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Restrictions from $\mathbb{R}^n$ to $\mathbb{Z}^n$ of weak type $(1,1)$ multipliers

by

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Abstract. Suppose that $\{\phi_j\}_{j=1}^{\infty}$ is a sequence of weak type $(1,1)$ multipliers for $L^1(\mathbb{R}^n)$ such that for each j, ϕ_j is continuous at every point of $\mathbb{Z}^n$. We show that the restrictions $\phi_j|\mathbb{Z}^n$, $j \ge 1$, are weak type $(1,1)$ multipliers for $L^1(\mathbb{T}^n)$. Moreover, the weak type $(1,1)$ norm of the maximal operator defined by the sequence $\{\phi_j\}_{j=1}^{\infty}$ controls that of the maximal operator defined by the sequence $\{\phi_j|\mathbb{Z}^n\}_{j=1}^{\infty}$. This de Leeuw type restriction theorem for maximal estimates of weak type $(1,1)$ answers in the affirmative a question about single multipliers posed by A. Pełczyński. Our central result, from which this restriction theorem follows by suitable regularization arguments, is another maximal theorem regarding convolution of a function in $L^1(\mathbb{R}^n)$ with weak type $(1,1)$ multipliers.

1. Introduction. Let n be a positive integer, and let G be either the additive group $\mathbb{R}^n$ or the multiplicative group $\mathbb{T}^n$. Denote by Γ the dual group of G. For $\phi \in L^\infty(\Gamma)$, we symbolize by T_ϕ the corresponding multiplier transform on $L^2(G)$: $T_\phi f = (\phi \widehat{f})^\vee$. The function ϕ is said to be a *multiplier of weak type* $(1,1)$ (in symbols $\phi \in M_1^{(w)}(\Gamma)$) provided that T_ϕ is of weak type $(1,1)$ on $L^1(G) \cap L^2(G)$. Given a sequence $\{\phi_j\}_{j\ge1} \subseteq M_1^{(w)}(\Gamma)$, we denote by $N_1^{(w)}(\{\phi_j\}_{j\ge1})$ the weak type $(1,1)$ norm of the maximal operator on $L^1(G) \cap L^2(G)$ defined by $\{T_{\phi_j}\}_{j\ge1}$.

In [4, Problem 5, p. 412], A. Pełczyński posed the following question, which seeks an analogue for weak type $(1,1)$ multipliers of a de Leeuw restriction theorem for strong type multipliers [3, Proposition 3.3]:

if $\phi \in M_1^{(w)}(\mathbb{R}^n)$, and ϕ is continuous at each point of $\mathbb{Z}^n$, is it necessarily true that the restriction $\phi|\mathbb{Z}^n$ belongs to $M_1^{(w)}(\mathbb{Z}^n)$?

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Special methods must be devised in order to treat this question, since standard tools such as Lorentz space duality do not fit weak type $(1,1)$ estimates.

In §3 we show the following maximal theorem, which includes an affirmative answer to Pełczyński's question.

(1.1) THEOREM. *Suppose that* $\{\phi_j\}_{j=1}^{\infty} \subseteq M_1^{(w)}(\mathbb{R}^n)$. *Suppose also that for each* $j \geq 1$, ϕ_j *is continuous at each point of* $\mathbb{Z}^n$. *Then* $\{\phi_j|\mathbb{Z}^n\}_{j=1}^{\infty} \subseteq M_1^{(w)}(\mathbb{Z}^n)$, *and*

$$N_1^{(w)}(\{\phi_j|\mathbb{Z}^n\}_{j=1}^{\infty}) \leq \zeta_n N_1^{(w)}(\{\phi_j\}_{j=1}^{\infty}),$$

where ζ_n *is a real constant depending only on* n.

Theorem (1.1) follows from our central result, which concerns convolutions with weak type $(1,1)$ multipliers. This is established in §2, and is stated as follows.

(1.2) THEOREM. *Suppose that* $k \in L^1(\mathbb{R}^n)$ *and* $\{\psi_j\}_{j=1}^{\infty} \subseteq M_1^{(w)}(\mathbb{R}^n)$. *Then*

$$\{k * \psi_j\}_{j=1}^{\infty} \subseteq M_1^{(w)}(\mathbb{R}^n),$$

and

$$N_1^{(w)}(\{k * \psi_j\}_{j=1}^{\infty}) \leq K_n \|k\|_{L^1(\mathbb{R}^n)} N_1^{(w)}(\{\psi_j\}_{j=1}^{\infty}),$$

where K_n *is a real constant depending only on* n.

In what follows, $\mathbb{N}$ will denote the set of positive integers, and λ will be Lebesgue measure on $\mathbb{R}^n$. Applications of Khinchin's Inequality [$6_{\rm I}$, Theorem V.8.4, p. 213] will be expressed in terms of the J-fold direct product D^J ($J \in \mathbb{N}$) of the multiplicative group $D = \{-1, 1\}$. The general element of D^J will be written $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_J)$, and $d\varepsilon$ will be Haar measure on D^J normalized to have total mass 1. The symbol "C" with a set of subscripts will be used to indicate a positive constant depending only on the subscripts which may change its value from one occurrence to another.

2. Proof of the convolution theorem.

We begin with three lemmas which provide the necessary framework for the demonstration of Theorem (1.2). The first of these is a variation on the method used to prove [5, Lemma 1, pp. 146, 147]. As noted in [5], this method traces back to the proof of Calderón's Lemma in [$6_{\rm II}$, pp. 165, 166].

(2.1) LEMMA. *Let* $\mathcal{G}$ *be a compact abelian group, and let* m *be normalized Haar measure on* $\mathcal{G}$. *Suppose that* A *is a measurable subset of* $\mathcal{G}$ *such that* $m(A) > 0$. *Let*

$$N = \max\{k \in \mathbb{N} : m(A) \leq k^{-1}\}.$$

Then there are elements $z_1, \ldots, z_N$ *of* $\mathcal{G}$ *such that the corresponding translates satisfy*

$$m\Big(\bigcup_{j=1}^{N} A + z_j\Big) > 1 - 2e^{-1}, \tag{2.2}$$

where e *is the base of natural logarithms.*

Proof. Let B denote the complement $\mathcal{G} \setminus A$, and let χ be the characteristic function of B. We have

$$\begin{aligned}(2.3)\quad &\int_{\mathcal{G}} dm(s) \int_{\mathcal{G}} dm(x_1) \ldots \\ &\ldots \int_{\mathcal{G}} dm(x_{N-1}) \int_{\mathcal{G}} \chi(x_1+s)\ldots\chi(x_N+s)\, dm(x_N) \\ &= [m(\mathcal{G}\setminus A)]^N < [1-(N+1)^{-1}]^N < 2e^{-1}.\end{aligned}$$

By Fubini's Theorem the left hand side is equal to

$$\begin{aligned}&\int_{\mathcal{G}} dm(x_1) \ldots \int_{\mathcal{G}} dm(x_N) \int_{\mathcal{G}} \chi(-x_1+s)\ldots\chi(-x_N+s)\, dm(s) \\ &= \int_{\mathcal{G}} dm(x_1) \ldots \int_{\mathcal{G}} dm(x_{N-1}) \int_{\mathcal{G}} m((B+x_1)\cap\ldots\cap(B+x_N))\, dm(x_N).\end{aligned}$$

Combining this with (2.3), we see that there are $z_1, \ldots, z_N$ in $\mathcal{G}$ such that

$$m\Big(\mathcal{G} \setminus \bigcup_{j=1}^{N} A + z_j\Big) = m((B+z_1)\cap\ldots\cap(B+z_N)) < 2e^{-1}. \ \blacksquare$$

Lemma (2.1) has the following adaptation to $\mathbb{R}^n$.

(2.4) LEMMA. *Suppose that* $b > 0$, *and* A *is a Lebesgue-measurable subset of* $[-b, b)^n$ *such that* $\lambda(A) > 0$. *Put*

$$N = \max\{k \in \mathbb{N} : \lambda(A) \leq (2b)^n k^{-1}\}.$$

Then there are points $\{u_\nu\}_{\nu=1}^{N2^n} \subseteq [-2b, 2b)^n$ *such that*

$$(2b)^n(1-2e^{-1}) < \lambda\Big(\bigcup_{\nu=1}^{N2^n} A + u_\nu\Big).$$

Proof. Let $Q = A + (b, \ldots, b) \subseteq [0, 2b)^n$. Denote by $\oplus$ the group operation on $[0, 2b)^n$ of addition $\bmod (2b)$ in each coordinate: for $x = (x_1, \ldots, x_n) \in [0, 2b)^n$, $y = (y_1, \ldots, y_n) \in [0, 2b)^n$, $1 \leq m \leq n$,

$$(x \oplus y)_m = \begin{cases} x_m + y_m & \text{if } x_m + y_m < 2b, \\ x_m + y_m - 2b & \text{if } x_m + y_m \geq 2b. \end{cases} \tag{2.5}$$

Apply Lemma (2.1) to the normalized Haar measure $(2b)^{-n}\lambda$ of the group $[0,2b)^n$ and the set Q. This gives points $\{z_k\}_{k=1}^N \subseteq [0,2b)^n$ such that

$$\lambda\Big(\bigcup_{k=1}^{N} Q \oplus z_k\Big) > (2b)^n(1-2e^{-1}). \tag{2.6}$$

For $1 \le k \le N$, put $z_k = (z_{k,1},\dots,z_{k,n})$, and let $W_{k,m,1} = [0, 2b - z_{k,m})$, $W_{k,m,2} = [2b - z_{k,m}, 2b)$, $1 \le m \le n$. It is readily seen that, for $1 \le k \le N$, we have

$$Q \oplus z_k = \bigcup\Big\{\Big[Q \cap \prod_{m=1}^{n} W_{k,m,j_m}\Big] \oplus z_k : (j_1,\dots,j_n) \in \{1,2\}^n\Big\}. \tag{2.7}$$

For each choice of $(j_1,\dots,j_n)$ in (2.7), we see from (2.5) that there is some $w_{k,(j_1,\dots,j_n)} \in [-2b,2b)^n$ such that

$$\Big[Q \cap \prod_{m=1}^{n} W_{k,m,j_m}\Big] \oplus z_k \subseteq Q + w_{k,(j_1,\dots,j_n)}.$$

Using this in (2.7), we now obtain the desired conclusion from (2.6). ∎

The circle of ideas in the next lemma is well-known. We include a version convenient for our purposes.

(2.8) LEMMA. *Suppose that (Ω,μ) is a measure space, $J \in \mathbb{N}$, and T_k, $1 \le k \le J$, is a linear mapping of $L^1(\Omega,\mu)$ into the complex-valued measurable functions on Ω such that T_k is of weak type $(1,1)$. Let $\mathcal{M}$ be the maximal operator on $L^1(\Omega,\mu)$ defined by the operators $\{T_k\}_{k=1}^J$, and denote the weak type $(1,1)$ norm of $\mathcal{M}$ on $L^1(\Omega,\mu)$ by $N_1^{(w)}(\mathcal{M})$. If $0<p<1$, and S is a subset of Ω such that $\mu(S) < \infty$, then for each $g \in L^1(\Omega,\mu)$,*

$$\|\mathcal{M}g\|_{L^p(S,\mu)} \le \mu(S)^{(1-p)/p}(1-p)^{-1/p}N_1^{(w)}(\mathcal{M})\|g\|_{L^1(\Omega,\mu)}.$$

Proof. Put $\mathcal{N} = (N_1^{(w)}(\mathcal{M}))^{-1}\mathcal{M}$, and let f be a unit vector in $L^1(\Omega,\mu)$. Writing $\delta(y) = \mu\{s \in S : (\mathcal{N}f)(s) > y\}$ for all $y > 0$, we have

$$\|\mathcal{N}f\|^p_{L^p(S,\mu)} = \int_0^{1/\mu(S)} py^{p-1}\delta(y)\,dy + \int_{1/\mu(S)}^{\infty} py^{p-1}\delta(y)\,dy.$$

Since $\delta(y) \le \mu(S)$, the first term on the right does not exceed $\mu(S)^{1-p}$. It is also clear that $\delta(y) \le y^{-1}$, and consequently the second term on the right does not exceed $p(1-p)^{-1}\mu(S)^{1-p}$. The desired conclusion is now evident. ∎

Proof of Theorem (1.2). Throughout what follows, the Schwartz class of functions on $\mathbb{R}^n$ will be symbolized by $\mathcal{S}$, and the characteristic function of a subset $B \subseteq \mathbb{R}^n$ will be written χ_B. Suppose that $J \in \mathbb{N}$,

$\{\psi_j\}_{j=1}^J \subseteq M_1^{(w)}(\mathbb{R}^n)$, $k \in \mathcal{S}$, $f \in \mathcal{S}$, and $y > 0$. Put

$$A = \{x \in \mathbb{R}^n : \max_{1\le j\le J} |(T_{k*\psi_j} f)(x)| > y\},$$

and suppose that $\lambda(A) > 0$. Pick and fix $M \in \mathbb{N}$ such that $A_M \equiv A \cap [-M,M)^n$ satisfies $\lambda(A_M) > \lambda(A)/2$. Let $N = \max\{k \in \mathbb{N} : \lambda(A_M) \le (2M)^n k^{-1}\}$. By Lemma (2.4) there are points $\{z_\nu\}_{\nu=1}^{N2^n} \subseteq [-2M,2M)^n$ such that

$$(2M)^n(1-2e^{-1}) < \lambda\Big(\bigcup_{\nu=1}^{N2^n} A_M + z_\nu\Big). \tag{2.9}$$

For $1 \le \nu \le N2^n$, let f_{-z_ν} denote the translate of f by $-z_\nu$. It is clear that

$$A_M + z_\nu \subseteq \{x \in [-3M,3M)^n : \max_{1\le j\le J} |(T_{k*\psi_j}(f_{-z_\nu}))(x)| > y\}.$$

Consequently, we see that, pointwise on $\mathbb{R}^n$,

$$\max_{1\le\nu\le N2^n} \chi_{A_M+z_\nu} \le y^{-1}\chi_{[-3M,3M]^n} \max_{1\le j\le J}\Big\{\sum_{\nu=1}^{N2^n} |T_{k*\psi_j}(f_{-z_\nu})|^2\Big\}^{1/2}. \tag{2.10}$$

Now fix an index p such that $0<p<1$. Applying Khinchin's Inequality for p to the majorant in (2.10), and then taking L^p-norms, we find with the aid of (2.9) that

$$(2M)^{n/p}C_p < y^{-1}\Big\{\int_{D^{N2^n}} \|\max_{1\le j\le J} |T_{k*\psi_j}F_\varepsilon|\|^p_{L^p([-3M,3M]^n)}\,d\varepsilon\Big\}^{1/p}, \tag{2.11}$$

where, for each ε belonging to D^{N2^n}, we define $F_\varepsilon \in \mathcal{S}$ by writing

$$F_\varepsilon = \sum_{\nu=1}^{N2^n} \varepsilon_\nu f_{-z_\nu}. \tag{2.12}$$

Temporarily fix ε, and for each $t \in \mathbb{R}^n$, denote by γ_t the corresponding character on $\mathbb{R}^n$. Then for $1 \le j \le J$, we can arrange to have for all $x \in \mathbb{R}^n$,

$$(T_{k*\psi_j}F_\varepsilon)(x) = \int_{\mathbb{R}^n} k(t)\gamma_t(x)(T_{\psi_j}(\gamma_{-t}F_\varepsilon))(x)\,dt.$$

Apply the Cauchy–Schwarz Inequality to the right-hand side to get

$$|(T_{k*\psi_j}F_\varepsilon)(x)| \le \|k\|^{1/2}_{L^1(\mathbb{R}^n)}\Big\{\int_{\mathbb{R}^n} |k(t)||(T_{\psi_j}(\gamma_{-t}F_\varepsilon))(x)|^2\,dt\Big\}^{1/2}. \tag{2.13}$$

It is easy to see from standard considerations that for each $x \in \mathbb{R}^n$, $\mathcal{S}$ includes the function of t given by $k(t)|(T_{\psi_j}(\gamma_{-t}F_\varepsilon))(x)|^2$. Putting $I_M = [-3M,3M]^n$, we see from (2.13) and Fatou's Lemma that

$$(2.14)\quad \Big\| \max_{1\le j\le J} |T_{k*\psi_j} F_\varepsilon| \Big\|_{L^p(I_M)} \le \|k\|^{1/2}_{L^1(\mathbb{R}^n)} \liminf_m 2^{-mn/2} \times \Big\| \max_{1\le j\le J} \Big\{ \sum_{\sigma\in\mathbb{Z}^n} |k(2^{-m}\sigma)| |T_{\psi_j}(\gamma_{-2^{-m}\sigma} F_\varepsilon)|^2 \Big\}^{1/2} \Big\|_{L^p(I_M)}.$$

We now focus attention on estimating the majorant in (2.14). For $m \in \mathbb{N}$, $L \in \mathbb{N}$, it is straightforward to see with the aid of Khinchin's Inequality (by suitable labelling of indices) that

$$(2.15)\quad \Big\| \max_{1\le j\le J} \Big\{ \sum_{\substack{\sigma\in\mathbb{Z}^n\\ \|\sigma\|_\infty\le L}} |k(2^{-m}\sigma)| |T_{\psi_j}(\gamma_{-2^{-m}\sigma} F_\varepsilon)|^2 \Big\}^{1/2} \Big\|_{L^p(I_M)}$$
$$\le C_p \Big\| \Big\{ \int_{D^{(2L+1)^n}} \max_{1\le j\le J} \Big[\Big| T_{\psi_j} \sum_{\substack{\sigma\in\mathbb{Z}^n\\ \|\sigma\|_\infty\le L}} \eta_\sigma |k(2^{-m}\sigma)|^{1/2} \times \gamma_{-2^{-m}\sigma} F_\varepsilon \Big|^p \Big] d\eta \Big\}^{1/p} \Big\|_{L^p(I_M)}$$
$$\le C_p \int_{D^{(2L+1)^n}} \Big\| \max_{1\le j\le J} \Big| T_{\psi_j} \sum_{\substack{\sigma\in\mathbb{Z}^n\\ \|\sigma\|_\infty\le L}} \eta_\sigma |k(2^{-m}\sigma)|^{1/2} \times \gamma_{-2^{-m}\sigma} F_\varepsilon \Big| \Big\|_{L^p(I_M)} d\eta.$$

From Lemma (2.8) we infer that

$$(2.16)\quad \int_{D^{(2L+1)^n}} \Big\| \max_{1\le j\le J} \Big| T_{\psi_j} \sum_{\substack{\sigma\in\mathbb{Z}^n\\ \|\sigma\|_\infty\le L}} \eta_\sigma |k(2^{-m}\sigma)|^{1/2} \gamma_{-2^{-m}\sigma} F_\varepsilon \Big| \Big\|_{L^p(I_M)} d\eta$$
$$\le C_{n,p} M^{n(1-p)/p} N_1^{(w)}(\{\psi_j\}_{j=1}^J) \times \int_{\mathbb{R}^n} \int_{D^{(2L+1)^n}} \Big| \sum_{\substack{\sigma\in\mathbb{Z}^n\\ \|\sigma\|_\infty\le L}} \eta_\sigma |k(2^{-m}\sigma)|^{1/2} \gamma_{-2^{-m}\sigma}(x) F_\varepsilon(x) \Big| \, d\eta\, dx.$$

After applying Khinchin's Inequality to the inner integral in the majorant of (2.16), we readily find that

$$\int_{D^{(2L+1)^n}} \Big\| \max_{1\le j\le J} \Big| T_{\psi_j} \sum_{\substack{\sigma\in\mathbb{Z}^n\\ \|\sigma\|_\infty\le L}} \eta_\sigma |k(2^{-m}\sigma)|^{1/2} \gamma_{-2^{-m}\sigma} F_\varepsilon \Big| \Big\|_{L^p(I_M)} d\eta$$

$$\le C_{n,p} M^{n(1-p)/p} N_1^{(w)}(\{\psi_j\}_{j=1}^J) \Big\{ \sum_{\substack{\sigma\in\mathbb{Z}^n\\ \|\sigma\|_\infty\le L}} |k(2^{-m}\sigma)| \Big\}^{1/2} \|F_\varepsilon\|_{L^1(\mathbb{R}^n)}.$$

Using this in (2.15), we get

$$(2.17)\quad \Big\| \max_{1\le j\le J} \Big\{ \sum_{\substack{\sigma\in\mathbb{Z}^n\\ \|\sigma\|_\infty\le L}} |k(2^{-m}\sigma)| |T_{\psi_j}(\gamma_{-2^{-m}\sigma} F_\varepsilon)|^2 \Big\}^{1/2} \Big\|_{L^p(I_M)}$$
$$\le C_{n,p} M^{n(1-p)/p} N_1^{(w)}(\{\psi_j\}_{j=1}^J) \Big\{ \sum_{\substack{\sigma\in\mathbb{Z}^n\\ \|\sigma\|_\infty\le L}} |k(2^{-m}\sigma)| \Big\}^{1/2} \|F_\varepsilon\|_{L^1(\mathbb{R}^n)}.$$

Employing (2.17) in (2.14), we infer that for each ε in D^{N2^n},

$$(2.18)\quad \Big\| \max_{1\le j\le J} |T_{k*\psi_j} F_\varepsilon| \Big\|_{L^p(I_M)} \le C_{n,p} \|k\|_{L^1(\mathbb{R}^n)} M^{n(1-p)/p} N_1^{(w)}(\{\psi_j\}_{j=1}^J) \|F_\varepsilon\|_{L^1(\mathbb{R}^n)}.$$

Notice that by (2.12) and the definition of N,

$$\|F_\varepsilon\|_{L^1(\mathbb{R}^n)} \le N2^n \|f\|_{L^1(\mathbb{R}^n)} \le 2^{n+1} \|f\|_{L^1(\mathbb{R}^n)} (2M)^n (\lambda(A))^{-1}.$$

Hence we deduce from (2.18) that

$$\Big\| \max_{1\le j\le J} |T_{k*\psi_j} F_\varepsilon| \Big\|_{L^p(I_M)} \le C_{n,p} \|k\|_{L^1(\mathbb{R}^n)} M^{n/p} N_1^{(w)}(\{\psi_j\}_{j=1}^J) \|f\|_{L^1(\mathbb{R}^n)} (\lambda(A))^{-1}.$$

Applying this to (2.11), we find after some obvious simplifications that M cancels out, leaving us with

$$(2.19)\quad \lambda\{x \in \mathbb{R}^n : \max_{1\le j\le J} |(T_{k*\psi_j} f)(x)| > y\} \le y^{-1} C_{n,p} \|k\|_{L^1(\mathbb{R}^n)} N_1^{(w)}(\{\psi_j\}_{j=1}^J) \|f\|_{L^1(\mathbb{R}^n)},$$

for $J \in \mathbb{N}$, $y > 0$, $k \in \mathcal{S}$, $f \in \mathcal{S}$. Since the index p is fixed, the constant $C_{n,p}$ in (2.19) depends only on n, and standard approximations using (2.19) now complete the proof of Theorem (1.2). ■

3. Proof of Theorem (1.1). Let the sequence $\{k_\nu\}_{\nu=1}^\infty$ be the Fejér kernel for $\mathbb{R}^n$: $k_\nu(x_1,\ldots,x_n) = \prod_{\sigma=1}^n F_\nu(x_\sigma)$, where, for all $s \in \mathbb{R}$,

$$F_\nu(s) = (2\pi)^{-1}\nu \left(\frac{\sin(\nu s/2)}{\nu s/2} \right)^2.$$

Denote normalized Haar measure on $\mathbb{T}^n$ by m. We consider a finite sequence $\{\phi_j\}_{j=1}^J \subseteq M_1^{(w)}(\mathbb{R}^n)$ such that for $1 \le j \le J$, ϕ_j is continuous at each point of $\mathbb{Z}^n$, and we shall show that for each $f \in L^2(\mathbb{T}^n)$, and $y > 0$,

$$(3.1)\qquad m\{\omega \in \mathbb{T}^n : \max_{1\le j\le J} |(T_{\phi_j|\mathbb{Z}^n} f)(\omega)| > y\} \le K_n N_1^{(w)}(\{\phi_j\}_{j=1}^J) y^{-1} \|f\|_{L^1(\mathbb{T}^n)},$$

where K_n is the constant in Theorem (1.2). This will establish Theorem (1.1) with the constant K_n of (1.2) serving as ζ_n.

By considering $\{\widehat{k}_\nu \phi_j\}_{j=1}^J$, and then letting $\nu \to \infty$, we can easily reduce the proof of (3.1) to the special case where each ϕ_j has compact support. In this case, for each $\nu \in \mathbb{N}$, and $1 \le j \le J$, there is $h_{\nu,j} \in L^1(\mathbb{R}^n)$ such that $\widehat{h}_{\nu,j} = k_\nu * \phi_j$. For each $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ let $\theta(x) = (\exp(ix_1), \ldots, \exp(ix_n))$, and let R_x be the translation operator on $L^1(\mathbb{T}^n)$ corresponding to $\theta(x)$. For $\nu \in \mathbb{N}$, and $1 \le j \le J$, we use $L^1(\mathbb{T}^n)$-valued Bochner integration to define the *transferred convolution operator* $H_{\nu,j}$ on $L^1(\mathbb{T}^n)$ by writing

$$(3.2)\qquad H_{\nu,j} f = \int_{\mathbb{R}^n} h_{\nu,j}(x)\, R_{-x} f\, dx \quad \text{for all } f \in L^1(\mathbb{T}^n).$$

Now fix $f \in L^2(\mathbb{T}^n)$. Taking the $L^1(\mathbb{T}^n)$-Fourier transform on each side of (3.2), we easily deduce that

$$H_{\nu,j} f = T_{(h_{\nu,j})^\wedge|\mathbb{Z}^n} f = T_{(k_\nu * \phi_j)|\mathbb{Z}^n} f.$$

For $\nu \in \mathbb{N}$, we can use the representation R to transfer the weak type $(1,1)$ bound of the maximal convolution operator on $L^1(\mathbb{R}^n)$ defined by the kernels $\{h_{\nu,j}\}_{j=1}^J$ (see [2] for this kind of transference in the classical setting at hand, or [1, Theorem (4.14)] for the general abstract setting). This gives for all $y > 0$,

$$m\{\omega \in \mathbb{T}^n : \max_{1\le j\le J} |(H_{\nu,j} f)(\omega)| > y\} \le N_1^{(w)}(\{(h_{\nu,j})^\wedge\}_{j=1}^J) y^{-1} \|f\|_{L^1(\mathbb{T}^n)}.$$

In view of the preceding discussion this can be rewritten in the form

$$(3.3)\qquad m\{\omega \in \mathbb{T}^n : \max_{1\le j\le J} |(T_{(k_\nu*\phi_j)|\mathbb{Z}^n} f)(\omega)| > y\} \le N_1^{(w)}(\{k_\nu * \phi_j\}_{j=1}^J) y^{-1} \|f\|_{L^1(\mathbb{T}^n)}.$$

After an application of Theorem (1.2) in the majorant of (3.3) we arrive at:

$$(3.4)\qquad m\{\omega \in \mathbb{T}^n : \max_{1\le j\le J} |(T_{(k_\nu*\phi_j)|\mathbb{Z}^n} f)(\omega)| > y\} \le K_n N_1^{(w)}(\{\phi_j\}_{j=1}^J) y^{-1} \|f\|_{L^1(\mathbb{T}^n)},$$

for $\nu \in \mathbb{N}$, $f \in L^2(\mathbb{T}^n)$, and $y > 0$. Letting $\nu \to \infty$ we can use the continuity of ϕ_j, $1 \le j \le J$, at each point of $\mathbb{Z}^n$ to obtain (3.1) from (3.4) and thereby complete the proof of Theorem (1.1). ∎

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