$

where the infimum extends over all possible representations of a.

Remark 4.2. If $\overline{A}$ and S are as in Examples 3.8(I), (II), (III) or (IV), one can easily check that $(A)^S_{z_0,p} = (A_1,\ldots,A_N)^J_M$ where M indicates the corresponding method: Lions–Peetre, Sparr for N spaces, Fernández and Cobos–Peetre respectively. In this case we do not need the condition of density of the intersection space in every space A_j, for any of the examples.

We now want to study conditions which are necessary for $(A)^S_{z_0,p}$ to be, first, an intermediate and Banach space (when $p \ge 1$), and then an interpolation space.

To have the log-intersection space $\mathcal{A}$ contained in $(A)^S_{z_0,p}$ we shall assume that the space generated by

$$\widetilde{\mathcal{A}} = \{a \in \mathcal{A} : \text{there exists } \alpha \in S \text{ with } J(\alpha,a) < \infty\}$$

coincides with $\mathcal{A}$. To obtain the embedding of $(A)^S_{z_0,p}$ in $\mathcal{U}$ we need the following result.

PROPOSITION 4.3. *Let $p \geq 1$ and assume that, for every countable set $M = \{\alpha_n\}_n$ in S, there exists a compact set K in D and a positive constant C_M such that*

$$(5)\qquad \left(\sum_n \left(\frac{\alpha_n(z_0)}{\sup_{z\in K}\alpha_n(z)}\right)^{p'}\right)^{1/p'} < C_M, \qquad z_0 \in D.$$

Then $(A)^S_{z_0,p}$ is embedded in $\mathcal{U}$. If $p < 1$ then the embedding holds without any extra hypothesis on S.

Proof. We prove the result for $p \geq 1$. The case $p < 1$ follows the same steps except the last inequality where we do not need to use Hölder's inequality.

Given $\varepsilon > 0$, let $a = \sum_n a_n$ and $M = \{\alpha_n\}_n \subset S$ such that

$$\left(\sum_n \left(\frac{J(\alpha_n, a_n)}{\alpha_n(z_0)}\right)^p\right)^{1/p} \leq \|a\|_{(A)^S_{z_0,p}} + \varepsilon.$$

Then, for every $(z_n)_n \in D$,

$$\begin{aligned}
\|a\|_{\mathcal{U}} &\leq \sum_n \|a_n\|_{\mathcal{U}} \leq \sum_n \operatorname*{ess\,inf}_{\gamma\in\Gamma}(P(\gamma)\|a_n\|_\gamma) \\
&= \sum_n \operatorname*{ess\,inf}_{\gamma\in\Gamma}\left(\frac{P(\gamma)}{\alpha_n(\gamma)}\alpha_n(\gamma)\|a_n\|_\gamma\right) \\
&\leq \sum_n \exp\left(\int_\Gamma \log\frac{P(\gamma)}{\alpha_n(\gamma)}P_{z_n}(\gamma)\,d\gamma\right) \\
&\quad \times \exp\left(\int_\Gamma \log(\alpha_n(\gamma)\|a_n\|_\gamma)P_{z_n}(\gamma)\,d\gamma\right) \\
&\leq \|P_{z_n}\|_\infty \sup_n(P(z_n)) \sum_n \frac{J(\alpha_n, a_n)}{\alpha_n(z_n)}.
\end{aligned}$$

Let K be a compact set in D satisfying the hypothesis. Then, choosing $z_n \in K$ such that $\alpha_n(z_n) = \sup_{z\in K}\alpha_n(z)$, we conclude that there exists $C > 0$ such that

$$\begin{aligned}
\|a\|_{\mathcal{U}} &\leq C\sum_n \frac{J(\alpha_n, a_n)}{\sup_{z\in K}\alpha_n(z)} \leq C\sum_n \frac{\alpha_n(z_0)}{\sup_{z\in K}\alpha_n(z)}\cdot\frac{J(\alpha_n, a_n)}{\alpha_n(z_0)} \\
&\leq C(1+\varepsilon)\left(\sum_n\left(\frac{\alpha_n(z_0)}{\sup_{z\in K}\alpha_n(z)}\right)^{p'}\right)^{1/p'}\|a\|_{(A)^S_{z_0,p}} \\
&\leq CC_M(\|a\|_{(A)^S_{z_0,p}} + \varepsilon). \ \blacksquare
\end{aligned}$$

Remark 4.4. Observe that, in fact, we have proved that $\sum_n \|a_n\|_{\mathcal{U}} \leq CC_M\|a\|_{(A)^S_{z_0,p}}$.

THEOREM 4.5. *Let $p \geq 1$. Under the hypotheses of Proposition 4.3 the spaces $(A)^S_{z_0,p}$ are Banach spaces.*

Proof. Let $(a_n)_n \subset (A)^S_{z_0,p}$ such that $\sum_n \|a_n\|_{(A)^S_{z_0,p}} < \infty$. Then, given $\varepsilon > 0$, for every n there exist $(a_{n,m})_m$ in $\mathcal{A}$ and $\{\alpha_{n,m}\}_m$ in S such that $a_n = \sum_m a_{n,m}$ and

$$(6)\qquad \sum_n\left(\sum_m\left(\frac{J(\alpha_{n,m}, a_{n,m})}{\alpha_{n,m}(z_0)}\right)^p\right)^{1/p} \leq (1+\varepsilon)\sum_n \|a_n\|_{(A)^S_{z_0,p}} < \infty.$$

Hence, if we consider the countable set $M = \{\alpha_{n,m} : n, m\}$, we get, by the previous remark, $\sum_n\sum_m \|a_{n,m}\|_{\mathcal{U}} \leq C_M\sum_n \|a_n\|_{(A)^S_{z_0,p}} < \infty$ and thus, we can define $a = \sum_{n,m} a_{n,m}$ in the $\mathcal{U}$-norm. In fact, we can write $a = \sum_n(\sum_{\alpha\in M} a_{n,\alpha}) = \sum_{\alpha\in M}(\sum_n a_{n,\alpha})$ where, if we set $M_n = \{\alpha_{n,m}\}_m$ and $\alpha \notin M_n$ then $a_{n,\alpha} = 0$.

Now, since, for a fixed $\alpha \in M$, $\sum_n J(\alpha, a_{n,\alpha}) < \infty$ and $a_{n,\alpha} \in \mathcal{A}$, we get

$$\int_\Gamma \alpha(\gamma)\sum_n \|a_{n,\alpha}\|_\gamma\,d\gamma \leq \sum_n J(\alpha, a_{n,\alpha}) < \infty,$$

and, therefore, $\sum_n a_{n,\alpha} \in A(\gamma)$ for almost every $\gamma \in \Gamma$ and $\log^+\|\sum_n a_{n,\alpha}\|_\gamma$ is in $L^1(\Gamma)$; that is, $\sum_n a_{n,\alpha} \in \mathcal{A}$. Hence, by (6),

$$\begin{aligned}
\|a\|_{(A)^S_{z_0,p}} &\leq \left(\sum_{\alpha\in M}\left(\frac{J(\alpha, \sum_n a_{n,\alpha})}{\alpha(z_0)}\right)^p\right)^{1/p} \leq \sum_n\left(\sum_{\alpha\in M}\left(\frac{J(\alpha, a_{n,\alpha})}{\alpha(z_0)}\right)^p\right)^{1/p} \\
&\leq \sum_n\left(\sum_m\left(\frac{J(\alpha_{n,m}, a_{n,m})}{\alpha_{n,m}(z_0)}\right)^p\right)^{1/p} < \infty.
\end{aligned}$$

That is, $a \in (A)^S_{z_0,p}$ and $\|a\|_{(A)^S_{z_0,p}} \leq \sum_n \|a_n\|_{(A)^S_{z_0,p}}$. $\blacksquare$

THEOREM 4.6. *Let $\overline{A}$ and $\overline{B}$ be two i.f. and let $T : \overline{A} \to \overline{B}$ be an interpolation operator with norm $\|T\|_{A(\gamma)\to B(\gamma)} \leq M(\gamma) \in \mathcal{L}$.*

(a) *If $\|M\|_\infty < \infty$ then, for every $S \subset \mathcal{L}$,*

$$T : (A)^S_{z_0,p} \to (B)^S_{z_0,p}$$

is bounded with norm less than or equal to $\|M\|_\infty$.

(b) *If $S \subset MS$ then*

$$T : (A)^S_{z_0,p} \to (B)^S_{z_0,p}$$

is bounded with norm less than or equal to $M(z_0)$ (that is, with convexity).

Proof. The proof of both parts is obvious by property (5) of Proposition 2.5. ∎

The following remark gives us simple and useful properties of these spaces.

Remark 4.7. (i) If $p_0 < p_1$ then $(A)^S_{z_0,p_0}$ is embedded in $(A)^S_{z_0,p_1}$.

(ii) If $S \subset S'$ then $(A)^S_{z_0,p}$ is embedded in $(A)^{S'}_{z_0,p}$.

(iii) If $S = S_1 \cup S_2$ then $(A)^S_{z_0,p}$ is equivalent to the sum space $(A)^{S_1}_{z_0,p} + (A)^{S_2}_{z_0,p}$.

Since we already know (see [CJ], [F], [CP]) that the K-method and J-method do not coincide in general, we cannot expect an equivalence theorem for both methods.

The next theorem gives a necessary condition for $(A)^S_{z_0,p}$ to be embedded in $[A]^S_{z_0,p}$. This result holds in all the known examples for a finite collection of spaces.

THEOREM 4.8. *Let $(N,+)$ be an additive group and let S be the set $\{\alpha_n : n \in N\}$ and assume that, for every n, m, $\alpha_{n+m}(\cdot) \sim \alpha_n(\cdot)\alpha_m(\cdot)$ with constants independent of n, m. If $\sum_n \operatorname{ess\,inf} \alpha_n(\gamma)/\alpha_n(z_0) < \infty$ and $p \ge 1$, then $(A)^S_{z_0,p}$ is embedded in $[A]^S_{z_0,p}$; and if $p < 1$, the embedding holds if $\sum_n (\operatorname{ess\,inf} \alpha_n(\gamma)/\alpha_n(z_0))^p < \infty$.*

Proof. Let $a \in (A)^S_{z_0,p}$. Given $\varepsilon > 0$, take $(a_n)_n$ such that $a = \sum_n a_n$ and

$$\left(\sum_n \left(\frac{J(\alpha_n, a_n)}{\alpha_n(z_0)}\right)^p\right)^{1/p} \le \|a\|_{(A)^S_{z_0,p}} + \varepsilon .$$

Then we get

$$\begin{aligned} K(\alpha_n, a) &\le \sum_m K(\alpha_n, a_m) \\ &\le \sum_m \inf\Big\{ \int_\Gamma \alpha_n(\gamma)\varphi(\gamma)\|a_m\|_\gamma \, d\gamma : \int_\Gamma \varphi(\gamma)\, d\gamma = 1\Big\} \\ &\sim \sum_m \inf\Big\{ \int_\Gamma \alpha_{n-m}(\gamma)\varphi(\gamma)\alpha_m(\gamma)\|a_m\|_\gamma \, d\gamma : \int_\Gamma \varphi(\gamma)\, d\gamma = 1\Big\} \\ &\le \sum_m J(\alpha_m, a_m) \inf\Big\{ \int_\Gamma \alpha_{n-m}(\gamma)\varphi(\gamma) \, d\gamma : \int_\Gamma \varphi(\gamma)\, d\gamma = 1\Big\} \\ &\le \sum_m J(\alpha_m, a_m) \operatorname{ess\,inf} \alpha_{n-m}(\gamma) . \end{aligned}$$

Therefore, if $p \ge 1$ then

$$\begin{aligned} \|a\|_{[A]^S_{z_0,p}} &= \left(\sum_n \left(\frac{K(\alpha_n, a)}{\alpha_n(z_0)}\right)^p\right)^{1/p} \\ &\le C\left(\sum_n \left(\sum_m \frac{J(\alpha_m, a_m)}{\alpha_m(z_0)} \cdot \frac{\operatorname{ess\,inf} \alpha_{n-m}(\gamma)}{\alpha_{n-m}(z_0)}\right)^p\right)^{1/p} \\ &\le C'\left(\sum_n \sum_m \left(\frac{J(\alpha_m, a_m)}{\alpha_m(z_0)}\right)^p \frac{\operatorname{ess\,inf} \alpha_{n-m}(\gamma)}{\alpha_{n-m}(z_0)}\right)^{1/p} \\ &\le C''\|a\|_{(A)^S_{z_0,p}} + \varepsilon . \end{aligned}$$

If $p < 1$ we just make the obvious changes in the second inequality. ∎

Remark 4.9. An interesting example of a set S satisfying the previous hypotheses is given by $S = \{M(\cdot)^n : n \in \mathbb{Z}\}$. This type of set includes $S = \{2^{n\alpha(\cdot)} : n \in \mathbb{Z}\}$ and the sets S of Examples 2.7(I)–(IV).

Moreover, if we want to have the usual properties of the spaces $[A]^S_{z_0,p}$ and $(A)^S_{z_0,p}$, namely, to be intermediate Banach spaces with $(A)^S_{z_0,p} \subset [A]^S_{z_0,p}$, we need essentially that S is a multiplicative group such that

$$\sum_{\alpha \in S} \operatorname*{ess\,inf}_{\gamma \in \Gamma} \alpha(\gamma)/\alpha(z_0) < \infty .$$

COROLLARY 4.10. *If $(A)^S_{z_0,p} \equiv [A]^S_{z_0,p}$ then, for every $S \subset S'$, $(A)^{S'}_{z_0,p}$ is embedded in $[A]^{S'}_{z_0,p}$ if and only if both spaces are equivalent.*

Proof. This is an easy consequence of

$$(A)^S_{z_0,p} \subset (A)^{S'}_{z_0,p} \subset [A]^{S'}_{z_0,p} \subset [A]^S_{z_0,p} . \ ∎$$

In the same way, we have

COROLLARY 4.11. *If $(A)^{S'}_{z_0,p} \subset [A]^{S'}_{z_0,p}$, then $(A)^S_{z_0,p} \subset [A]^S_{z_0,p}$ for every $S \subset S'$.*

APPLICATIONS 4.12. Some of the following results are already known in the literature. However, we show here how all of them are trivial consequences of Remarks 3.11 and 4.7.

(1) Let $\{\Gamma_0, \Gamma_1\}$ be a partition of Γ. Let Γ_j^k with $j = 0, 1$ and $k = 1, 2$ be such that $\{\Gamma_j^k : k = 1, 2\}$ is a partition of Γ_j for each j. Consider

$$S = \left\{ \{\alpha_n(\cdot)\}_{n \in \mathbb{Z}} : \alpha_n(\gamma) = \begin{cases} 1 & \text{if } \gamma \in \Gamma_0 \\ 2^n & \text{if } \gamma \in \Gamma_1 \end{cases} \right\}$$

and

$$S^{\text{Sparr}} = \left\{ \{\alpha_{n_1,n_2,n_3}(\cdot)\}_{n_j\in\mathbb{Z}} : \alpha_{n_1,n_2,n_3}(\gamma) = \begin{cases} 1 & \text{if } \gamma \in \Gamma_0^1 \\ 2^{n_1} & \text{if } \gamma \in \Gamma_0^2 \\ 2^{n_2} & \text{if } \gamma \in \Gamma_1^1 \\ 2^{n_3} & \text{if } \gamma \in \Gamma_1^2 \end{cases} \right\}.$$

Then, obviously, $S \subset S^{\text{Sparr}}$ and hence

$$(A_0, A_1, A_2, A_3)^{K,\text{Sparr}}_{(\theta_1,\theta_2,\theta_3),p} \subset (A_0 + A_1, A_2 + A_3)_{\theta_2+\theta_3,p}$$

and

$$(A_0 \cap A_1, A_2 \cap A_3)_{\theta_2+\theta_3,p} \subset (A_0, A_1, A_2, A_3)^{J,\text{Sparr}}_{(\theta_1,\theta_2,\theta_3),p}.$$

(2) With S as before and

$$S^{\text{Fernández}} = \left\{ \{\alpha_{n,k}(\cdot)\}_{n,k\in\mathbb{Z}} : \alpha_{n,k}(\gamma) = \begin{cases} 1 & \text{if } \gamma \in \Gamma_0^1 \\ 2^{n} & \text{if } \gamma \in \Gamma_0^2 \\ 2^{k} & \text{if } \gamma \in \Gamma_1^1 \\ 2^{n+k} & \text{if } \gamma \in \Gamma_1^2 \end{cases} \right\}$$

we get the analogous result for the Fernández spaces; that is,

$$(A_0, A_1, A_2, A_3)^{K,\text{Fernández}}_{(\alpha,\beta),p} \subset (A_0 + A_1, A_2 + A_3)_{\beta,p}$$

and

$$(A_0 \cap A_1, A_2 \cap A_3)_{\beta,p} \subset (A_0, A_1, A_2, A_3)^{J,\text{Fernández}}_{(\alpha,\beta),p}.$$

(3) *Relation between Fernández and Sparr spaces.* Since $S^{\text{Fernández}} \subset S^{\text{Sparr}}$, we get

$$(A_0, A_1, A_2, A_3)^{K,\text{Sparr}}_{(\theta_1,\theta_2,\theta_3),p} \subset (A_0, A_1, A_2, A_3)^{K,\text{Fernández}}_{(\alpha,\beta),p}$$

and

$$(A_0, A_1, A_2, A_3)^{J,\text{Fernández}}_{(\alpha,\beta),p} \subset (A_0, A_1, A_2, A_3)^{J,\text{Sparr}}_{(\theta_1,\theta_2,\theta_3),p}$$

whenever $\alpha = \theta_1 + \theta_2$, $\beta = \theta_1 + \theta_3$ and $\theta_1 + \theta_2 + \theta_3 < 1$. In particular, taking $\theta_3 = \alpha\beta$ we get a result proved by Cobos–Peetre in [CP].

We now give an example of an i.f. and a set S such that the space $(A)^S_{z_0,p}$ is not embedded in $[A]^S_{z_0,p}$.

EXAMPLE 4.13. Let $\overline{A} = \{A(\gamma) = A_j : \gamma \in \Gamma_j,\ j = 0, 1\}$ be an i.f. with $A_0 = L^1$ and $A_1 = L^\infty$. Let S be the s.b.s. of Example 2.7(I) and let

$$S' = \left\{ \alpha_n(\gamma) = \begin{cases} 1 & \text{if } \gamma \in \Gamma_0 \\ n^2 & \text{if } \gamma \in \Gamma_1 \end{cases} \right\}.$$

Assume that $(A)^{S\cup S'}_{z_0,p} \subset [A]^{S\cup S'}_{z_0,p}$. Then by Remarks 3.11 and 4.7, we would have, for $p = p(\theta) = 1/(1-\theta)$, with $\theta = |\Gamma_1|_{z_0}$,

$$L^p + (A)^{S'}_{z_0,p} = (A)^{S\cup S'}_{z_0,p} \subset [A]^{S\cup S'}_{z_0,p} = L^p \cap [A]^{S'}_{z_0,p},$$

and consequently $L^p \subset [A]^{S'}_{z_0,p}$.

Now, if $f \in [A]^{S'}_{z_0,p}$ and $f^{**}(t) = t^{-1}\int_0^t f^*(s)\,ds$ with f^* the non-increasing rearrangement function, then

$$\|f\|_{[A]^{S'}_{z_0,p}} \equiv \left(\sum_n \left(\frac{K(n^2,f)}{n^{2\theta}}\right)^p\right)^{1/p} = \left(\sum_n \left(\frac{n^2 f^{**}(n^2)}{n^{2\theta}}\right)^p\right)^{1/p}$$
$$= \left(\sum_n (n^{2(1-\theta)} f^{**}(n^2))^p\right)^{1/p} = \left(\sum_n n^2 (f^{**}(n^2))^{1/(1-\theta)}\right)^{1-\theta} < \infty.$$

But, if

$$f(t) = \begin{cases} 1 & \text{if } 0 \le t \le 1, \\ 1/\sqrt{t} & \text{if } t > 1, \end{cases}$$

one can easily check that the above series diverges, for say $\theta = 3/4$. Therefore L^p is not embedded in $[A]^{S'}_{z_0,p}$.

5. Relation with the complex method for families. In this section, we shall use the notation of [CCRSW]. Let us recall that for an i.f. $\overline{A}$, the St. Louis group defines the function spaces

$$\mathcal{G}(A(\cdot),\Gamma) = \left\{ g(z) = \sum{}' \varphi_j a_j : \ \varphi_j \in H^\infty(D),\ a_j \in \mathcal{A},\ \|g\|_{\mathcal{G}(A(\cdot),\Gamma)} < \infty \right\}$$

where $\|g\|_{\mathcal{G}(A(\cdot),\Gamma)} = \operatorname{ess\,sup}_{\gamma\in\Gamma} \|g(\gamma)\|_{A(\gamma)}$, and $\mathcal{F}(A(\cdot),\Gamma)$ is the completion of $\mathcal{G}(A(\cdot),\Gamma)$ with respect to the norm $\|\cdot\|_{\mathcal{G}}$. However, as is shown in [CCRSW], one can substitute the above norm by $\|g\| = (\int_\Gamma \|g(\gamma)\|^p_\gamma\, d\gamma)^{1/p}$ for every $p \ge 1$ and the interpolation spaces $A\{z_0\}$ and $A[z_0]$ remain unchanged. In this paper, we shall denote the norm with index p by $\|\cdot\|_{\mathcal{G}^p}$ and the corresponding spaces by $\mathcal{G}^p(A(\cdot),\Gamma)$ and $\mathcal{F}^p(A(\cdot),\Gamma)$.

In this section, we show that the usual embeddings known for the classical method:

$$(A_0, A_1)_{\theta,1} \subset [A_0, A_1]_\theta \subset (A_0, A_1)_{\theta,\infty}$$

(see [BL]) are now given by

$$(A)^S_{z_0,1} \subset A[z_0] \subset [A]^S_{z_0,\infty}.$$

PROPOSITION 5.1. *Let $z_0 \in D$ and $S \subset \mathcal{L}$. Then the following embeddings hold.*

(a) *$(A)^S_{z_0,1}$ is embedded in $A[z_0]$.*

(b) *$A[z_0]$ is embedded in $[A]^S_{z_0,\infty}$.*

Proof. (a) Let $a \in (A)^S_{z_0,1}$. Then, for every $\varepsilon > 0$, there exist two countable sets $\{\alpha_n\}_n \subset S$ and $\{a_n\}_n \subset \mathcal{A}$ such that $a = \sum_n a_n$ and

$$\sum_n \frac{J(\alpha_n, a_n)}{\alpha_n(z_0)} \le \|a\|_{(A)^S_{z_0,1}} + \varepsilon\,.$$

Now, if $a \in \mathcal{A}$, then (see [CCRSW])

$$\|a\|_{A[z_0]} \le \exp \int_\Gamma \log \|a\|_\gamma P_{z_0}(\gamma)\,d\gamma$$

$$= \exp \int_\Gamma \log(\alpha(\gamma)\|a\|_\gamma/\alpha(\gamma)) P_{z_0}(\gamma)\,d\gamma \le \frac{J(\alpha, a)}{\alpha(z_0)}\,,$$

and hence

$$\|a\|_{A[z_0]} \le \sum_n \|a_n\|_{A[z_0]} \le \|a\|_{(A)^S_{z_0,1}} + \varepsilon\,.$$

(b) Let $a \in A[z_0]$ and let $f \in \mathcal{F}^1(A(\cdot), \Gamma)$ be such that $f(z_0) = a$. Since there exists $(g_n)_n$ in $\mathcal{G}^1(A(\cdot), \Gamma)$ such that g_n converges to f, and we can always approximate each g_n by functions in $\mathcal{G}$, we see that f is a suitable function to compute $K(\alpha, a)$. More precisely, given $\alpha \in \mathcal{S}$, write $\widetilde{\alpha}(z) = \exp \int_\Gamma \log \alpha(\gamma) H_z(\gamma)\,d\gamma$ with H_z the Herglotz kernel (see [G]) and consider $f_\alpha = \widetilde{\alpha}(z_0) f/\widetilde{\alpha}$. Then $a = \int_\Gamma f_\alpha(\gamma) P_{z_0}(\gamma)\,d\gamma$ and

$$\int_\Gamma \alpha(\gamma)\|f_\alpha(\gamma)\|_\gamma P_{z_0}(\gamma)\,d\gamma \le C\alpha(z_0) \int_\Gamma \|f(\gamma)\|_\gamma\,d\gamma \le C\alpha(z_0)\|f\|^1_{\mathcal{F}}\,.$$

Thus, $K(\alpha, a) \le C\alpha(z_0)\|f\|_{\mathcal{F}^1}$ for every $f \in \mathcal{F}^1$ such that $f(z_0) = a$. That is,

$$\sup_{\alpha \in S} \frac{K(\alpha, a)}{\alpha(z_0)} \le C\|a\|_{A[z_0]}\,,$$

and consequently there is a constant C such that $\|a\|_{[A]^S_{z_0,\infty}} \le C\|a\|_{A[z_0]}$. ∎

Appendix. (1) We could also define, for $q \ge 1$, the interpolation spaces $[A]^S_{z_0,p,q}$ and $(A)^S_{z_0,p,q}$ as in 3.1 and 4.1 respectively but with the K- and J-functionals substituted by

$$K_q(\alpha, a) = \inf \Big\{ \Big(\int_\Gamma (\alpha(\gamma)\|a(\gamma)\|_\gamma)^q\,d\gamma \Big)^{1/q} \Big\}$$

and

$$J_q(\alpha, a) = \Big(\int_\Gamma (\alpha(\gamma)\|a\|_\gamma)^q\,d\gamma \Big)^{1/q}.$$

Although some slight modifications are needed, the theory can be developed similarly. In general these spaces do not coincide with the ones described in the paper, which obviously correspond to the case $q = 1$ in the K-method and $q = \infty$ in the J-method. These spaces will be studied in a forthcoming paper where duality and reiteration results will also be given.

(2) According to an observation of the referee, our method could also have been defined for families of Banach spaces indexed by points in an arbitrary measure space instead of the special case of the unit circle equipped with the harmonic measure. The latter is the natural setting to compare with the complex method for families but for the case of a finite family it is more natural to consider a finite set. The author thanks the referee for this remark.

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