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Pointwise multiplicative inequalities and Nirenberg type estimates in weighted Sobolev spaces

by

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Abstract. We prove pointwise multiplicative inequalities on bounded domains with the cone property and on infinite cones, and derive a certain class of multiplicative inequalities in weighted Sobolev spaces with Muckenhoupt weights. We also find some new formulas to represent functions by their derivatives.

1. Introduction and statement of results. The investigation of inequalities for intermediate derivatives originated in the 1914 papers of Landau and Hadamard (see [La], [H]). It was developed in 1939 by Kolmogorov ([K]), who first derived them in a multiplicative form (supremum norms, one variable). Nirenberg ([N1], [N2]) and independently Gagliardo ([G]) extended this result to the case of nonweighted L^p norms in several variables. The nonweighted case is well understood by now (see e.g. [A], [BIN], [I], [So], [Mi]). Weighted inequalities for functions defined on $\mathbb{R}^n$ in which the integrability exponent of the intermediate derivative is the same as that of the function and of the second order derivatives were proved by Gutiérrez and Wheeden [GW] (see also the references given there).

In this paper we derive pointwise multiplicative inequalities, where a pointwise value of the maximal function of an intermediate derivative is estimated in terms of the maximal functions of the function and of its relevant derivatives. Those estimates easily imply L^p multiplicative inequalities with Muckenhoupt weights. The idea of pointwise multiplicative inequalities is presented in Theorem 1. In the rest of the paper we extend it to the vector-valued case. Our estimates involve certain families of differential operators. We also obtain integral representations in terms of this class of operators.

As far as I know such representations are missing in the literature, although there are many ways to represent functions by their derivatives (see [BIN], Sec. 7, [S]).

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The inequalities presented in this paper correspond to those of Bojarski and Hajłasz (see [BH]); however, the approach is different and independent.

2. Preliminaries. Let $\varrho \in L^1_{\mathrm{loc}}(\Omega)$. We denote by $L^p_\varrho(\Omega)$ the space of functions f for which $\int_\Omega |f|^p \varrho\, dx < \infty$. If $\Omega = \mathbb{R}^n$ then we simply write L^p_ϱ. If $\varrho \equiv 1$ then ϱ will be omitted in notation.

Our basic tool is the following Sobolev integral representation formula ([Ma], Th. 1.1.10/1).

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, starshaped with respect to a ball B with $\overline{B} \subseteq \Omega$. Choose $\omega \in C_0^\infty(B)$ such that $\int_B \omega\, dx = 1$. Then for any $f \in W^{m,1}(\Omega)$,

$$(*)\qquad f(x) = \mathcal{P}^{m-1}_\omega f(x) + \sum_{|\alpha|=m} \int_\Omega K_\alpha(x,y) D^\alpha f(y)\, dy \quad \text{a.e. on } \Omega$$

where

$$\mathcal{P}^{m-1}_\omega f(x) = \int_\Omega \Big(\sum_{|\beta|<m} D^\beta_y \Big(\frac{(y-x)^\beta}{\beta!} \omega(y) \Big) \Big) f(y)\, dy$$

($\mathcal{P}^{m-1}_\omega f(x)$ is a polynomial of degree less than m) and

$$K_\alpha(x,y) = \frac{(-1)^m m}{\alpha!} \frac{(y-x)^\alpha}{|y-x|^n} \int_{|y-x|}^{\infty} \omega\Big(x + t \frac{y-x}{|y-x|} \Big) t^{n-1}\, dt\,.$$

Note that $K_\alpha \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{x = y\})$ and it depends only on ω.

The formula is obtained by multiplying Taylor's expansion of f by ω and integrating over Ω.

We will also use another representation formula holding for any $f \in C_0^\infty$:

$$(**)\qquad f = \sum_{|\alpha|=m} H_\alpha * D^\alpha f$$

where

$$H_\alpha(x) = \frac{(-1)^m m}{\alpha! n \omega_n} \frac{x^\alpha}{|x|^n}$$

and ω_n denotes the volume of the unit ball in $\mathbb{R}^n$ ([Ma], Th. 1.1.10/2).

If f is a locally integrable function then Mf denotes the Hardy–Littlewood maximal function of f (see e.g. [T]). Note that Mf is a function well defined at every point and may sometimes be infinite. For f vector-valued, the maximal function is defined as the sum of the maximal functions of the coordinates. It was proved by Muckenhoupt that if $\varrho \in A_p$ (i.e. ϱ is a Muckenhoupt weight), $p > 1$, then the maximal operator is bounded in L^p_ϱ (see [T] or [M]). If Ω is a bounded domain and $f \in L^1_{\mathrm{loc}}$ then $⨍_\Omega f\, dx$ is the average of f on Ω, that is $|\Omega|^{-1} \int_\Omega f\, dx$.

Let $\mathcal{P}_j = (\mathcal{P}_{j1}, \ldots, \mathcal{P}_{jk})$ $(j = 1, \ldots, N)$ be scalar differential operators acting on vector-valued functions $f = (f_1, \ldots, f_k)$ by

$$\mathcal{P}_j f = \sum_{i=1}^k \mathcal{P}_{ji} f_i\,.$$

Denote by $P_{ji}(x,\xi)$ the corresponding characteristic polynomials. We write simply $P_{ji}(\xi)$ if $\mathcal{P}_j$ has constant coefficients.

We say that the family $\{\mathcal{P}_j\}$ satisfies the *condition* (C) if

(i) all the $\mathcal{P}_j$ have constant coefficients,

(ii) $\mathcal{P}_j$ is homogeneous of order m_j for some natural number m_j (that is, all nonzero components $\mathcal{P}_{ji}$ are homogeneous of order m_j),

(iii) the matrix $\{P_{ji}(\xi)\}^{j=1,\ldots,N}_{i=1,\ldots,k}$ has rank k for any complex ξ_i $(i = 1, \ldots, n)$, $(\xi_1, \ldots, \xi_n) \neq (0, \ldots, 0)$.

In the scalar case (iii) simply means that the $P_j(\xi)$ with ξ_i complex have no common zeros except $\xi = (0, \ldots, 0)$. Note that the family $\{\mathcal{P}_\alpha\}_{|\alpha|=m}$, where $\mathcal{P}_\alpha f = D^\alpha f$ (acting on scalar functions), satisfies the condition (C).

We will be interested in the Sobolev type spaces of vector-valued functions

$$L^{\{\mathcal{P}_j, p_j\}}_\varrho(\Omega) = \{f = (f_1, \ldots, f_k) : f_i \in D', \mathcal{P}_j f \in L^{p_j}_\varrho(\Omega)\}$$

where the family $\{\mathcal{P}_j\}$ satisfies the condition (C) and $\varrho \in A_{\min_j\{p_j\}}$.

An easy calculation shows that if $p_1 \le p_2$ then $A_{p_1} \cap A_{p_2} = A_{p_1}$. Therefore $A_{\min_j\{p_j\}} = \bigcap_j A_{p_j}$.

If $\varrho \equiv 1$ then we omit ϱ in our notation. If $\Omega = \mathbb{R}^n$ then we write simply $L^{\{\mathcal{P}_j, p_j\}}_\varrho$.

We define the related subspace

$$\mathcal{R} = \{f = (f_1, \ldots, f_k) : f_i \in D',\ \mathcal{P}_j f = 0 \text{ for } j = 1, \ldots, N\}\,.$$

In the case of the space

$$L^{m,p}_\varrho(\Omega) = \{f = (f_1, \ldots, f_k) : f_i \in D',\ \nabla^m f \in L^p_\varrho(\Omega)\}\,,$$

$\mathcal{R}$ is the set of functions whose coordinates are polynomials of order less than m (here $\nabla^m f$ stands for the vector with components $D^\alpha f$, $|\alpha| = m$).

We denote by $L^{\{\mathcal{P}_j, p_j\}}_{\mathrm{loc}}(\Omega)$ the space of functions which are in $L^{\{\mathcal{P}_j, p_j\}}$ on every compact subset of Ω.

By C we denote a general constant. It can vary from line to line.

3. Pointwise estimates in terms of gradients. Let us start from

LEMMA 1. *Let $\phi \in L^1$ be a radial-decreasing function, and $f \in L^1_{\mathrm{loc}}$. Then for almost every x the convolution $\phi * f$ satisfies*
$$|\phi * f(x)| \le C\|\phi\|_{L^1} Mf(x)$$
with C independent of f.

Proof. This follows from easy calculations based on dividing $\mathbb{R}^n$ into disjoint subsets $\{2^k \le |x| < 2^{k+1}\}$, for $k \in \mathbb{Z}$. ∎

LEMMA 2. *If Ω is a bounded domain starshaped with respect to a ball B then for all $x \in \Omega$ the function $y \mapsto K_\alpha(x,y)$ is identically zero near the boundary of Ω.*

Proof. Let $x \in \Omega$. Denote by V_x the convex hull of x and B. It follows from the assumptions on Ω that $\overline{V}_x$ is a compact subset of Ω. Moreover, if $y \notin V_x$ then $x + t\frac{y-x}{|y-x|} \notin B$ for any $t \ge |y-x|$. The assertion follows easily from this observation. ∎

COROLLARY 1. *The representation formula $(*)$ holds for every $f \in W^{m,1}_{\mathrm{loc}}(\Omega)$.*

Proof. Let $f \in W^{m,1}_{\mathrm{loc}}(\Omega)$. Choose $\Omega' \subseteq \Omega$ such that $\overline{\Omega'} \subseteq \Omega$, $\overline{B} \subseteq \Omega'$ and Ω' is starshaped with respect to B. Since $f \in W^{m,1}(\Omega')$ we have
$$f(x) = \mathcal{P}^{m-1}_\omega f(x) + \sum_{|\alpha|=m} \int_{\Omega'} K_\alpha(x,y) D^\alpha f(y)\,dy$$
almost everywhere on Ω'. Since K_α is supported in $V_x \subseteq \Omega'$ we can write as well
$$f(x) = \mathcal{P}^{m-1}_\omega f(x) + \sum_{|\alpha|=m} \int_{\Omega} K_\alpha(x,y) D^\alpha f(y)\,dy$$
almost everywhere on Ω'. But Ω' was almost arbitrary, so the inequality is satisfied almost everywhere on Ω. ∎

LEMMA 3. *Let $\gamma > 0$. Then for any $y, z \in \mathbb{R}^n$ and any $r, \varepsilon > 0$ we have*

(i)
$$\int_{B(z,r)} \frac{1}{|x-y|^{n+\gamma}} \chi_{\{|x-y|\ge\varepsilon\}}\,dx \le C\bigg\{\frac{1}{|z-y|^{n+\gamma}}\chi_{\{|z-y|\ge 2\varepsilon/3\}} + \varepsilon^{-\gamma}\frac{1}{r^n}\chi_{\{|z-y|\le 2r\}}\bigg\},$$

(ii)
$$\int_{B(z,r)} \frac{1}{|x-y|^{n-\gamma}} \chi_{\{|x-y|\le\varepsilon\}}\,dx \le C\bigg\{\frac{1}{|z-y|^{n-\gamma}}\chi_{\{|z-y|\le 2\varepsilon\}} + \varepsilon^{\gamma}\frac{1}{r^n}\chi_{\{|z-y|\le 2r\}}\bigg\}.$$

Proof. By a simple change of variable we may assume that $y = 0$. Let us start from inequality (i). Denote by I its left hand side. We distinguish two cases: $r < \frac{1}{2}|z|$ and $r \ge \frac{1}{2}|z|$. In the first case we have $\frac{1}{2}|z| < |x| < \frac{3}{2}|z|$ and $I \le C/|z|^{n+\gamma}$. But if $|z| < \frac{2}{3}\varepsilon$ then $B(z,r) \cap \{|x| \ge \varepsilon\} = \emptyset$. Therefore
$$I \le \frac{C}{|z|^{n+\gamma}} \chi_{\{|z|\ge 2\varepsilon/3\}}.$$
If $r \ge \frac{1}{2}|z|$ then
$$I \le \frac{1}{r^n}\int_{\mathbb{R}^n} \frac{1}{|x|^{n+\gamma}}\chi_{\{|x|\ge\varepsilon\}}\,dx \le \frac{C}{r^n}\varepsilon^{-\gamma}.$$
Thus in any case,
$$I \le C\bigg(\frac{1}{|z|^{n+\gamma}}\chi_{\{|z|\ge 2\varepsilon/3\}} + \varepsilon^{-\gamma}\frac{1}{r^n}\chi_{\{|z|\le 2r\}}\bigg).$$

Now let us prove (ii). Denoting by I the left hand side of the inequality we find that if $r < \frac{1}{2}|z|$ then
$$I \le \frac{C}{|z|^{n-\gamma}}\chi_{\{|z|\le 2\varepsilon\}}.$$
On the other hand, if $r \ge \frac{1}{2}|z|$ then
$$I \le \frac{1}{r^n}\int_{B(0,\varepsilon)} \frac{1}{|x|^{n-\gamma}}\,dx \le \frac{C}{r^n}\varepsilon^{\gamma}.$$
Thus in any case I can be estimated by
$$C\bigg(\frac{1}{|z|^{n-\gamma}}\chi_{\{|z|\le 2\varepsilon\}} + \varepsilon^{\gamma}\frac{1}{r^n}\chi_{\{|z|\le 2r\}}\bigg). \ ∎$$

Similar estimates are obtained in Lemma 2 of [BH].

If h is a function defined on Ω then $h\chi_\Omega$ stands for h extended by zero outside Ω.

THEOREM 1. *Let Ω be a bounded domain, starshaped with respect to a ball B. Choose $\omega \in C_0^\infty(B)$ with $\int \omega\,dx = 1$. Then there exists a constant C such that for every $z \in \mathbb{R}^n$, every $f \in W^{m,1}_{\mathrm{loc}}(\Omega)$ and every multiindex β with $|\beta| = k$, $0 < k < m$,*
$$M(D^\beta(f - \mathcal{P}^{m-1}_\omega f)\chi_\Omega)(z) \le C(M((f-P)\chi_\Omega)(z))^{1-k/m}\Big(\sum_{|\alpha|=m} M(D^\alpha f\chi_\Omega)(z)\Big)^{k/m}$$
where P is any polynomial of degree less than m.

Proof. Let $f \in W^{m,1}_{\mathrm{loc}}(\Omega)$, and let β be any multiindex of order k. From the Sobolev representation formula $(*)$ and Corollary 1,

$$|D^\beta(f-\mathcal{P}^{m-1}_\omega f)(x)| = \Big|\sum_{|\alpha|=m}\int_\Omega D^\beta_x K_\alpha(x,y)D^\alpha f(y)\,dy\Big|$$

for almost every $x \in \Omega$. Let $\phi \in C_0^\infty$ be a radial function such that $\phi \equiv 0$ for $|x| \le 1/2$ and $\phi \equiv 1$ for $|x| > 1$. Define $\phi_\varepsilon(x) = \phi(x/\varepsilon)$. We have

$$\begin{aligned} D^\beta_x K_\alpha(x,y) &= \phi_\varepsilon(x-y)D^\beta_x K_\alpha(x,y) + (1-\phi_\varepsilon(x-y))D^\beta_x K_\alpha(x,y)\\ &= A_\varepsilon(x,y) + B_\varepsilon(x,y)\,. \end{aligned}$$

Since $A_\varepsilon(x,\cdot) \in C_0^\infty(\Omega)$ (Lemma 2) and $D^\alpha f$ are distributional derivatives we have

$$\int_\Omega A_\varepsilon(x,y)D^\alpha f(y)\,dy = (-1)^{|\alpha|}\int_\Omega D^\alpha_y A_\varepsilon(x,y)(f-P)(y)\,dy$$

where P is any polynomial of degree less than m. An easy computation shows that

$$|D^\alpha_y A_\varepsilon(x,y)| \le \frac{C}{|x-y|^{n+k}}\chi_{\{|x-y|\ge\varepsilon/2\}}\,,$$

$$|B_\varepsilon(x,y)| \le \frac{C}{|x-y|^{n+k-m}}\chi_{\{|x-y|<\varepsilon\}}\,.$$

Thus, for almost every $x \in \mathbb{R}^n$ we have the estimate

$$\begin{aligned} &|D^\beta(f-\mathcal{P}^{m-1}_\omega f)\chi_\Omega(x)|\\ &\quad\le C\Big(\int_\Omega \frac{1}{|x-y|^{n+k}}\chi_{\{|x-y|\ge\varepsilon/2\}}(f-P)(y)\,dy\\ &\qquad + \sum_{|\alpha|=m}\int_\Omega \frac{1}{|x-y|^{n+k-m}}\chi_{\{|x-y|<\varepsilon\}}D^\alpha f(y)\,dy\Big)\,. \end{aligned}$$

Now let us estimate the average of $|D^\beta(f-\mathcal{P}^{m-1}_\omega f)\chi_\Omega(x)|$ over a ball $B(z,r)$. Integrating the last inequality over the ball, and applying the Fubini theorem and Lemma 3 we obtain

$$\begin{aligned} &\fint_{B(z,r)} |D^\beta(f-\mathcal{P}^{m-1}_\omega f)\chi_\Omega(x)|\,dx\\ &\quad\le C\Big[\int_\Omega\Big(\fint_{B(z,r)}\frac{1}{|x-y|^{n+k}}\chi_{\{|x-y|\ge\varepsilon/2\}}\,dx\Big)(f-P)(y)\,dy\\ &\qquad + \sum_{|\alpha|=m}\int_\Omega\Big(\fint_{B(z,r)}\frac{1}{|x-y|^{n+k-m}}\chi_{\{|x-y|<\varepsilon\}}\,dx\Big)D^\alpha f(y)\,dy\Big] \end{aligned}$$

$$\begin{aligned} &\le C\Big[\int_\Omega \frac{1}{|y-z|^{n+k}}\chi_{\{|y-z|\ge\varepsilon/3\}}(f-P)(y)\,dy\\ &\quad + \frac{\varepsilon^{-k}}{r^n}\int_\Omega \chi_{\{|y-z|\le 2r\}}(f-P)(y)\,dy\\ &\quad + \sum_{|\alpha|=m}\int_\Omega \frac{1}{|y-z|^{n+k-m}}\chi_{\{|y-z|\le 2\varepsilon\}}D^\alpha f(y)\,dy\\ &\quad + \frac{\varepsilon^{m-k}}{r^n}\int_\Omega \chi_{\{|y-z|\le 2r\}}D^\alpha f(y)\,dy\Big]\,. \end{aligned}$$

Obviously

$$\frac{\varepsilon^{-k}}{r^n}\int_\Omega \chi_{\{|y-z|\le 2r\}}(f-P)(y)\,dy \le C\varepsilon^{-k}M((f-P)\chi_\Omega)(z)\,,$$

$$\frac{\varepsilon^{m-k}}{r^n}\int_\Omega \chi_{\{|y-z|\le 2r\}}D^\alpha f(y)\,dy \le C\varepsilon^{m-k}M(D^\alpha f\chi_\Omega)(z)\,.$$

To the remaining parts of the sum we may apply Lemma 1. Namely, let

$$\phi_1(y) = \min\Big\{\frac{1}{|y|^{n+k}},\frac{1}{\varepsilon^{n+k}}\Big\},\qquad \phi_2(y) = \frac{1}{|y|^{n-m+k}}\chi_{\{|y|\le\varepsilon\}}\,.$$

Obviously both functions satisfy the assumptions of Lemma 1 and $\int\phi_1\,dy \le C\varepsilon^{-k}$, $\int\phi_2\,dy \le C\varepsilon^{m-k}$. Moreover, since

$$\frac{1}{|y|^{n+k}}\chi_{\{|y|\ge\varepsilon/3\}} \le C\phi_1(y)\quad\text{and}\quad \frac{1}{|y|^{n+k-m}}\chi_{\{|y|\le 2\varepsilon\}} \le C\phi_2(y)$$

it follows that

$$\begin{aligned} &\fint_{B(z,r)} |D^\beta(f-\mathcal{P}^{m-1}_\omega f)\chi_\Omega(x)|\,dx\\ &\qquad\le C\Big(\varepsilon^{-k}M((f-P)\chi_\Omega)(z) + \varepsilon^{m-k}\sum_{|\alpha|=m}M(D^\alpha f\chi_\Omega)(z)\Big)\,. \end{aligned}$$

The assertion follows immediately from the last inequality by choosing appropriate ε and r. ∎

Remarks. 1) Since K_α depends only on the choice of ω we see that the constant C depends only on ω.

2) The inequality of Theorem 1 may be written in the form

$$\begin{aligned} &M(\nabla^k(f-\mathcal{P}^{m-1}_\omega f)\chi_\Omega)(z)\\ &\qquad\le C(M((f-P)\chi_\Omega)(z))^{1-k/m}(M(\nabla^m f\chi_\Omega)(z))^{k/m} \end{aligned}$$

where $\nabla^l f\chi_\Omega$ stands for the vector $(D^\alpha f)_{|\alpha|=l}$ extended by zero outside Ω. We will sometimes also use this notation.

It is known (see e.g. [Ma], Lemma 1.1.9/1) that bounded domains with the cone property are finite sums of starshaped domains ("starshaped" means starshaped with respect to some ball). This together with Theorem 1 implies

THEOREM 2. *Let Ω be a bounded domain with the cone property. Then there exists a constant C such that for every $z \in \mathbb{R}^n$, every $f \in W^{m,1}_{\rm loc}(\Omega)$ and every $0 < k < m$ we have*

$$M(\nabla^k f\chi_\Omega)(z) \le C[\|f\|_{L^1(\Omega')} + (M((f-P)\chi_\Omega)(z))^{1-k/m}(M(\nabla^m f\chi_\Omega)(z))^{k/m}]$$

where P is any polynomial of degree less than m and Ω' is some set such that $\overline{\Omega'} \subseteq \Omega$.

THEOREM 3. *Let $f \in W^{m,1}_{\rm loc}$, $a \in \mathbb{R}^n$, and $r > 0$. Then for any $z \in \mathbb{R}^n$ and any $0 < k < m$,*

$$M(\nabla^k f)(z) \le C\Big[\sup_{R>0}\frac{1}{R^k}\fint_{B(aR,rR)}|f|\,dx + (M(f-P)(z))^{1-k/m}(M(\nabla^m f)(z))^{k/m}\Big]$$

where P is any polynomial of degree less than m.

If additionally $\lim_{R\to\infty} R^{-k}\fint_{B(aR,rR)}|f|\,dx = 0$ then

$$M(\nabla^k f)(z) \le C(M(f-P)(z))^{1-k/m}(M(\nabla^m f)(z))^{k/m}.$$

The constant C does not depend on f.

Proof. By a simple change of variables we may assume that $B(a,r) \subseteq B(1)$ (where $B(1)$ is the unit ball with center at zero). Since $B(1)$ is starshaped with respect to the ball $B(a,r)$ it follows from Theorem 1 that

$$M(\nabla^k f\chi_{B(1)})(z) \le C[\|f\|_{L^1(B(a,r))} + (M((f-P)\chi_{B(1)})(z))^{1-k/m}(M(\nabla^m f\chi_{B(1)})(z))^{k/m}].$$

Rescaling, that is, substituting for f the function $f_R(x) = f(Rx)$ we obtain

$$M(\nabla^k f\chi_{B(R)})(z) \le C\Big[\frac{1}{R^k}\fint_{B(aR,rR)}|f|\,dx + (M((f-P)\chi_{B(R)})(z))^{1-k/m}(M(\nabla^m f\chi_{B(R)})(z))^{k/m}\Big]$$

with the same constant.

That implies the assertion since if $h \in L^1_{\rm loc}$ and $h \ge 0$ we have

$$M(h\chi_{B(R)})(z) \xrightarrow{R\to\infty} Mh(z) \quad \text{for every } z \in \mathbb{R}^n. \ \blacksquare$$

Remarks. 1) The assumption $\lim_{R\to\infty} R^{-k}\fint_{B(aR,rR)}|f|\,dx = 0$ of Theorem 3 is satisfied for every $f \in W^{m,p}$ $(1 \le p \le \infty)$.

2) The same arguments as in the proof of Theorem 3 show that inequalities of the form

$$M(\nabla^k f\chi_\Omega)(z) \le C(M((f-P)\chi_\Omega)(z))^{1-k/m}(M(\nabla^m f\chi_\Omega)(z))^{k/m}$$

hold on every infinite cone, provided $\lim_{R\to\infty} R^{-k}\fint_{\Omega\cap B(R)}|f|\,dx = 0$.

3) If f is integrable with any power then $M(f-P)$ is finite only if $P = $ constant.

4. Integral representations

LEMMA 4. *Let $\{P_j(\xi)\}_{j=1,\dots,N}$ be a family of polynomials of n complex variables $\xi = (\xi_1,\dots,\xi_n)$ which have no common zeros except $\xi = (0,\dots,0)$. Then there exists a positive integer l such that for any multiindex α of order l we can choose polynomials $a_{\alpha,j}$ satisfying*

$$\xi^\alpha = \sum_{j=1}^N a_{\alpha,j}(\xi)P_j(\xi).$$

Proof. It follows from Hilbert's Nullstellensatz (see e.g. [L], Chapt. 10.2) that there exist numbers $k_1,\dots,k_n$ and polynomials $a_{1,j},\dots,a_{n,j}$ which satisfy $\xi_i^{k_i} = \sum_{j=1}^N a_{i,j}(\xi)P_j(\xi)$. Multiplying both sides by a suitable power of ξ_i we may assume that $k_1 = \dots = k_n = k$. Set $l = nk$. If α is a multiindex of order l, then there exists at least one coordinate α_i with $\alpha_i \ge k$. That easily implies the assertion. ■

THEOREM 4. *Let Ω be a bounded, starshaped domain, and $\{\mathcal{P}_j\}_{j=1,\dots,N}$ be a family of differential operators acting on vector-valued functions $f = (f_1,\dots,f_k)$ and satisfying the condition* (C) *for some positive integers $m_1,\dots,m_N$. Then there exist vector-valued functions $K_j(x,y)$ $(j = 1,\dots,N)$, $K_j = (K_{j1},\dots,K_{jk})$, satisfying the following conditions:*

(i) *$K_{ji} \in C^\infty(\mathbb{R}^n\times\mathbb{R}^n \setminus \{x = y\})$,*

(ii) *$K_{ji}(x,\cdot) \equiv 0$ near the boundary of Ω, for $x \in \Omega$,*

(iii) *$|D_x^\alpha D_y^\beta K_{ji}(x,y)| \le C/|x-y|^{n-m_j+|\alpha|+|\beta|}$ for any $x,y \in \Omega$,*

(iv) *there exists a positive integer l no smaller than $\max_j m_j$ and scalar differential operators $\mathcal{P}_{j,i,\alpha}$ $(j = 1,\dots,N,\ i = 1,\dots,k,\ |\alpha| = l)$, homogeneous of order $l - m_j$, with constant coefficients such that*

$$K_{ji}(x,y) = \sum_{|\alpha|=l}(\mathcal{P}_{j,i,\alpha})_y K_\alpha(x,y)$$

where $(\mathcal{P}_{j,i,\alpha})_y$ indicates that the operator $\mathcal{P}_{j,i,\alpha}$ acts on functions of y,

(v) *if l is as in* (iv) *then for any $f \in C^\infty(\Omega)$,*

$$f_i(x) = \mathcal{P}_\omega^{l-1} f_i(x) + \sum_{j=1}^{N} \int_\Omega K_{ji}(x,y)\mathcal{P}_j f(y)\,dy$$

where $\mathcal{P}_\omega^{l-1} f_i$ is as in (*).

Proof. We will consider separately the cases 1) $k = 1$ and 2) $k > 1$. Let us start from Case 1. Let $f \in C^\infty(\Omega)$.

It follows from Lemma 4 that there exists an integer l and polynomials $a_{\alpha,j}(\xi)$ homogeneous of order $l - m_j$ such that for any multiindex α of order l,

$$\xi^\alpha = \sum_{j=1}^{N} a_{\alpha,j}(\xi) P_j(\xi).$$

Denote by $a_{\alpha,j}$ the differential operator corresponding to $a_{\alpha,j}(\xi)$. Representing D^α in terms of $\mathcal{P}_j$ (where $|\alpha| = l$) and substituting it into (*) (see Corollary 1) we obtain

$$f(x) = \mathcal{P}_\omega^{l-1} f(x) + \sum_{|\alpha|=l} \sum_{j=1}^{N} \int_\Omega K_\alpha(x,y) a_{\alpha,j} \mathcal{P}_j f(y)\,dy$$

for any $x \in \Omega$.

But since $K_\alpha(x,\cdot)$ vanishes near $\partial\Omega$ (Lemma 2) and $|D_y^\gamma K_\alpha(x,y)| \le C/|x-y|^{n-l+|\gamma|}$ it follows that in every integral we can integrate by parts (substitute first $K_\alpha^\varepsilon(x,y) = \phi_\varepsilon(x-y)K_\alpha(x,y)$ for K_α, where ϕ_ε is as in the proof of Theorem 1, and let $\varepsilon \to 0$). That gives

$$f(x) = \mathcal{P}_\omega^{l-1} f(x) + \sum_{j=1}^{N} \int_\Omega \Big(\sum_{|\alpha|=l} (-1)^{l-m_j} (a_{\alpha,j})_y K_\alpha(x,y) \Big) \mathcal{P}_j f(y)\,dy.$$

Set

$$K_j(x,y) = \sum_{|\alpha|=l} (-1)^{l-m_j} (a_{\alpha,j})_y K_\alpha(x,y).$$

It is easy to see that the K_j have the required properties.

Now we will prove the second case. Let $f = (f_1,\ldots,f_k)$, $f_i \in C^\infty(\Omega)$. We will use the method described by Smith ([S], Th. 1.2). Let $J = (j_1,\ldots,j_k)$, $1 \le j_r \le N$, and $D_J(\xi) = \{d_{ri}(\xi)\}_{r,i=1,\ldots,k}$ where $d_{ri}(\xi) = P_{j_r,i}(\xi)$. Denote by $d_J(\xi)$ the determinant of the matrix $D_J(\xi)$. Let $d_J^{ri}(\xi)$ be the algebraic complement of $d_{ri}(\xi)$ in $D_J(\xi)$. Since the $d_J(\xi)$ have no common nontrivial zeros and are homogeneous, by the previous case there exist kernels K_J which satisfy

$$f_i(x) = \mathcal{P}_\omega^{l-1} f_i(x) + \sum_J \int_\Omega K_J(x,y) d_J f_i(y)\,dy$$

where d_J is the differential operator with characteristic polynomial $d_J(\xi)$. Define $\mathcal{L} = \sum_J \int_\Omega K_J(x,y) d_J f_i(y)\,dy = \sum_{m,J} \int_\Omega K_J(x,y)\delta_{im} d_J f_i(y)\,dy$, where δ_{im} is the Kronecker delta. Obviously

$$\delta_{im} d_J(\xi) = \sum_{r=1}^{k} P_{j_r,m}(\xi) d_J^{ri}(\xi).$$

That gives

$$\mathcal{L} = \sum_{m,J,r} \int_\Omega K_J(x,y) d_J^{ri} \mathcal{P}_{j_r,m} f_m(y)\,dy$$

where d_J^{ri} is the differential operator corresponding to the polynomial $d_J^{ri}(\xi)$. By the same reasons as in the previous case we can integrate by parts in every integral to obtain

$$\mathcal{L} = \sum_{m,J,r} \int_\Omega (-1)^{s(d_J^{ri})} (d_J^{ri})_y K_J(x,y) \mathcal{P}_{j_r,m} f_m(y)\,dy$$

where $s(P)$ denotes the degree of the polynomial $P(\xi)$. We group expressions with $j_r = j$:

$$\begin{aligned}\mathcal{L} &= \sum_{j=1}^{N} \sum_{r,J:j_r=j} \sum_m \int_\Omega (-1)^{s(d_J^{ri})} (d_J^{ri})_y K_J(x,y) \mathcal{P}_{jm} f_m(y)\,dy \\ &= \sum_{j=1}^{N} \int_\Omega \Big(\sum_{r,J:j_r=j} (-1)^{s(d_J^{ri})} (d_J^{ri})_y K_J(x,y) \Big) \mathcal{P}_j f(y)\,dy.\end{aligned}$$

Set

$$K_{ji}(x,y) = \sum_{r,J:j_r=j} (-1)^{s(d_J^{ri})} (d_J^{ri})_y K_J(x,y).$$

Obviously the functions $K_j = (K_{j1},\ldots,K_{jk})$ have property (v).

Let us look at the construction more closely. For some $l \in \mathbb{N}$ we have $\xi^\alpha = \sum_J a_{\alpha,J} d_J$ for any multiindex α of length l. From Case 1 we have $K_J = \sum_{|\alpha|=l} (-1)^{s(a_{\alpha,J})} (a_{\alpha,J})_y K_\alpha$. Thus, since $s(d_J^{ri} a_{\alpha,J}) = l - m_j$ we obtain

$$K_{ji} = (-1)^{l-m_j} \sum_{|\alpha|=l} \sum_{r,J:j_r=j} (d_J^{ri} a_{\alpha,J})_y K_\alpha.$$

That easily implies the assertion. ∎

Almost the same arguments as in Theorem 4 but applied to the representation formula (**) give a result similar to [S], Th. 4.1.

THEOREM 5. *Let $\{\mathcal{P}_j\}_{j=1,\dots,N}$ be a family of differential operators acting on vector-valued functions $f=(f_1,\dots,f_k)$ and satisfying the condition* (C) *for some positive integers $m_1,\dots,m_N$. Then there exist vector-valued functions $H_j(x), H_j=(H_{j1},\dots,H_{jk})$, satisfying the following conditions:*

(i) *$H_{ji}(x)$ is smooth except at $x=0$,*
(ii) *$H_{ji}(x)$ is homogeneous of order $-(n-m_j)$,*
(iii) *$|D^\beta H_{ji}(x)|\le C/|x|^{n-m_j+|\beta|}$ for all $x\in\mathbb{R}^n$,*
(iv) *for any $f=(f_1,\dots,f_k)$, $f_i\in C_0^\infty$, and any $x\in\mathbb{R}^n$ we have*

$$f_i(x)=\sum_{j=1}^N\int\limits_{\mathbb{R}^n}H_{ji}(x-y)\mathcal{P}_jf(y)\,dy\,.$$

The condition (iv) of Theorem 5 may be written as $\delta_i=\sum_{j=1}^N\mathcal{P}_jH_{ji}$ ($i=1,\dots,k$), where $\delta_i(f_1,\dots,f_k)=f_i(0)$. Using arguments similar to [Ma], Th. 1.1.2/1, but with the fundamental solution of the polyharmonic equation replaced by the family $\{H_{ji}\}$, we derive

COROLLARY 2. *For any domain Ω the space $L^{\{\mathcal{P}_j,1\}}(\Omega)$ consists of locally integrable functions.*

Next, standard methods, similar to [Ma], Th. 1.1.5/1, give

COROLLARY 3. *Let Ω be any domain, and Ω' any compact set contained in Ω. Then for any $f\in L^{\{\mathcal{P}_j,1\}}(\Omega)$ there exists a sequence $f_n\in L^{\{\mathcal{P}_j,1\}}(\Omega)\cap C^\infty(\Omega)$ such that*

$$f_n\to f \text{ in } L^1(\Omega') \quad\text{and}\quad \mathcal{P}_jf_n\to\mathcal{P}_jf \text{ in } L^1(\Omega)\,.$$

Remark. If Ω is a bounded domain with continuous boundary then the functions f_n can be taken smooth in a certain neighbourhood of Ω (see [Ma], Th. 1.1.6/2).

As a consequence of Corollary 3 and methods from Corollary 1 we obtain

THEOREM 6. *Let Ω be a bounded, starshaped domain, and $\{\mathcal{P}_j\}_{j=1,\dots,N}$ a family of operators satisfying the condition* (C). *Then the representation formula* (v) *of Theorem 4 holds almost everywhere for every $f\in L^{\{\mathcal{P}_j,1\}}_{\rm loc}(\Omega)$.*

Remark. It follows immediately from Theorem 6 that if Ω is a bounded, starshaped domain and $\{\mathcal{P}_j\}$ satisfies the condition (C) then the space $\mathcal{R}$ corresponding to $\{\mathcal{P}_j\}$ consists of polynomials.

5. Pointwise estimates in terms of nongradient operators. Having the integral representations from Theorem 4 it is a matter of routine to extend the results of Theorem 2 and 3 to a more general form.

THEOREM 7. *Let Ω be a bounded domain with the cone property, and $\{\mathcal{P}_j\}_{j=1,\dots,N}$ a family of operators satisfying the condition* (C). *Then there exists a constant C such that for every $f\in L^{\{\mathcal{P}_j,1\}}_{\rm loc}(\Omega)$, $S\in\mathcal{R}$, and $z\in\mathbb{R}^n$,*

$$M(\nabla^kf_i\chi_\Omega)(z)$$
$$\le C\Big[\|f_i\|_{L^1(\Omega')}+\sum_{j=1}^N(M((f-S)\chi_\Omega)(z))^{1-k/m_j}(M(\mathcal{P}_jf\chi_\Omega)(z))^{k/m_j}\Big]$$

where $0<k<\min_j\{m_j\}$ and Ω' is some set such that $\overline{\Omega'}\subset\Omega$.

THEOREM 8. *If $a\in\mathbb{R}^n$ and $r>0$ then there exists a constant C such that for every $f\in L^{\{\mathcal{P}_j,1\}}_{\rm loc}$, $S\in\mathcal{R}$, and $z\in\mathbb{R}^n$,*

$$M(\nabla^kf_i)(z)\le C\Big[\sup_{R>0}\frac{1}{R^k}\int\limits_{B(aR,rR)}|f_i|\,dx$$
$$+\sum_{j=1}^N(M(f-S)(z))^{1-k/m_j}(M(\mathcal{P}_jf)(z))^{k/m_j}\Big]$$

where $0<k<\min_j\{m_j\}$.

If additionally $\lim_{R\to\infty}R^{-k}\int_{B(aR,rR)}|f_i|\,dx=0$ then

$$M(\nabla^kf_i)(z)\le C\Big[\sum_{j=1}^N(M(f-S)(z))^{1-k/m_j}(M(\mathcal{P}_jf)(z))^{k/m_j}\Big]\,.$$

Remark. Inequalities of the same type as in Theorem 8 hold in an infinite cone (see Remark after Theorem 3).

6. Nirenberg type estimates. The inclusion $L^{\{\mathcal{P}_j,p\}}_\varrho(\Omega)\subseteq L^{\{\mathcal{P}_j,1\}}(\Omega)$ for bounded domains, Theorems 1, 2, 3, 7 and 8, Hölder's inequality and Muckenhoupt's theorem immediately imply

THEOREM 9. *Let Ω be a bounded domain, starshaped with respect to a ball B. Choose $\omega\in C_0^\infty(B)$ with $\int\omega\,dx=1$. Assume that $1<r,p<\infty$ and $\varrho\in A_{\min\{p,r\}}$. Then there exists a constant C such that for every $z\in\mathbb{R}^n$, every $f\in L^{m,p}_\varrho(\Omega)\cap L^r_\varrho(\Omega)$ and every multiindex β with $|\beta|=k$, $0<k<m$,*

$$\|D^\beta(f-\mathcal{P}^{m-1}_\omega f)\|_{L^q_\varrho(\Omega)}\le C(\|f-P\|_{L^r_\varrho(\Omega)})^{1-k/m}\Big(\sum_{|\alpha|=m}\|D^\alpha f\|_{L^p_\varrho(\Omega)}\Big)^{k/m}$$

where

$$\frac1q=\frac1p\frac km+\Big(1-\frac km\Big)\frac1r$$

and P is any polynomial of degree less than m.

THEOREM 10. *Let Ω be a bounded domain with the cone property, let $\{\mathcal{P}_j\}_{j=1,\dots,N}$ satisfy the condition* (C), $1<r,\ p_j<\infty$, $\varrho\in A_{\min\{p_1,\dots,p_N,r\}}$, *and for any* $j,\ l$,

$$\frac{1}{m_j}\left(\frac{1}{p_j}-\frac{1}{r}\right)=\frac{1}{m_l}\left(\frac{1}{p_l}-\frac{1}{r}\right).$$

Then there exists a constant C *such that for every* $f\in L^r_\varrho(\Omega)\cap L^{\{\mathcal{P}_j,p_j\}}_\varrho(\Omega)$, $S\in\mathcal{R}$ *and any multiindex* β *of order* k *with* $0<k<\min_j m_j$,

$$\|D^\beta f_i\|_{L^q_\varrho(\Omega)} \le C\Big[\|f_i\|_{L^r_\varrho(\Omega)}+\sum_{j=1}^N(\|f-S\|_{L^r_\varrho(\Omega)})^{1-k/m_j}(\|\mathcal{P}_jf\|_{L^{p_j}_\varrho(\Omega)})^{k/m_j}\Big]$$

where

$$\frac{1}{q}=\frac{k}{m_j}\frac{1}{p_j}+\left(1-\frac{k}{m_j}\right)\frac{1}{r}.$$

THEOREM 11. *Let* $\{\mathcal{P}_j\}$ *satisfy condition* (C), $1<r,p_j<\infty$, $\varrho\in A_{\min\{p_1,\dots,p_N,r\}}$ *and for any* $j,\ l$,

$$\frac{1}{m_j}\left(\frac{1}{p_j}-\frac{1}{r}\right)=\frac{1}{m_l}\left(\frac{1}{p_l}-\frac{1}{r}\right).$$

Then there exists a constant C *such that for every* $f\in L^r_\varrho\cap L^{\{\mathcal{P}_j,p_j\}}_\varrho$, $S\in\mathcal{R}$ *and any multiindex* β *of order* k *with* $0<k<\min_j m_j$,

$$\|D^\beta f_i\|_{L^q_\varrho}\le C\Big[\sum_{j=1}^N(\|f-S\|_{L^r_\varrho})^{1-k/m_j}(\|\mathcal{P}_jf\|_{L^{p_j}_\varrho})^{k/m_j}\Big]$$

where

$$\frac{1}{q}=\frac{k}{m_j}\frac{1}{p_j}+\left(1-\frac{k}{m_j}\right)\frac{1}{r}.$$

Proof. We use Theorem 8. The only nontrivial thing is to show that if $f\in L^r_\varrho$ then

$$\lim_{R\to\infty}\frac{1}{R^k}\fint_{B(aR,rR)}|f|\,dx=0\quad\text{for some } a\in\mathbb{R}^n, r>0.$$

That is a consequence of the following property of A_r weights:

$$\fint_Q|f|\,dx\le C\left(\frac{1}{\int_Q\varrho\,dx}\int_Q|f|^r\varrho\,dx\right)^{1/r}$$

where Q is any cube in $\mathbb{R}^n$ (C does not depend on f and Q, see [T], Th. IX.2.1). ∎

Remarks. 1) The inequality $|f(x)|\le Mf(x)\le\|f\|_{L^\infty}$ implies that we have the corresponding supremum norm estimates.

2) If we want to prove the inequalities of Theorems 1, 2, 3, 7 and 8 with the maximal function of an intermediate derivative replaced by the intermediate derivative itself (this is possible since $|f(x)|\le Mf(x)$), then the proofs much simplify. In particular, we do not need to apply Lemma 3.

3) Similar methods imply the analogous estimates in L^1-norm. We will show that for $\Omega=\mathbb{R}^n$. Applying the methods from Theorem 1 to the representation formulae of Theorem 5 we deduce that if $f\in L^{\{\mathcal{P}_j,1\}}$ we have

$$|\nabla^k f(x)|\le C\Big[\int_{|x-y|>\varepsilon}\frac{1}{|x-y|^{n+k}}|f(y)|\,dy+\sum_j\int_{|x-y|<\varepsilon}\frac{1}{|x-y|^{n-m_j+k}}|\mathcal{P}_jf(y)|\,dy\Big]$$

almost everywhere. Now it is enough to integrate over $\mathbb{R}^n$.

4) It is known that there are estimates of the form

$$\|D^\gamma f\|_{L^q}\le C(\|D^{\alpha_1}f\|_{L^{p_1}})^{\mu_1}\dots(\|D^{\alpha_N}f\|_{L^{p_N}})^{\mu_N}$$

where $\gamma=\sum_{j=1}^N\alpha_j\mu_j$, $0<\mu_j<1$, $\sum_j\mu_j=1$, $1/q=\sum_{j=1}^N(1/p_j)\mu_j$ and $1<p_j<\infty$ (that is, γ is an intermediate derivative of $\{\alpha_j\}$, see [BIN], Th. 15.7). We cannot expect inequalities of the form

$$M(D^\gamma f)(z)\le C(M(D^{\alpha_1}f)(z))^{\mu_1}\dots(M(D^{\alpha_N}f)(z))^{\mu_N}$$

with γ,α_j and μ_j as above. If such inequalities existed they would imply the corresponding inequalities in supremum norms. But Boman [Bo] found necessary and sufficient conditions for supremum norm estimates to hold. They are not always satisfied if γ is any intermediate derivative of $\{\alpha_j\}$.

5) Obviously, a purely pointwise inequality, that is, the inequality $|\nabla^k f(x)|\le C|f(x)|^{1-k/m}|\nabla^m f(x)|^{k/m}$, does not hold. To see that, it is enough to take $f(x)=\sin x$ and $m=2$, $k=1$.

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Restrictions from $\mathbb{R}^n$ to $\mathbb{Z}^n$ of weak type $(1,1)$ multipliers

by

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Abstract. Suppose that $\{\phi_j\}_{j=1}^{\infty}$ is a sequence of weak type $(1,1)$ multipliers for $L^1(\mathbb{R}^n)$ such that for each j, ϕ_j is continuous at every point of $\mathbb{Z}^n$. We show that the restrictions $\phi_j|\mathbb{Z}^n$, $j \ge 1$, are weak type $(1,1)$ multipliers for $L^1(\mathbb{T}^n)$. Moreover, the weak type $(1,1)$ norm of the maximal operator defined by the sequence $\{\phi_j\}_{j=1}^{\infty}$ controls that of the maximal operator defined by the sequence $\{\phi_j|\mathbb{Z}^n\}_{j=1}^{\infty}$. This de Leeuw type restriction theorem for maximal estimates of weak type $(1,1)$ answers in the affirmative a question about single multipliers posed by A. Pełczyński. Our central result, from which this restriction theorem follows by suitable regularization arguments, is another maximal theorem regarding convolution of a function in $L^1(\mathbb{R}^n)$ with weak type $(1,1)$ multipliers.

1. Introduction. Let n be a positive integer, and let G be either the additive group $\mathbb{R}^n$ or the multiplicative group $\mathbb{T}^n$. Denote by Γ the dual group of G. For $\phi \in L^\infty(\Gamma)$, we symbolize by T_ϕ the corresponding multiplier transform on $L^2(G)$: $T_\phi f = (\phi\widehat{f})^\vee$. The function ϕ is said to be a *multiplier of weak type* $(1,1)$ (in symbols $\phi \in M_1^{(w)}(\Gamma)$) provided that T_ϕ is of weak type $(1,1)$ on $L^1(G) \cap L^2(G)$. Given a sequence $\{\phi_j\}_{j\ge1} \subseteq M_1^{(w)}(\Gamma)$, we denote by $N_1^{(w)}(\{\phi_j\}_{j\ge1})$ the weak type $(1,1)$ norm of the maximal operator on $L^1(G) \cap L^2(G)$ defined by $\{T_{\phi_j}\}_{j\ge1}$.

In [4, Problem 5, p. 412], A. Pełczyński posed the following question, which seeks an analogue for weak type $(1,1)$ multipliers of a de Leeuw restriction theorem for strong type multipliers [3, Proposition 3.3]:

if $\phi \in M_1^{(w)}(\mathbb{R}^n)$, and ϕ is continuous at each point of $\mathbb{Z}^n$, is it necessarily true that the restriction $\phi|\mathbb{Z}^n$ belongs to $M_1^{(w)}(\mathbb{Z}^n)$?

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