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Triebel–Lizorkin spaces on spaces of homogeneous type

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Abstract. In [HS] the Besov and Triebel–Lizorkin spaces on spaces of homogeneous type were introduced. In this paper, the Triebel–Lizorkin spaces on spaces of homogeneous type are generalized to the case where $p_0 < p \le 1 \le q < \infty$, and a new atomic decomposition for these spaces is obtained. As a consequence, we give the Littlewood–Paley characterization of Hardy spaces on spaces of homogeneous type which were introduced by the maximal function characterization in [MS2].

1. Introduction. We begin by recalling the definitions necessary for introducing the Triebel–Lizorkin spaces on spaces of homogeneous type. A *quasi-metric* d on a set X is a function $d : X \times X \to [0,\infty)$ satisfying:

(1.1)
(i) $d(x,y) = 0$ if and only if $x = y$,
(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$,
(iii) there exists a constant $A < \infty$ such that for all $x, y, z \in X$,
$$d(x,y) \le A[d(x,z) + d(z,y)]\,.$$

Any quasi-metric defines a topology, for which the balls $B(x,r) = \{y \in X : d(y,x) < r\}$ form a base. However, the balls themselves need not be open when $A > 1$.

DEFINITION (1.2) ([CW1]). A *space of homogeneous type* (X, d, μ) is a set X together with a quasi-metric d and a nonnegative measure μ on X such that $\mu(B(x,r)) < \infty$ for all $x \in X$ and all $r > 0$, and there exists $A' < \infty$ such that for all $x \in X$ and all $r > 0$,

$$\mu(B(x,2r)) \le A'\mu(B(x,r))\,. \tag{1.3}$$

Here μ is assumed to be defined on a σ-algebra which contains all Borel sets and all balls $B(x,r)$.

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Macías and Segovia [MS1] have shown that one can replace d by another quasi-metric ϱ such that there exist $c < \infty$ and some $\theta, 0 < \theta < 1$, with

$$\varrho(x,y) \approx \inf\{\mu(B) : B \text{ is a ball containing } x \text{ and } y\}, \tag{1.4}$$

$$|\varrho(x,y) - \varrho(x',y)| \le c\varrho(x,x')^\theta[\varrho(x,y)+\varrho(x',y)]^{1-\theta} \quad \text{for all } x, x' \text{ and } y \in X. \tag{1.5}$$

We will suppose that $\mu(X) = \infty$ and $\mu(\{x\}) = 0$ for all $x \in X$. These hypotheses allow us to construct an approximation to the identity (see [H2]).

DEFINITION (1.6). A sequence $(S_k)_{k\in\mathbb{Z}}$ of operators is called an *approximation to the identity* if $S_k(x,y)$, the kernels of S_k, are functions from $X \times X$ into $\mathbb{C}$ such that there exists a constant C such that for all $k \in \mathbb{Z}$ and all x, x', y, $y' \in X$, and some $0 < \varepsilon \le \theta$, and some $c < \infty$,

(i) $S_k(x,y) = 0$ if $\varrho(x,y) \ge c2^{-k}$ and $\|S_k\|_\infty \le C2^k$,

(ii) $|S_k(x,y) - S_k(x',y)| \le C2^{k(1+\varepsilon)}\varrho(x,x')^\varepsilon$,

(iii) $|S_k(x,y) - S_k(x,y')| \le C2^{k(1+\varepsilon)}\varrho(y,y')^\varepsilon$,

(iv) $|[S_k(x,y) - S_k(x,y')] - [S_k(x',y) - S_k(x',y')]| \le C\varrho(x,x')^\varepsilon\varrho(y,y')^\varepsilon 2^{k(1+2\varepsilon)}$,

(v) $\int_X S_k(x,y)\,d\mu(y) = 1$,

(vi) $\int_X S_k(x,y)\,d\mu(x) = 1$.

For the existence of such a sequence of operators, see [DJS] where all conditions are introduced and checked except the condition (iv) in (1.6). It is easy to see that the construction in [DJS] satisfies (iv).

To define the Triebel–Lizorkin spaces on spaces of homogeneous type we need the following definition (see [HS]).

DEFINITION (1.7). Fix two exponents $0 < \beta \le \theta$ and $\gamma > 0$. A function f defined on X is said to be a *strong smooth molecule* of type (β,γ) centered at $x_0 \in X$ with width $d > 0$ if f satisfies the following conditions:

(1.8)

(i) $|f(x)| \le c\dfrac{d^\gamma}{(d+\varrho(x,x_0))^{1+\gamma}}$,

(ii) $|f(x) - f(x')| \le c\left[\dfrac{\varrho(x,x')}{d+\varrho(x,x_0)}\right]^\beta \dfrac{d^\gamma}{(d+\varrho(x,x_0))^{1+\gamma}}$ for $\varrho(x,x') \le \dfrac{1}{2A}(d+\varrho(x,x_0))$,

(iii) $\int_X f(x)\,d\mu(x) = 0$.

This definition was first introduced in [M] for the case $X = \mathbb{R}^n$ with the condition (ii) in (1.8) replaced by

$$|f(x) - f(x')| \le c\left[\frac{\varrho(x,x')}{d}\right]^\beta \left[\frac{d^\gamma}{(d+\varrho(x,x_0))^{1+\gamma}} + \frac{d^\gamma}{(d+\varrho(x',x_0))^{1+\gamma}}\right]. \tag{1.9}$$

The collection of all strong smooth molecules of type (β,γ) centered at $x_0 \in X$ with width $d > 0$ will be denoted by $M^{(\beta,\gamma)}(x_0,d)$. If $f \in M^{(\beta,\gamma)}(x_0,d)$, the norm of f in $M^{(\beta,\gamma)}(x_0,d)$ is then defined by

$$\|f\|_{M^{(\beta,\gamma)}(x_0,d)} = \inf\{c \ge 0 : \text{(i) and (ii) in (1.8) hold}\}. \tag{1.10}$$

Now we fix a point $x_0 \in X$ and denote the class of all $f \in M^{(\beta,\gamma)}(x_0,1)$ by $M^{(\beta,\gamma)}$. It is easy to see that $M^{(\beta,\gamma)}$ is a Banach space under the norm $\|f\|_{M^{(\beta,\gamma)}} < \infty$. Just as the space of distributions S' is defined on $\mathbb{R}^n$, the dual space $(M^{(\beta,\gamma)})'$ consists of all linear functionals L from $M^{(\beta,\gamma)}$ to $\mathbb{C}$ with the property that there exists a finite constant c such that for all $f \in M^{(\beta,\gamma)}$, $|L(f)| \le c\|f\|_{M^{(\beta,\gamma)}}$. We denote the natural pairing of elements $h \in (M^{(\beta,\gamma)})'$ and $f \in M^{(\beta,\gamma)}$ by $\langle h, f\rangle$. It is also easy to see that for $x_1 \in X$ and $d > 0$, $M^{(\beta,\gamma)}(x_1,d) = M^{(\beta,\gamma)}$ with equivalent norms. Thus, $\langle h, f\rangle$ is well defined for all $h \in (M^{(\beta,\gamma)})'$ and all $f \in M^{(\beta,\gamma)}(x_1,d)$ with $x_1 \in X$ and $d > 0$.

In [HS] the Besov and Triebel–Lizorkin spaces on spaces of homogeneous type were introduced by use of the family of operators $(D_k)_{k\in\mathbb{Z}}$ where $D_k = S_k - S_{k-1}$ and $(S_k)_{k\in\mathbb{Z}}$ is an approximation to the identity defined in (1.6). More precisely, the *Besov space* $\dot{B}_p^{\alpha,q}$ for $-\varepsilon < \alpha < \varepsilon, 1 \le p, q \le \infty$, is the collection of all $f \in (M^{(\beta,\gamma)})'$ with $0 < \beta, \gamma < \varepsilon$ such that

$$\|f\|_{\dot{B}_p^{\alpha,q}} = \Big\{\sum_{k\in\mathbb{Z}}(2^{k\alpha}\|D_k(f)\|_p)^q\Big\}^{1/q} < \infty. \tag{1.11}$$

The *Triebel–Lizorkin space* $\dot{F}_p^{\alpha,q}$ for $-\varepsilon < \alpha < \varepsilon, 1 < p, q < \infty$, is the collection of all $f \in (M^{(\beta,\gamma)})'$ with $0 < \beta, \gamma < \varepsilon$ such that

$$\|f\|_{\dot{F}_p^{\alpha,q}} = \Big\|\Big\{\sum_{k\in\mathbb{Z}}(2^{k\alpha}|D_k(f)|)^q\Big\}^{1/q}\Big\|_p < \infty. \tag{1.12}$$

In this paper, our concern is to define the Triebel–Lizorkin spaces $\dot{F}_p^{\alpha,q}$ on spaces of homogeneous type for $-\varepsilon < \alpha < \varepsilon$ and $p_0 < p \le 1 \le q < \infty$. The key fact to do this is that the Littlewood–Paley g-function in the right hand side of (1.12) will be replaced by the Littlewood–Paley S-function. To be precise, we introduce the following definition.

DEFINITION (1.13). Suppose that $(D_k)_{k\in\mathbb{Z}}$ is as in (1.12), $0 < a < \infty$, $-\varepsilon < \alpha < \varepsilon$, and $1 \le q < \infty$. The *generalized Littlewood–Paley S-function* of $f \in (M^{(\beta,\gamma)})'$ with $0 < \beta, \gamma < \varepsilon$ is defined by

$$(1.14) \qquad S_{q,a}^{\alpha}(f)(x) = \Big\{ \sum_{k\in\mathbb{Z}} \int_{\varrho(x,y)\le a2^{-k}} 2^k (2^{k\alpha} |D_k(f)(y)|)^q \, d\mu(y) \Big\}^{1/q}.$$

By a standard argument, for $0 < p < \infty, 1 \le q < \infty$, and $0 < a < \infty$,

$$(1.15) \qquad \|S_{q,a}^{\alpha}(f)\|_p \approx \|S_{q,1}^{\alpha}(f)\|_p .$$

We write $S_q^{\alpha}(f)(x) = S_{q,1}^{\alpha}(f)(x)$ and introduce the following norm of $f \in (M^{(\beta,\gamma)})'$ with $0 < \beta, \gamma < \varepsilon$.

Definition (1.16). Suppose that $-\varepsilon < \alpha < \varepsilon$, $0 < p < \infty$, and $1 \le q < \infty$. We define the "norm" of $f \in (M^{(\beta,\gamma)})'$ with $0 < \beta, \gamma < \varepsilon$ by

$$(1.17) \qquad \|f\|_{\dot F_p^{\alpha,q}} = \|S_q^{\alpha}(f)\|_p .$$

The first main result in this paper is the following.

Theorem A. *Suppose that $(S_k)_{k\in\mathbb{Z}}$ and $(P_k)_{k\in\mathbb{Z}}$ are approximations to the identity. Set $D_k = S_k - S_{k-1}$ and $E_k = P_k - P_{k-1}$. Then for $f \in (M^{(\beta,\gamma)})'$ with $\max(0,\alpha) < \beta < \varepsilon$ and $\max(0,-\alpha) < \gamma < \varepsilon$, there exist constants c_1 and c_2 such that for $-\varepsilon < \alpha < \varepsilon, 1/(1+\varepsilon) < p < \infty$ and $1 \le q < \infty$,*

$$(1.18) \qquad c_1 \|\widetilde S_q^{\alpha}(f)\|_p \le \|S_q^{\alpha}(f)\|_p \le c_2 \|\widetilde S_q^{\alpha}(f)\|_p$$

where

$$\widetilde S_q^{\alpha}(f)(x) = \Big\{ \sum_{k\in\mathbb{Z}} \int_{\varrho(x,y)\le 2^{-k}} 2^k (2^{k\alpha} |E_k(f)(y)|)^q \, d\mu(y) \Big\}^{1/q}.$$

In [HS], when $1 < p, q < \infty$ the similar result for the Littlewood–Paley g-function defined in the right hand side of (1.12) was proved. The idea of the proof of this theorem is to use molecular decomposition. This molecular decomposition will follow from the Calderón-type reproducing formula which was established in [HS] (see also [H2]). The proof of Theorem A also shows that the L^p norm of $S_q^{\alpha}(f)$ is equivalent to the L^p norm of $\{\sum_{k\in\mathbb{Z}} (2^{k\alpha}|D_k(f)|)^q\}^{1/q}$ for $1 < p, q < \infty$. This allows us to generalize the Triebel–Lizorkin spaces $\dot F_p^{\alpha,q}$ on spaces of homogeneous type to the case where $-\varepsilon < \alpha < \varepsilon, 1/(1+\varepsilon) < p < \infty$, and $1 \le q < \infty$.

Definition (1.19). Suppose that $(S_k)_{k\in\mathbb{Z}}$ is an approximation to the identity. Set $D_k = S_k - S_{k-1}$. The *Triebel–Lizorkin space* $\dot F_p^{\alpha,q}$ on a space of homogeneous type for $-\varepsilon < \alpha < \varepsilon, 1/(1+\varepsilon) < p < \infty$, and $1 \le q < \infty$, is the collection of all $f \in (M^{(\beta,\gamma)})'$ with $\max(0,\alpha) < \beta < \varepsilon$ and $\max(0,-\alpha) < \gamma < \varepsilon$ such that

$$\|f\|_{\dot F_p^{\alpha,q}} = \|S_q^{\alpha}(f)\|_p < \infty .$$

To state our next result we need the following definition.

Definition (1.20). Suppose that $-\varepsilon < \alpha < \varepsilon$ and $0 < p \le 1 \le q < \infty$. A distribution $a \in (M^{(\beta,\gamma)})'$ with $0 < \beta, \gamma < \varepsilon$ is said to be a (p,q,α) *atom for* $\dot F_q^{\alpha,q}$ if

(i) $\operatorname{supp} a \subset B = B(x_0, r)$,

(ii) $\|a\|_{\dot F_q^{\alpha,q}} \le \mu(B)^{1/q-1/p}$ where the norm of $\dot F_q^{\alpha,q}$ is taken in the sense of (1.12),

(iii) a satisfies the following cancellation condition: for any $g \in M^{(\beta,\gamma)}$ with $\max(0,-\alpha) < \beta < \varepsilon$ and $\max(0,\alpha) < \gamma < \varepsilon$,

$$\langle a, g\rangle = \langle a, [g - g_{\widetilde B}]\eta_{\widetilde B}\rangle$$

where $\widetilde B = B(x_0, 2Ar)$,

$$g_{\widetilde B} = \frac{1}{\int_{\widetilde B} \eta_{\widetilde B}(x)\, d\mu(x)} \int_{\widetilde B} g(x)\eta_{\widetilde B}(x)\, d\mu(x)$$

and $\eta_{\widetilde B} \in C_0^1(X)$ with $\eta_{\widetilde B}(x) = 1$ for $x \in B$ and $\eta_{\widetilde B}(x) = 0$ for $x \in \widetilde B^c$.

Remark. This definition was first introduced in [H1] for the case $X = \mathbb{R}^n, \alpha = 0$, and $1 \le q \le 2$ with the cancellation condition (iii) replaced by the following: $\int a(x)x^{\gamma}\, dx = 0$ for $0 \le |\gamma| \le [n(1/p - 1)]$, the greatest integer less than or equal to $n(1/p-1)$. In general, the original cancellation condition does not make sense for $a \in \dot F_q^{\alpha,q}$ since a could be a distribution. However, by a duality result in [HS], $(\dot F_q^{\alpha,q})^* = \dot F_{q'}^{-\alpha,q'}$ for $1 \le q < \infty$ with $1/q + 1/q' = 1$, and by the fact that if $g \in M^{(\beta,\gamma)}$ with $\max(0,-\alpha) < \beta < \varepsilon$ and $\max(0,\alpha) < \gamma < \varepsilon$ then g and $[g(x) - g_{\widetilde B}]\eta_{\widetilde B}(x)$ are in $\dot F_{q'}^{-\alpha,q'}$, the cancellation condition (iii) in (1.20) makes sense. Furthermore, if a is an integrable function then (iii) is equivalent to the original cancellation condition: $\int a\, d\mu(x) = 0$. Recall that a function a defined on $\mathbb{R}^n$ is a $(p,2,0)$ *atom for* H^p, $1/2 < p \le 1$, if a satisfies (i) supp $a \subseteq Q$, Q is a cube in $\mathbb{R}^n$, (ii) $\|a\|_2 \le |Q|^{1/2}$, (iii) $\int a(x)\, dx = 0$. It is easy to see that when $X = \mathbb{R}^n, \alpha = 0$ and $q = 2$, any $(p,2,0)$ atom, $1/2 < p \le 1$, for $\dot F_q^{\alpha,q}$ is a $(p,2,0)$ atom, $1/2 < p \le 1$, for H^p (see [CW2] and [TW]).

The next main result in this paper is the following atomic decomposition of $\dot F_p^{\alpha,q}$.

Theorem B. *Suppose that $-\varepsilon < \alpha < \varepsilon$,*

$$\max\Big(\frac{1}{1+\alpha+\varepsilon}, \frac{1}{1+\varepsilon}\Big) < p \le 1 \le q < \infty ,$$

and $f \in (M^{(\beta,\gamma)})'$ with $\max(0,\alpha) < \beta < \varepsilon$ and $\max(0,-\alpha) < \gamma < \varepsilon$. Then $f \in \dot F_p^{\alpha,q}$ if and only if there exist a sequence $\{\lambda_k\}_{k=1}^{\infty}$ of numbers and a collection $\{a_k\}_{k=1}^{\infty}$ of (p,q,α) atoms for $\dot F_q^{\alpha,q}$ such that

$$f = \sum_k \lambda_k a_k \tag{1.21}$$

where the series converges in $(M^{(\varepsilon,\varepsilon)})'$ *and* $\sum |\lambda_k|^p < \infty$. *Moreover,*

$$\|f\|_{\dot F_p^{\alpha,q}} \sim \inf \Big(\sum |\lambda_k|^p \Big)^{1/p} \tag{1.22}$$

where the infimum is taken over all representations $f = \sum_k \lambda_k a_k$ *as above.*

As a consequence of Theorem B, we give a new Littlewood–Paley characterization of the Hardy spaces H^p on spaces of homogeneous type, which were introduced by Macías and Segovia [MS2].

THEOREM C. *For* $1/(1+\varepsilon) < p \le 1$, $\dot F_p^{0,2} = H^p$ *with equivalent norms, where* H^p *is defined in* [MS2] *by means of a maximal function.*

We prove Theorem A in Section 2, and Theorems B and C in Section 3.

2. Proof of Theorem A. We first recall a result of Christ [Ch] which gives an analogue of the Euclidean dyadic cubes.

THEOREM (2.1). *There exist a collection of open subsets* $\{Q_\tau^k \subset X : k \in \mathbb{Z}, \tau \in I_k\}$, *where* I_k *denotes some (possibly finite) index set depending on* k, *and constants* $\delta \in (0,1), a_0 > 0, \eta > 0$ *and* $0 < c_1, c_2 < \infty$ *such that*

(i) $\mu(X \setminus \bigcup Q_\tau^k) = 0$ *for all* $k \in \mathbb{Z}$,
(ii) *if* $j \ge k$ *then either* $Q_{\tau'}^j \subset Q_\tau^k$ *or* $Q_{\tau'}^j \cap Q_\tau^k = \emptyset$,
(iii) *for all* (k,τ) *and* $j < k$, *there is a unique* τ' *such that* $Q_\tau^k \subset Q_{\tau'}^j$,
(iv) $\mathrm{diam}(Q_\tau^k) \le c_1 \delta^k$,
(v) *each* Q_τ^k *contains some ball* $B(z_\tau^k, a_0\delta^k)$.

We fix such a collection of open subsets as in Theorem (2.1) and call all Q_τ^k in (2.1) the "dyadic cubes" in X. Without loss of generality, we may assume $\delta = 1/2$ in Theorem (2.1). It is easy to check that our results and proofs are independent of the choice of open subsets which satisfy the hypotheses of Theorem (2.1).

Now we define smooth molecules.

DEFINITION (2.2). Fix two exponents $0 < \beta \le \theta$ and $\gamma > 0$. We say that $m_{Q_\tau^k}$ is a (β,γ)-*smooth molecule* for a "dyadic cube" Q_τ^k with $\mu(Q_\tau^k) \sim 2^{-k}$ if

(i) $$\int m_{Q_\tau^k}(x)\, d\mu(x) = 0,$$

(ii) $$|m_{Q_\tau^k}(x)| \le \mu(Q_\tau^k)^{-1/2}(1 + 2^k \varrho(x, z_\tau^k))^{-(1+\gamma)},$$

(iii) $$|m_{Q_\tau^k}(x) - m_{Q_\tau^k}(y)| \le \mu(Q_\tau^k)^{-1/2-\beta}\varrho(x,y)^\beta[(1+2^k\varrho(x,z_\tau^k))^{-(1+\gamma)} + (1+2^k\varrho(y,z_\tau^k))^{-(1+\gamma)}]$$

where z_τ^k is the "center" of Q_τ^k.

As in the case $X = \mathbb{R}^n$ (see [FJ]), we also need certain spaces of sequences indexed by "dyadic cubes" $\{Q_\tau^k\}$ in X, which will characterize the coefficients in our decomposition of $\dot F_p^{\alpha,q}$. For $-\varepsilon < \alpha < \varepsilon, 0 < p < \infty$ and $1 \le q < \infty$, $\dot f_p^{\alpha,q}$ is the collection of sequences $s = \{s_Q\}_{Q \in \{Q_\tau^k\}}$ such that

$$\|s\|_{\dot f_p^{\alpha,q}} = \Big\| \Big\{ \sum_{Q_\tau^k} [\mu(Q_\tau^k)^{-\alpha-1/2} |s_{Q_\tau^k}| \chi_{Q_\tau^k}]^q \Big\}^{1/q} \Big\|_p \tag{2.3}$$

is finite, where $\chi_{Q_\tau^k}$ is the characteristic function of Q_τ^k.

To prove Theorem A, we need the Calderón-type reproducing formula on spaces of homogeneous type which was established in [HS].

THEOREM (2.4). *Suppose that* $\{S_k\}_{k\in\mathbb{Z}}$ *is an approximation to the identity. Set* $D_k = S_k - S_{k-1}$. *Then there exist families of operators* $\{\widetilde D_k\}_{k\in\mathbb{Z}}$ *and* $\{\widehat D_k\}_{k\in\mathbb{Z}}$ *such that for all* $f \in (M^{(\beta,\gamma)})'$,

$$f = \sum_{k\in\mathbb{Z}} \widetilde D_k D_k(f) = \sum_{k\in\mathbb{Z}} D_k \widehat D_k(f) \tag{2.5}$$

where the series converge in the sense that for all $g \in M^{(\beta',\gamma')}$ *with* $\beta' > \beta$ *and* $\gamma' > \gamma$,

$$\lim_{M\to\infty} \Big\langle \sum_{|k|\le M} \widetilde D_k D_k(f), g \Big\rangle = \langle f, g\rangle = \lim_{M\to\infty} \Big\langle \sum_{|k|\le M} D_k \widehat D_k(f), g \Big\rangle. \tag{2.6}$$

Moreover, $\widetilde D_k(x,y)$ *and* $\widehat D_k(x,y)$, *the respective kernels of* $\widetilde D_k$ *and* $\widehat D_k$, *satisfy the following conditions: for* $0 < \varepsilon' < \varepsilon$, *there exists a constant* c *such that*

(i) $$|\widetilde D_k(x,y)| \le c\frac{2^{-k\varepsilon'}}{(2^{-k}+\varrho(x,y))^{1+\varepsilon'}}, \quad |\widehat D_k(x,y)| \le c\frac{2^{-k\varepsilon'}}{(2^{-k}+\varrho(x,y))^{1+\varepsilon'}},$$

(ii) $$|\widetilde D_k(x,y) - \widetilde D_k(x',y)| \le c\Big(\frac{\varrho(x,x')}{2^{-k}+\varrho(x,y)}\Big)^{\varepsilon'} \frac{2^{-k\varepsilon'}}{(2^{-k}+\varrho(x,y))^{1+\varepsilon'}} \quad \text{for } \varrho(x,x') \le \frac{1}{2A}[2^{-k}+\varrho(x,y)],$$

$$|\widehat D_k(x,y) - \widehat D_k(x,y')| \le c\Big(\frac{\varrho(y,y')}{2^{-k}+\varrho(x,y)}\Big)^{\varepsilon'} \frac{2^{-k\varepsilon'}}{(2^{-k}+\varrho(x,y))^{1+\varepsilon'}} \quad \text{for } \varrho(y,y') \le \frac{1}{2A}[2^{-k}+\varrho(x,y)],$$

(iii) $$\int \widetilde D_k(x,y)\,d\mu(x) = \int \widetilde D_k(x,y)\,d\mu(y) = 0,$$
$$\int \widehat D_k(x,y)\,d\mu(x) = \int \widehat D_k(x,y)\,d\mu(y) = 0.$$

Using this Calderón-type reproducing formula, we have the following smooth molecular decomposition.

THEOREM (2.7). *Suppose that* $-\varepsilon<\alpha<\varepsilon$, $1/(1+\varepsilon)<p<\infty$ *and* $1\le q<\infty$. *If* $S_q^\alpha(f)\in L^p$ *for* $f\in(M^{(\beta,\gamma)})'$ *with* $0<\beta,\gamma<\varepsilon$, *then there exist a sequence* $s=\{s_{Q_\tau^k}\}$ *and* $(\varepsilon',\varepsilon')$-*smooth molecules* $m_{Q_\tau^k}$, $0<\varepsilon'<\varepsilon$, *such that* $f=\sum_{Q_\tau^k}s_{Q_\tau^k}m_{Q_\tau^k}$ *in* $(M^{(\beta',\gamma')})'$ *with* $\beta'>\beta$ *and* $\gamma'>\gamma$, *and*

$$\|s\|_{\dot f_p^{\alpha,q}}\le c\|S_q^\alpha(f)\|_p. \tag{2.8}$$

Proof. By Theorem (2.4),

$$f(x)=\sum_k\widetilde D_kD_k(f)(x)=\sum_k\int\widetilde D_k(x,y)D_k(f)(y)\,d\mu(y) \tag{2.9}$$
$$=\sum_k\sum_\tau\int_{Q_\tau^k}\widetilde D_k(x,y)D_k(f)(y)\,d\mu(y)=\sum_k\sum_\tau s_{Q_\tau^k}m_{Q_\tau^k}$$

where $s_{Q_\tau^k}=c\mu(Q_\tau^k)^{-1/2}\int_{Q_\tau^k}|D_k(f)(y)|\,d\mu(y)$ and

$$m_{Q_\tau^k}=c^{-1}\mu(Q_\tau^k)^{1/2}\Big[\int_{Q_\tau^k}|D_k(f)(y)|\,d\mu(y)\Big]^{-1}\int_{Q_\tau^k}\widetilde D_k(x,y)D_k(f)(y)\,d\mu(y)$$

and the series above converges in $(M^{(\beta',\gamma')})'$ with $\beta'>\beta$ and $\gamma'>\gamma$. It is easy to check that $m_{Q_\tau^k}$ satisfy the conditions (i)–(iii) of (2.2) with $\beta=\gamma=\varepsilon'$, $0<\varepsilon'<\varepsilon$. To see (2.8), we have

$$\|s\|_{\dot f_p^{\alpha,q}}=\Big\|\Big\{\sum_{Q_\tau^k}(\mu(Q_\tau^k)^{-\alpha-1/2}|s_{Q_\tau^k}|\chi_{Q_\tau^k})^q\Big\}^{1/q}\Big\|_p \tag{2.10}$$
$$=c\Big\|\Big\{\sum_{Q_\tau^k}\Big(\mu(Q_\tau^k)^{-\alpha-1}\int_{Q_\tau^k}|D_k(f)(y)|\,d\mu(y)\,\chi_{Q_\tau^k}\Big)^q\Big\}^{1/q}\Big\|_p$$
$$\le c\Big\|\Big\{\sum_k\sum_\tau\mu(Q_\tau^k)^{-1}\int_{Q_\tau^k}(2^{k\alpha}|D_k(f)(y)|)^q\,d\mu(y)\,\chi_{Q_\tau^k}\Big\}^{1/q}\Big\|_p$$
$$\le c\Big\|\Big\{\sum_k\int_{\varrho(x,y)\le c2^{-k}}2^k(2^{k\alpha}|D_k(f)(y)|)^q\,d\mu(y)\Big\}^{1/q}\Big\|_p$$
$$=c\|S_{q,c}^\alpha(f)\|_p\le c\|S_q^\alpha(f)\|_p,$$

which proves (2.8) and, hence, Theorem (2.7).

Conversely, we have

THEOREM (2.11). *If* $f=\sum_{Q_\tau^k}s_{Q_\tau^k}m_{Q_\tau^k}$ *where the series converges in* $(M^{(\varepsilon,\varepsilon)})'$ *and* $m_{Q_\tau^k}$ *are* (β,γ)-*smooth molecules with* $\max(0,\alpha)<\beta<\varepsilon$ *and* $\max(0,-\alpha)<\gamma<\varepsilon$, *then for* $1/(1+\gamma)<p<\infty$ *and* $1\le q<\infty$,

$$\Big\|\Big\{\sum_k\int_{\varrho(x,y)\le 2^{-k}}2^k(2^{k\alpha}|E_k(f)(y)|)^q\,d\mu(y)\Big\}^{1/q}\Big\|_p\le c\|s\|_{\dot f_p^{\alpha,q}} \tag{2.12}$$

where $E_k(x,y)$, *the kernels of the operators* E_k, *satisfy the estimates* (i) *and* (iii) *of* (1.7) *and* $\int E_k(x,y)\,d\mu(y)=0$.

To prove Theorem (2.11), we need a lemma of [FJ].

LEMMA (2.13). *Suppose that* $\mu,\eta\in\mathbb Z$ *with* $\eta\le\mu$ *and for "dyadic cubes"* Q_τ^μ *with* $\mu(Q_\tau^\mu)\sim2^{-\mu}$,

$$|f_{Q_\tau^\mu}(x)|\le(1+2^\eta\varrho(x,z_\tau^\mu))^{-(1+\sigma)},$$

where z_τ^μ *is the center of* Q_τ^μ *and* $\sigma>0$. *Then*

$$\sum_\tau|s_{Q_\tau^\mu}||f_{Q_\tau^\mu}(x)|\le c2^{\mu-\eta}\Big(M\Big(\sum_\tau|s_{Q_\tau^\mu}|\chi_{Q_\tau^\mu}\Big)^r\Big)^{1/r}(x) \tag{2.14}$$

where M *is the Hardy–Littlewood maximal function and* $r>1/(1+\sigma)$.

The proof of this lemma on spaces of homogeneous type is similar to the case $X=\mathbb R^n$ (see [FJ]). We leave the details to the reader.

Now we return to the proof of Theorem (2.11). Suppose that $m_{Q_\tau^\mu}$ is a (β,γ)-smooth molecule with $\max(0,\alpha)<\beta<\varepsilon$ and $\max(0,-\alpha)<\gamma<\varepsilon$, and $\mu(Q_\tau^\mu)\sim2^{-\mu}$. Then

$$|E_k(m_{Q_\tau^\mu})(x)|=\Big|\int[E_k(x,y)-E_k(x,z_\tau^\mu)]m_{Q_\tau^\mu}(y)\,d\mu(y)\Big|$$
$$\le\int_{\varrho(y,z_\tau^\mu)\le\frac{1}{2A}(2^{-k}+\varrho(x,y))}|E_k(x,y)-E_k(x,z_\tau^\mu)||m_{Q_\tau^\mu}(y)|\,d\mu(y)$$
$$+\int_{\varrho(y,z_\tau^\mu)>\frac{1}{2A}(2^{-k}+\varrho(x,y))}|E_k(x,y)||m_{Q_\tau^\mu}(y)|\,d\mu(y)$$
$$+\int_{\varrho(y,z_\tau^\mu)>\frac{1}{2A}(2^{-k}+\varrho(x,y))}|E_k(x,z_\tau^\mu)||m_{Q_\tau^\mu}(y)|\,d\mu(y)$$
$$=I+II+III.$$

If $k\le\mu$ and $2^k\varrho(x,z_\tau^\mu)\le c$, then

$$I\le c\int_{\varrho(y,z_\tau^\mu)\le\frac{1}{2A}(2^{-k}+\varrho(x,y))}\Big(\frac{\varrho(y,z_\tau^\mu)}{2^{-k}+\varrho(x,y)}\Big)^{\gamma'}$$
$$\times\frac{2^{-k\gamma'}}{(2^{-k}+\varrho(x,y))^{1+\gamma'}}\,\frac{\mu(Q_\tau^\mu)^{-1/2}}{(1+2^\mu\varrho(y,z_\tau^\mu))^{1+\gamma}}\,d\mu(y)$$
$$\le c\mu(Q_\tau^\mu)^{-1/2}2^{-(\mu-k)(1+\gamma')}$$

$$\leq c\mu(Q_\tau^\mu)^{-1/2}2^{-(\mu-k)(1+\gamma')}(1+2^k\varrho(x,z_\tau^\mu))^{-(1+\gamma')}$$

where $0<\gamma'<\gamma$ and the last inequality follows from the fact that $2^k\varrho(x,z_\tau^\mu)\leq c$. Next,

$$\begin{aligned} II &\leq \int\limits_{\varrho(y,z_\tau^\mu)>\frac{1}{2A}(2^{-k}+\varrho(x,y))} \frac{2^{-k\varepsilon}}{(2^{-k}+\varrho(x,y))^{1+\varepsilon}}\,\frac{\mu(Q_\tau^\mu)^{-1/2}}{(1+2^\mu\varrho(y,z_\tau^\mu))^{1+\gamma}}\,d\mu(y)\\ &\leq c\mu(Q_\tau^\mu)^{-1/2}2^{-(\mu-k)(1+\gamma)}\\ &\leq c\mu(Q_\tau^\mu)^{-1/2}2^{-(\mu-k)(1+\gamma')}(1+2^k\varrho(x,z_\tau^\mu))^{-(1+\gamma')}\end{aligned}$$

since $0<\gamma'<\gamma$ and $\mu\geq k$. Similarly,

$$\begin{aligned} III &\leq c\int\limits_{\varrho(y,z_\tau^\mu)>\frac{1}{2A}(2^{-k}+\varrho(x,y))} \frac{2^{-k\varepsilon}}{(2^{-k}+\varrho(x,z_\tau^\mu))^{1+\varepsilon}}\,\frac{\mu(Q_\tau^\mu)^{-1/2}}{(1+2^\mu\varrho(y,z_\tau^\mu))^{1+\gamma}}\,d\mu(y)\\ &\leq c\mu(Q_\tau^\mu)^{-1/2}2^{-(\mu-k)(1+\gamma)}\\ &\leq c\mu(Q_\tau^\mu)^{-1/2}2^{-(\mu-k)(1+\gamma')}(1+2^k\varrho(x,z_\tau^\mu))^{-(1+\gamma')}.\end{aligned}$$

If $k\leq\mu$ and $2^k\varrho(x,z_\tau^\mu)>c$, then

$$\begin{aligned} I \leq &\int\limits_{\substack{\varrho(y,z_\tau^\mu)\leq\frac{1}{2A}\varrho(x,z_\tau^\mu)\\ \varrho(y,z_\tau^\mu)\leq\frac{1}{2A}(2^{-k}+\varrho(x,y))}} \left(\frac{\varrho(y,z_\tau^\mu)}{2^{-k}+\varrho(x,y)}\right)^{\gamma'}\\ &\qquad\times\frac{2^{-k\gamma'}}{(2^{-k}+\varrho(x,y))^{1+\gamma'}}\,\frac{\mu(Q_\tau^\mu)^{-1/2}}{(1+2^\mu\varrho(y,z_\tau^\mu))^{1+\gamma}}\,d\mu(y)\\ &+\int\limits_{\substack{\varrho(y,z_\tau^\mu)>\frac{1}{2A}\varrho(x,z_\tau^\mu)\\ \varrho(y,z_\tau^\mu)\leq\frac{1}{2A}(2^{-k}+\varrho(x,y))}} \left(\frac{\varrho(y,z_\tau^\mu)}{2^{-k}+\varrho(x,y)}\right)^{\gamma'}\\ &\qquad\times\frac{2^{-k\gamma'}}{(2^{-k}+\varrho(x,y))^{1+\gamma'}}\,\frac{\mu(Q_\tau^\mu)^{-1/2}}{(1+2^\mu\varrho(y,z_\tau^\mu))^{1+\gamma}}\,d\mu(y)\\ \leq\ & c\int\limits_{\varrho(x,y)\geq\frac{1}{2A}\varrho(x,z_\tau^\mu)} \frac{\varrho(y,z_\tau^\mu)^{\gamma'}}{(2^{-k}+\varrho(x,y))^{1+\gamma'}}\,\frac{\mu(Q_\tau^\mu)^{-1/2}}{(1+2^\mu\varrho(y,z_\tau^\mu))^{1+\gamma}}\,d\mu(y)\\ &+c\int\limits_{\varrho(y,z_\tau^\mu)>\frac{1}{2A}\varrho(x,z_\tau^\mu)} \frac{2^{-k\gamma'}}{(2^{-k}+\varrho(x,y))^{1+\gamma'}}\,\frac{\mu(Q_\tau^\mu)^{-1/2}}{(1+2^\mu\varrho(y,z_\tau^\mu))^{1+\gamma}}\,d\mu(y)\\ \leq\ & c\mu(Q_\tau^\mu)^{-1/2}2^{-\mu(1+\gamma')}\varrho(x,z_\tau^\mu)^{-(1+\gamma')}\\ \leq\ & c\mu(Q_\tau^\mu)^{-1/2}2^{-(\mu-k)(1+\gamma')}(1+2^k\varrho(x,z_\tau^\mu))^{-(1+\gamma')}.\end{aligned}$$

These estimates imply that if $k\leq\mu$ and $0<\gamma'<\gamma$, then

$$(2.15)\qquad |E_k(m_{Q_\tau^\mu})(x)|\leq c\mu(Q_\tau^\mu)^{-1/2}2^{-(\mu-k)(1+\gamma')}(1+2^k\varrho(x,z_\tau^\mu))^{-(1+\gamma')}.$$

If $k\geq\mu$ and $2^k\varrho(x,z_\tau^\mu)\leq c$, then

$$\begin{aligned} |E_k(m_{Q_\tau^\mu})(x)| &= \Big|\int E_k(x,y)[m_{Q_\tau^\mu}(y)-m_{Q_\tau^\mu}(x)]\,d\mu(y)\Big| \qquad \text{since } E_k(1)=0\\ &\leq c\int\frac{2^{-k\varepsilon}}{(2^{-k}+\varrho(x,y))^{1+\varepsilon}}\mu(Q_\tau^\mu)^{-1/2-\beta}\varrho(x,y)^\beta\\ &\qquad\times[(1+2^\mu\varrho(y,z_\tau^\mu))^{-(1+\gamma)}+(1+2^\mu\varrho(x,z_\tau^\mu))^{-(1+\gamma)}]\,d\mu(y)\\ &\leq c\mu(Q_\tau^\mu)^{-1/2-\beta}2^{-k\beta} \qquad \text{since } \beta<\varepsilon\\ &\leq c\mu(Q_\tau^\mu)^{-1/2}2^{-(k-\mu)\beta}(1+2^\mu\varrho(x,z_\tau^\mu))^{-(1+\gamma)}.\end{aligned}$$

Similarly, if $k\geq\mu$ and $2^\mu\varrho(x,z_\tau^\mu)>c$, then

$$\begin{aligned} &|E_k(m_{Q_\tau^\mu})(x)|\\ &\quad\leq c\int\limits_{\varrho(y,z_\tau^\mu)>\frac{1}{2A}\varrho(x,z_\tau^\mu)}\frac{2^{-k\varepsilon}}{(2^{-k}+\varrho(x,y))^{1+\varepsilon}}\mu(Q_\tau^\mu)^{-1/2-\beta}\varrho(x,y)^\beta\\ &\qquad\times[(1+2^\mu\varrho(y,z_\tau^\mu))^{-(1+\gamma)}+(1+2^\mu\varrho(x,z_\tau^\mu))^{-(1+\gamma)}]\,d\mu(y)\\ &\qquad+c\int\limits_{\varrho(y,z_\tau^\mu)\leq\frac{1}{2A}\varrho(x,z_\tau^\mu)}\frac{2^{-k\varepsilon}}{(2^{-k}+\varrho(x,y))^{1+\varepsilon}}\mu(Q_\tau^\mu)^{-1/2}\\ &\qquad\times[(1+2^\mu\varrho(y,z_\tau^\mu))^{-(1+\gamma)}+(1+2^\mu\varrho(x,z_\tau^\mu))^{-(1+\gamma)}]\,d\mu(y)\\ &\quad\leq c\mu(Q_\tau^\mu)^{-1/2-\beta}2^{-k\beta}(1+2^\mu\varrho(x,z_\tau^\mu))^{-(1+\gamma)}\\ &\qquad+c\mu(Q_\tau^\mu)^{-1/2}2^{-k\varepsilon}\varrho(x,z_\tau^\mu)^{-(1+\varepsilon)}2^{-\mu}\\ &\qquad+c\mu(Q_\tau^\mu)^{-1/2}\varrho(x,z_\tau^\mu)^{-\varepsilon}2^{-k\varepsilon}(1+2^\mu\varrho(x,z_\tau^\mu))^{-(1+\gamma)}\\ &\quad\leq c\mu(Q_\tau^\mu)^{-1/2}2^{-(k-\mu)\beta}(1+2^\mu\varrho(x,z_\tau^\mu))^{-(1+\gamma)}\\ &\qquad\qquad\qquad\text{since } \gamma<\varepsilon \text{ and } \beta<\varepsilon.\end{aligned}$$

These estimates imply that if $k\geq\mu$ and $0<\beta,\gamma<\varepsilon$, then

$$(2.16)\qquad |E_k(m_{Q_\tau^\mu})(x)|\leq c\mu(Q_\tau^\mu)^{-1/2}2^{-(k-\mu)\beta}(1+2^\mu\varrho(x,z_\tau^\mu))^{-(1+\gamma)}.$$

Note that if $\varrho(z,x)\leq 2^{-k}$, then (2.15) and (2.16) yield

$$(2.17)\qquad |E_k(m_{Q_\tau^\mu})(x)|\leq c\mu(Q_\tau^\mu)^{-1/2}2^{-(\mu-k)(1+\gamma')}(1+2^k\varrho(z,z_\tau^\mu))^{-(1+\gamma')}$$
$$\text{for } k\leq\mu \text{ and } 0<\gamma'<\gamma,$$

$$(2.18)\qquad |E_k(m_{Q_\tau^\mu})(x)|\leq c\mu(Q_\tau^\mu)^{-1/2}2^{-(k-\mu)\beta}(1+2^\mu\varrho(z,z_\tau^\mu))^{-(1+\gamma)}$$
$$\text{for } k\geq\mu.$$

Now we have

$$\Big\{\sum_k \int_{\varrho(x,y)\le 2^{-k}} 2^k(2^{k\alpha}|E_k(f)(y)|)^q\,d\mu(y)\Big\}^{1/q}$$

$$\le c\Big\{\sum_k \int_{\varrho(x,y)\le 2^{-k}} 2^k\Big(2^{k\alpha}\sum_{\mu=-\infty}^{k}\sum_\tau |s_{Q^\mu_\tau}||E_k(m_{Q^\mu_\tau})(y)|\Big)^q d\mu(y)\Big\}^{1/q}$$

$$+ c\Big\{\sum_k \int_{\varrho(x,y)\le 2^{-k}} 2^k\Big(2^{k\alpha}\sum_{\mu=k+1}^{\infty}\sum_\tau |s_{Q^\mu_\tau}||E_k(m_{Q^\mu_\tau})(y)|\Big)^q d\mu(y)\Big\}^{1/q}$$

$$\le c\Big\{\sum_k\Big(2^{k\alpha}\sum_{\mu=-\infty}^{k}\sum_\tau |s_{Q^\mu_\tau}|\mu(Q^\mu_\tau)^{-1/2}2^{-(k-\mu)\beta} \times(1+2^k\varrho(x,z^\mu_\tau))^{-(1+\gamma)}\Big)^q\Big\}^{1/q} \quad \text{by (2.18)}$$

$$+ c\Big\{\sum\Big(2^{k\alpha}\sum_{\mu=k+1}^{\infty}\sum_\tau |s_{Q^\mu_\tau}|\mu(Q^\mu_\tau)^{-1/2}2^{-(\mu-k)(1+\gamma')} \times(1+2^k\varrho(x,z^\mu_\tau))^{-(1+\gamma')}\Big)^q\Big\}^{1/q} \quad \text{by (2.17)}$$

$$\le c\Big\{\sum_k\Big(2^{k\alpha}\sum_{\mu=-\infty}^{k}2^{-(k-\mu)\beta}\mu(Q^\mu_\tau)^{-1/2}\Big(M\Big(\sum_\tau|s_{Q^\mu_\tau}|\chi_{Q^\mu_\tau}\Big)^r\Big)^{1/r}(x)\Big)^q\Big\}^{1/q}$$

$$+ c\Big\{\sum_k\Big(2^{k\alpha}\sum_{\mu=k+1}^{\infty}2^{-(\mu-k)\gamma'}\mu(Q^\mu_\tau)^{-1/2} \times\Big(M\Big(\sum_\tau|s_{Q^\mu_\tau}|\chi_{Q^\mu_\tau}\Big)^r\Big)^{1/r}(x)\Big)^q\Big\}^{1/q} \quad \text{by (2.13) and } r>\frac{1}{1+\gamma'}$$

$$\le c\Big\{\sum_k\Big(\sum_{\mu=-\infty}^{k}2^{-(k-\mu)(\beta-\alpha)}\mu(Q^\mu_\tau)^{-1/2-\alpha} \times\Big(M\Big(\sum_\tau|s_{Q^\mu_\tau}|\chi_{Q^\mu_\tau}\Big)^r\Big)^{1/r}(x)\Big)^q\Big\}^{1/q}$$

$$+ c\Big\{\sum_k\Big(\sum_{\mu=k+1}^{\infty}2^{-(\mu-k)(\gamma'+\alpha)}\mu(Q^\mu_\tau)^{-1/2-\alpha} \times\Big(M\Big(\sum_\tau|s_{Q^\mu_\tau}|\chi_{Q^\mu_\tau}\Big)^r\Big)^{1/r}(x)\Big)^q\Big\}^{1/q}$$

$$\le c\Big\{\sum_\mu\Big(\mu(Q^\mu_\tau)^{-1/2-\alpha}\Big(M\Big(\sum_\tau|s_{Q^\mu_\tau}|\chi_{Q^\mu_\tau}\Big)^r\Big)^{1/r}(x)\Big)^q\Big\}^{1/q}$$

since $\sum_{k\ge\mu}2^{-(k-\mu)(\beta-\alpha)}+\sum_{k\le\mu}2^{-(\mu-k)(\gamma'+\alpha)}<\infty$ for $\beta>\alpha$ and $\gamma'>-\alpha$. Integrating yields

$$\Big\|\Big\{\sum_k\int_{\varrho(x,y)\le 2^{-k}}2^k(2^{k\alpha}|E_k(f)(y)|)^q\,d\mu(y)\Big\}^{1/q}\Big\|_p \tag{2.19}$$

$$\le c\Big\|\Big\{\sum_\mu\Big(M\Big(\sum_\tau(\mu(Q^\mu_\tau)^{-1/2-\alpha}|s_{Q^\mu_\tau}|\chi_{Q^\mu_\tau})^r\Big)^{1/r}(x)\Big)^q\Big\}^{1/q}\Big\|_p$$

(since for $p>1/(1+\gamma)$ we can choose $\gamma'<\gamma$ and $r>1/(1+\gamma')$ such that $p/r>1$, and then apply the Fefferman and Stein vector-valued maximal function inequality with $p/r>1$ and $q/r>1$)

$$\le c\Big\|\Big\{\sum_\mu\sum_\tau(\mu(Q^\mu_\tau)^{-1/2-\alpha}|s_{Q^\mu_\tau}|\chi_{Q^\mu_\tau})^q\Big\}^{1/q}\Big\|_p=c\|s\|_{\dot f^{\alpha,q}_p},$$

which shows Theorem (2.11).

Now we prove Theorem A. Suppose that $f\in(M^{(\beta,\gamma)})'$ with $\max(0,\alpha)<\beta<\varepsilon$ and $\max(0,-\alpha)<\gamma<\varepsilon$, and $\|\widetilde S^\alpha_q(f)\|_p<\infty$. By Theorem (2.7), $f=\sum_{Q^k_\tau}s_{Q^k_\tau}m_{Q^k_\tau}$ where the series converges in $(M^{(\beta',\gamma')})'$ with $\beta'>\beta$ and $\gamma'>\gamma$, and $m_{Q^k_\tau}$ are $(\varepsilon',\varepsilon')$-smooth molecules with $0<\varepsilon'<\varepsilon$, and $\|s\|_{\dot f^{\alpha,q}_p}\le c\|\widetilde S^\alpha_q(f)\|_p$. Without loss of generality, we may choose $|\alpha|<\varepsilon'<\varepsilon$, and $p>1/(1+\varepsilon')$. By Theorem (2.11),

$$\|S^\alpha_q(f)\|_p\le c\|s\|_{\dot f^{\alpha,q}_p}\le c\|\widetilde S^\alpha_q(f)\|_p\,,$$

which shows one inequality of (1.18). The other inequality of (1.18) can be proved in the same manner.

R e m a r k. It was proved in [HS] that the Triebel–Lizorkin spaces $\dot F^{\alpha,q}_p$ on spaces of homogeneous type for $-\varepsilon<\alpha<\varepsilon$ and $1<p,q<\infty$ have the same smooth molecular decomposition as in Theorems (2.7) and (2.11). Thus, the Triebel–Lizorkin spaces $\dot F^{\alpha,q}_p$ defined in (1.19) generalize $\dot F^{\alpha,q}_p$ defined in (1.12) to the case where $-\varepsilon<\alpha<\varepsilon$, $1/(1+\varepsilon)<p\le 1\le q<\infty$.

3. Proofs of Theorems B and C. To prove the "if" part of Theorem B, we need the following lemma.

LEMMA (3.1). *Suppose that* $-\varepsilon<\alpha<\varepsilon$, $1/(1+\alpha+\varepsilon)<p\le1\le q<\infty$, *and* $\{a_k\}_{k=1}^\infty$ *is a collection of* (p,q,α) *atoms for* $\dot F^{\alpha,q}_p$ *and* $\{\lambda_k\}_{k=1}^\infty$ *is a sequence of numbers with* $\sum|\lambda_k|^p<\infty$. *Then there exists a pair* (β_0,γ_0) *with* $\max(0,-\alpha)<\beta_0<\varepsilon$ *and* $\max(0,\alpha)<\gamma_0<\varepsilon$ *such that the series* $\sum\lambda_k a_k$ *converges in* $(M^{(\beta_0,\gamma_0)})'$.

P r o o f. First consider the case where $-\varepsilon<\alpha<\varepsilon$ and $1/(1+\alpha+\varepsilon)<p<1$. It suffices to show that there exists a constant $c>0$ such that for all

$g \in M^{(\beta_0,\gamma_0)}$ with $\max(0,-\alpha) < \beta_0 < \varepsilon$ and $\max(0,\alpha) < \gamma_0 < \varepsilon$,

$$|\langle a, g\rangle| \le c \tag{3.2}$$

for all (p,q,α) atoms a for $\dot F_p^{\alpha,q}$.

To see (3.2), suppose that a is a (p,q,α) atom for $\dot F_p^{\alpha,q}$ with support $B = B(x_0,r)$; then, by the cancellation condition on a,

$$\langle a, g\rangle = \langle a, [g - g_{\widetilde B}]\eta_{\widetilde B}\rangle \tag{3.3}$$

where $\widetilde B = B(x_0, 2Ar)$,

$$g_{\widetilde B} = \frac{1}{\int_{\widetilde B}\eta_{\widetilde B}(x)\,d\mu(x)}\int_{\widetilde B} g(x)\eta_{\widetilde B}(x)\,d\mu(x)$$

and $\eta_{\widetilde B} \in C_0^1(X)$ with $\eta_{\widetilde B}(x) = 1$ for $x \in B$ and $\eta_{\widetilde B}(x) = 0$ for $x \in \widetilde B^c$. We claim that there exists a constant $c > 0$ such that

$$\|[g(\cdot) - g_{\widetilde B}]\eta_{\widetilde B}(\cdot)\|_{\dot F_{q'}^{-\alpha,q'}} \le c\mu(B)^{1/p-1/q}. \tag{3.4}$$

Assuming (3.4) for the moment, we then have

$$\begin{aligned}|\langle a, g\rangle| &\le \|a\|_{\dot F_q^{\alpha,q}}\|[g(\cdot) - g_{\widetilde B}]\eta_{\widetilde B}(\cdot)\|_{\dot F_{q'}^{-\alpha,q'}} \\ &\qquad \text{(since } (\dot F_q^{\alpha,q})' = \dot F_{q'}^{-\alpha,q'}\text{; see [HS] for this result and the proof)} \\ &\le c\mu(B)^{1/q-1/p}\mu(B)^{1/p-1/q} \qquad \text{by the size condition on } a \text{ and (3.4)} \\ &\le c,\end{aligned}$$

which proves (3.2).

To prove (3.4), we first consider the case where $-\varepsilon < \alpha < \varepsilon$, $1/(1+\alpha+\varepsilon) < p < 1/(1+\alpha)$ and $p < 1$. It suffices to show (3.4) for $g \in M^{(\beta_0,\gamma_0)}$ with $\max(0,-\alpha) < \beta_0 = (1/p-1) - \alpha < \varepsilon$ and $\max(0,\alpha) < \gamma_0 < \varepsilon$. Set $f(x) = [g(x) - g_{\widetilde B}]\eta_{\widetilde B}(x)$. It is easy to check that f satisfies the following estimates:

$$|f(x)| \le c\mu(B)^{\beta_0}, \tag{3.5}$$

$$|f(x_1) - f(x_2)| \le c\varrho(x_1,x_2)^{\beta_0} \tag{3.6}$$

where c is a constant depending only on g. Set $8AB = B(x_0, 8Ar)$. Thus, by the definition of the Triebel–Lizorkin spaces $\dot F_q^{\alpha,q}$ in (1.12),

$$\begin{aligned}\|[g(\cdot) - g_{\widetilde B}]\eta_{\widetilde B}(\cdot)\|^{q'}_{\dot F_{q'}^{-\alpha,q'}} &= \sum_{k\in\mathbb Z}\int 2^{-k\alpha q'}|D_k(f)(x)|^{q'}\,d\mu(x) \qquad (3.7)\end{aligned}$$

$$\begin{aligned}&= \Big(\sum_{2^{-k}\le 4Ar}\int_{8AB} + \sum_{2^{-k}\le 4Ar}\int_{(8AB)^c} \\ &\quad + \sum_{2^{-k}>4Ar}\int_{8AB} + \sum_{2^{-k}>4Ar}\int_{(8AB)^c}\Big)2^{-k\alpha q'}|D_k(f)(x)|^{q'}\,d\mu(x) \\ &= I + II + III + IV.\end{aligned}$$

Note first that $II = 0$, since if $x \in (8AB)^c$ and $2^{-k} \le 4Ar$, then $\varrho(x,y) > 4Ar \ge 2^{-k}$ for $y \in B(x_0,r)$, and, hence, $D_k(f)(x) = 0$. Using (3.6) and the fact that $D_k(1) = 0$, we get

$$|D_k(f)(x)| = \Big|\int D_k(x,y)[f(y) - f(x)]\,d\mu(y)\Big| \le c2^{-k\beta_0}.$$

Thus, if $\beta_0 = (1/p-1) - \alpha > -\alpha$, then

$$I \le c\sum_{2^{-k}\le 4Ar} 2^{-k\alpha q'}2^{-k\beta_0 q'}\mu(B) \le c\mu(B)r^{(\alpha+\beta_0)q'} \le c\mu(B)^{1+(1/p-1)q'}.$$

By (3.5),

$$|D_k(f)(x)| = \Big|\int D_k(x,y)f(y)\,d\mu(y)\Big| \le c2^k\mu(B)^{1+\beta_0}.$$

Thus,

$$\begin{aligned}III &\le c\sum_{2^{-k}>4Ar} 2^{-k\alpha q'}2^{kq'}\mu(B)^{1+(1+\beta_0)q'} \\ &\le c\mu(B)^{1+(1+\beta_0)q'}r^{-(1-\alpha)q'} \qquad \text{since } 1-\alpha > 0 \\ &\le c\mu(B)^{1+(1/p-1)q'}.\end{aligned}$$

To estimate IV, note that if $x \in (8AB)^c$, then

$$\begin{aligned}|D_k(f)(x)| &= \Big|\int [D_k(x,y) - D_k(x,x_0)]f(y)\,d\mu(y)\Big| \qquad (3.8)\\ &\qquad\qquad \text{since } \int f(y)\,d\mu(y) = 0 \\ &\le c\int\Big(\frac{\varrho(y,x_0)}{2^{-k}+\varrho(x,x_0)}\Big)^{\varepsilon}\frac{2^{-k\varepsilon}}{(2^{-k}+\varrho(x,x_0))^{1+\varepsilon}}|f(y)|\,d\mu(y) \\ &\le c\mu(B)^{1+\beta_0+\varepsilon}\frac{2^{-k\varepsilon}}{(2^{-k}+\varrho(x,y))^{1+2\varepsilon}}.\end{aligned}$$

Thus,

$$\begin{aligned}IV &\le c\sum_{2^{-k}>4Ar} 2^{-k\alpha q'}\mu(B)^{(1+\beta_0+\varepsilon)q'} \\ &\quad \times \int_{\varrho(x,x_0)\ge 8Ar}\Big[\frac{2^{-k\varepsilon}}{(2^{-k}+\varrho(x,y))^{1+2\varepsilon}}\Big]^{q'}\,d\mu(x)\end{aligned}$$

$$\leq c\mu(B)^{(1+\beta_0+\varepsilon)q'} \sum_{2^{-k}>4Ar} 2^{-k\alpha q'} 2^{-k[1-(1+\varepsilon)q']}$$
$$\leq c\mu(B)^{(1+\beta_0+\varepsilon)q'} r^{\alpha q'+1-(1+\varepsilon)q'} \quad \text{since } \alpha q' + 1 - (1+\varepsilon)q' < 0$$
$$\leq c\mu(B)^{1+(1/p-1)q'} .$$

The estimates above imply

$$\|[g(\cdot) - g_{\widetilde{B}}]\eta_{\widetilde{B}}(\cdot)\|_{\dot{F}_{q'}^{-\alpha,q'}} \leq c\mu(B)^{1/q'+(1/p-1)} = c\mu(B)^{1/p-1/q},$$

which proves (3.4) for the case where $-\varepsilon < \alpha < \varepsilon, 1/(1+\alpha+\varepsilon) \leq p < 1/(1+\alpha)$ and $p < 1$.

Now we prove (3.4) for the case where $0 < \alpha < \varepsilon, 1/(1+\alpha) \leq p < 1$ and $g \in M^{(\beta_1,\gamma_1)}$ with $\max(0,-\alpha) < \beta_1 < \varepsilon$ and $\max(0,\alpha) < \gamma_1 < \varepsilon$. To do this, we need the following estimates:

(3.9) $|D_k(f)(x)| \leq c2^k$ since $\int |f(y)|\, d\mu(y) \leq c$,

(3.10) $|D_k(f)(x)| \leq c$ since $|f(y)| \leq c$ and $\|D_k(x,\cdot)\|_1 \leq c$,

(3.11) $|D_k(f)(x)| \leq c2^k\mu(B)^{1+\beta_1}$ since $|f(y)| \leq c\mu(B)^{\beta_1}$.

By taking the geometric mean between (3.9) and (3.10), for $0 \leq \sigma \leq 1$ we have

$$|D_k(f)(x)| \leq c2^{k\sigma} . \tag{3.12}$$

We estimate I, III and IV in (3.7) as follows. For $0 \leq \sigma_0 = \alpha - (1/p-1) < \alpha < 1$,

$$I \leq c \sum_{2^{-k}\leq 4Ar} 2^{-k\alpha q'} 2^{k\sigma_0 q'} \mu(B) \quad \text{by (3.12)}$$
$$\leq c\mu(B) r^{(1/p-1)q'} \quad \text{since } 1/p - 1 > 0$$
$$\leq c\mu(B)^{1+(1/p-1)q'} .$$

To estimate III, by taking the geometric mean between (3.9) and (3.11), we have

$$|D_k(f)(x)| \leq c2^k\mu(B)^{(1+\beta_1)\sigma} \quad \text{for } 0 \leq \sigma \leq 1 . \tag{3.13}$$

Thus, choosing β_1 with $0 \leq \sigma_0 = \frac{1/p-\alpha}{1+\beta_1} \leq 1$,

$$III \leq c \sum_{2^{-k}>4Ar} 2^{-k\alpha q'} 2^{kq'} \mu(B)^{(1+\beta_1)\sigma_0 q'+1} \quad \text{by (3.13)}$$
$$\leq c\mu(B)^{1+(1+\beta_1)\sigma_0 q'} r^{(\alpha-1)q'} \quad \text{since } \alpha < 1$$
$$\leq c\mu(B)^{1+(1/p-1)q'} .$$

To estimate IV, we need the following estimates: for $x \in (8AB)^c$,

$$|D_k(f)(x)| = \Big|\int [D_k(x,y) - D_k(x,x_0)]f(y)\, d\mu(y)\Big| \tag{3.14}$$

$$\leq c\mu(B)^{\varepsilon} \frac{2^{-k\varepsilon}}{(2^{-k} + \varrho(x,y))^{1+2\varepsilon}} \quad \text{since } \int |f(y)|\, d\mu(y) \leq c .$$

By taking the geometric mean between (3.8) and (3.14), we see that for $0 \leq \sigma_0 = \frac{1/p-\alpha}{1+\beta_1} \leq 1$,

$$|D_k(f)(x)| \leq c\mu(B)^{\varepsilon+(1+\beta_1)\sigma_0} \frac{2^{-k\varepsilon}}{(2^{-k} + \varrho(x,y))^{1+2\varepsilon}} \tag{3.15}$$
$$\leq c \sum_{2^{-k}>4Ar} \mu(B)^{[\varepsilon+(1+\beta_1)\sigma_0]q'} 2^{-k[\alpha q'+1-(1+\varepsilon)q']}$$
$$\leq c\mu(B)^{[\varepsilon+(1+\beta_1)\sigma_0]q'} r^{\alpha q'+1-(1+\varepsilon)q'} \leq c\mu(B)^{1+(1/p-1)q'} .$$

These estimates imply that for $-\varepsilon < \alpha < \varepsilon$, $p < 1$ and $\alpha \geq 1/p - 1$,

$$\|[g(\cdot) - g_{\widetilde{B}}]\eta_{\widetilde{B}}(\cdot)\|_{\dot{F}_{q'}^{-\alpha,q'}} \leq c\mu(B)^{1/p-1/q}$$

which proves (3.4) for the case where $0 < \alpha < \varepsilon$ and $1/(1+\alpha) \leq p < 1$.

It remains to prove Lemma (3.1) for the case $p = 1$. In the proof of the "if" part of Theorem B, we will see that for any $(1,q,\alpha)$ atom a for $\dot{F}_p^{\alpha,q}$,

$$\Big\|\Big\{\sum_{k\in\mathbb{Z}} (2^{k\alpha}|D_k(a)|)^q\Big\}^{1/q}\Big\|_1 \leq c \tag{3.16}$$

where c is a constant independent of a. By using the Calderón-type reproducing formula of Theorem (2.4) (see [HS] for the details of the proof),

$$|\langle a, g\rangle| \leq \int \Big\{\sum_{k\in\mathbb{Z}} (2^{k\alpha}|D_k(a)|)^q\Big\}^{1/q} \Big\{\sum_{k\in\mathbb{Z}} (2^{-k\alpha}|\widetilde{D}_k^*(g)|)^{q'}\Big\}^{1/q'} d\mu(x)$$
$$\leq \Big\|\Big\{\sum_{k\in\mathbb{Z}} (2^{k\alpha}|D_k(a)|)^q\Big\}^{1/q}\Big\|_1 \Big\|\Big\{\sum_{k\in\mathbb{Z}} (2^{-k\alpha}|\widetilde{D}_k^*(g)|)^{q'}\Big\}^{1/q'}\Big\|_\infty .$$

Therefore, it suffices to show that for $-\varepsilon < \alpha < \varepsilon$ and $1 < q' \leq \infty$, and $g \in M^{(\beta,\gamma)}$ with $\varepsilon > \beta > \max(0,-\alpha)$ and $\gamma > 0$, there exists a constant $c > 0$ such that

$$\Big\|\Big\{\sum_{k\in\mathbb{Z}} (2^{-k\alpha}|\widetilde{D}_k^*(g)|)^{q'}\Big\}^{1/q'}\Big\|_\infty \leq c \tag{3.17}$$

where $\widetilde{D}_k$ is as in Theorem (2.4) and $\widetilde{D}_k^*$ is the transpose of $\widetilde{D}_k$.

To see (3.17), suppose that $g \in M^{(\beta,\gamma)}$ with $\varepsilon > \beta > \max(0,-\alpha)$ and $\gamma > 0$. Then

$$|\widetilde{D}_k^*(g)(x)| \leq c2^k , \tag{3.18}$$

$$(3.19)\qquad |\widetilde{D}_k^*(g)(x)| = \Big|\int \widetilde{D}_k(y,x)g(y)\,d\mu(y)\Big| = \Big|\int \widetilde{D}_k(y,x)[g(y)-g(x)]\,d\mu(y)\Big| \quad \text{since } \widetilde{D}_k^*(1)=0$$
$$\le c2^{-k\beta}$$

where c is a constant depending only on g. Thus,

$$\Big\{\sum_{k\in\mathbb{Z}}(2^{-k\alpha}|\widetilde{D}_k^*(g)|)^{q'}\Big\}^{1/q'} \le c\Big[\Big\{\sum_{k<0}(2^{-k\alpha}|\widetilde{D}_k^*(g)|)^{q'}\Big\}^{1/q'} + \Big\{\sum_{k\ge 0}(2^{-k\alpha}|\widetilde{D}_k^*(g)|)^{q'}\Big\}^{1/q'}\Big]$$
$$\le c\Big(\sum_{k<0}2^{k(1-\alpha)q'}\Big)^{1/q'} + c\Big(\sum_{k\ge 0}2^{-k(\beta+\alpha)q'}\Big)^{1/q'}$$
$$\le c \quad \text{since } 1-\alpha>0 \text{ and } \beta+\alpha>0,$$

which proves (3.17); this completes the proof of Lemma (3.1).

Now we prove the "if" part of Theorem B. Suppose that $f=\sum_{k=1}^{\infty}\lambda_k a_k$ where $\{a_k\}_{k=1}^{\infty}$ is a collection of (p,q,α) atoms for $\dot{F}_p^{\alpha,q}$ and $\{\lambda_k\}_{k=1}^{\infty}$ is a sequence of numbers with $\sum_{k=1}^{\infty}|\lambda_k|^p<\infty$. By Lemma (3.1), the series $\sum_{k=1}^{\infty}\lambda_k a_k$ converges in $(M^{(\beta,\gamma)})'$ for some $0<\beta,\gamma<\varepsilon$. Thus,

$$\|f\|^p_{\dot{F}_p^{\alpha,q}} = \|S_q^\alpha(f)\|_p^p \le \sum_{k=1}^{\infty}|\lambda_k|^p\|S_q^\alpha(a_k)\|_p^p.$$

It suffices to prove that there exists a constant c such that for each (p,q,α) atom a for $\dot{F}_p^{\alpha,q}$,

$$(3.20)\qquad \|S_q^\alpha(a)\|_p \le c.$$

To see (3.20), suppose that a is a (p,q,α) atom for $\dot{F}_p^{\alpha,q}$ supported on $Q=Q(x_Q,r)$. Set $2AQ=Q(x_Q,2Ar)$. Then

$$\|S_q^\alpha(a)\|_p^p = \int S_q^\alpha(a)^p(x)\,d\mu(x) \le \Big(\int_{2AQ} + \int_{(2AQ)^c}\Big)S_q^\alpha(a)^p(x)\,d\mu(x) = I + II.$$

By Hölder's inequality and the size condition on a,

$$I \le c\mu(Q)^{1-p/q}\Big(\int_{2AQ} S_q^\alpha(a)^q(x)\,d\mu(x)\Big)^{p/q} = c\mu(Q)^{1-p/q}\|a\|^p_{\dot{F}_q^{\alpha,q}}$$
$$\le c\mu(Q)^{1-p/q}\mu(Q)^{(1/q-1/p)p} = c.$$

To estimate II, we claim that there exists a constant c such that for $x\in(2AQ)^c$,

$$(3.21)\qquad |D_k(a)(y)| \le c\mu(Q)^{1-1/p+\alpha+\varepsilon}2^{k(1+\varepsilon)}.$$

Assume this claim for the moment; then for $x\in(2AQ)^c$,

$$\int_{\varrho(x,y)\le 2^{-k}} 2^k(2^{k\alpha}|D_k(a)(y)|)^q\,d\mu(y) \le c\mu(Q)^{(1-1/p+\alpha+\varepsilon)q}2^{k(1+\varepsilon+\alpha)q}.$$

If $x\in(2AQ)^c$ and $\varrho(x,y)\le 2^{-k}$, then $D_k(a)(y)=0$ for $2^k>c/\varrho(x,x_Q)$ where c is a constant. Thus,

$$\int_{(2AQ)^c} S_q^\alpha(a)^p(x)\,d\mu(x) \le c\int_{(2AQ)^c}\Big\{\sum_{2^k\le c/\varrho(x,x_Q)}2^{k(1+\alpha+\varepsilon)q}\mu(Q)^{(1+\alpha+\varepsilon-1/p)q}\Big\}^{p/q}d\mu(x)$$
$$\le c\int_{(2AQ)^c}\mu(Q)^{(1+\alpha+\varepsilon-1/p)p}\varrho(x,x_Q)^{-(1+\alpha+\varepsilon)p}\,d\mu(x)$$
$$\le c\mu(Q)^{(1+\alpha+\varepsilon-1/p)p}\mu(Q)^{1-(1+\alpha+\varepsilon)p} \quad \text{since } (1+\alpha+\varepsilon)p>1$$
$$= c,$$

which together with the estimate of I yields (3.20).

We now prove (3.16). Suppose that a is a $(1,q,\alpha)$ atom supported on $Q=Q(x_Q,r)$. Set again $2AQ=Q(x_Q,2Ar)$. Then

$$\Big\|\Big\{\sum_{k\in\mathbb{Z}}(2^{k\alpha}|D_k(a)|)^q\Big\}^{1/q}\Big\|_1 \le \Big(\int_{2AQ}+\int_{(2AQ)^c}\Big)\Big\{\sum_{k\in\mathbb{Z}}(2^{k\alpha}|D_k(a)(x)|)^q\Big\}^{1/q}d\mu(x) = I + II.$$

By Hölder's inequality and the size condition on a,

$$I \le c\mu(Q)^{1-p/q}\|a\|^p_{\dot{F}_q^{\alpha,q}} = c.$$

By (3.21) and the fact that if $x\in(2AQ)^c$ and $\varrho(x,y)\le 2^{-k}$, then $D_k(a)(y)=0$ for $2^k>c/\varrho(x,x_Q)$ where c is a constant,

$$II \le \int_{(2AQ)^c}\Big\{\sum_{k\in\mathbb{Z}}(2^{k\alpha}|D_k(a)(x)|)^q\Big\}^{1/q}d\mu(x)$$
$$\le c\int_{(2AQ)^c}\Big\{\sum_{2^k\le c/\varrho(x,x_Q)}2^{k(1+\alpha+\varepsilon)}\mu(Q)^{\alpha+\varepsilon}\Big\}\,d\mu(x)$$

$$\leq c \int_{(2AQ)^c} \mu(Q)^{\alpha+\varepsilon} \varrho(x, x_Q)^{-(1+\alpha+\varepsilon)}\, d\mu(x) \leq c\,,$$

which shows (3.16).

Now it remains to prove the claim (3.21). Set $\eta_Q \in C_0^1(X)$ with $0 \leq \eta_Q \leq 1, \eta_Q(x) = 1$ for $x \in Q$ and $\eta_Q(x) = 0$ for $x \in (2AQ)^c$. Then

$$(3.22)\qquad |D_k(a)(y)| = \Big|\int D_k(y,z)a(z)\, d\mu(z)\Big| = \Big|\int [D_k(y,z) - \widetilde{D}_k]\eta_Q(z)a(z)\, d\mu(z)\Big|$$

by the cancellation condition on a,

(where $\widetilde{D}_k = \frac{1}{\int \eta_Q(z)\, d\mu(z)} \int D_k(y,z)\eta_Q(z)\, d\mu(z)$)

$$\leq \|a\|_{\dot F_q^{\alpha,q}} \|[D_k(y,\cdot) - \widetilde{D}_k]\eta_Q(\cdot)|_{\dot F_{q'}^{-\alpha,q'}}$$
$$\leq \mu(Q)^{1/q-1/p} \|[D_k(y,\cdot) - \widetilde{D}_k]\eta_Q(\cdot)\|_{\dot F_{q'}^{-\alpha,q'}}\,.$$

Set $f(x) = [D_k(y,x) - \widetilde{D}_k]\eta_Q(x)$. By the definition of the Triebel–Lizorkin spaces in (1.12),

$$(3.23)\qquad \|f\|^{q'}_{\dot F_{q'}^{-\alpha,q'}} = \sum_{j\in\mathbb{Z}} \int 2^{-j\alpha q'} |D_j(f)(x)|^{q'}\, d\mu(x)$$
$$= \Big(\sum_{2^{-j}\leq 4Ar} \int_{8AQ} + \sum_{2^{-j}\leq 4Ar} \int_{(8AQ)^c} + \sum_{2^{-j}>4Ar} \int_{8AQ} + \sum_{2^{-j}>4Ar} \int_{(8AQ)^c}\Big) 2^{-j\alpha q'} |D_j(f)(x)|^{q'}\, d\mu(x)$$
$$= I + I\!I + I\!I\!I + IV\,.$$

As in the proof of Lemma (3.1), $I\!I = 0$. To estimate I, note that $|f(x) - f(y)| \leq c2^{k(1+\varepsilon)}\varrho(x,y)^{\varepsilon}$. Thus,

$$|D_j(f)(x)| = \Big|\int D_j(x,y)f(y)\, d\mu(y)\Big| = \Big|\int D_j(x,y)[f(y) - f(x)]\, d\mu(y)\Big|$$
$$\leq c\int |D_j(x,y)|2^{k(1+\varepsilon)}\varrho(x,y)^{\varepsilon}\, d\mu(y) \leq c2^{-j\varepsilon}2^{k(1+\varepsilon)}\,.$$

Hence,

$$I \leq c \sum_{2^{-j}\leq 4Ar} 2^{-j\alpha q'} \int_{8AQ} (2^{-j\varepsilon}2^{k(1+\varepsilon)})^{q'}\, d\mu(x)$$
$$\leq c\mu(Q) \sum_{2^{-j}\leq 4Ar} 2^{-j(\alpha+\varepsilon)q'} 2^{k(1+\varepsilon)q'}$$
$$\leq c\mu(Q) r^{(\alpha+\varepsilon)q'} 2^{k(1+\varepsilon)q'} \qquad \text{since } \varepsilon + \alpha > 0\,.$$

Notice that $|f(x)| = |[D_k(y,x) - \widetilde{D}_k]\chi_Q(x)| \leq c2^{k(1+\varepsilon)}r^{\varepsilon}$. Thus,

$$|D_j(f)(x)| \leq \int_{2AQ} |D_j(x,y)||f(y)|\, d\mu(y) \leq c2^j 2^{k(1+\varepsilon)} r^{\varepsilon}\mu(Q)$$

and, therefore,

$$I\!I\!I \leq c \sum_{2^{-j}>4Ar} 2^{-j\alpha q'} \int_{8AQ} |D_j(f)(x)|^{q'}\, d\mu(x)$$
$$\leq c \sum_{2^{-j}>4Ar} \mu(Q)^{1+q'} 2^{(1-\alpha)jq'} 2^{k(1+\varepsilon)q'} r^{\varepsilon q'}$$
$$\leq c\mu(Q)2^{k(1+\varepsilon)q'} r^{(\alpha+\varepsilon)q'} \qquad \text{since } (1-\alpha)q' > 0\,.$$

Finally, for $x \in (8AQ)^c$ we have

$$|D_j(f)(x)| = \Big|\int_{2AQ} [D_j(x,y) - D_j(x,x_Q)]f(y)\, d\mu(y)\Big|$$
$$\leq c\mu(Q)2^{k(1+\varepsilon)} r^{2\varepsilon} \frac{2^{-j\varepsilon}}{(2^{-j} + \varrho(x,x_Q))^{1+2\varepsilon}}\,.$$

Thus,

$$IV \leq c \sum_{2^{-j}>4Ar} \mu(Q)^{q'} 2^{k(1+\varepsilon)q'} r^{2\varepsilon q'} 2^{-j\alpha q'} \times \int_{\varrho(x,x_Q)\geq cr} \Big[\frac{2^{-j\varepsilon}}{(2^{-j} + \varrho(x,x_Q))^{1+2\varepsilon}}\Big]^{q'}\, d\mu(x)$$
$$\leq c \sum_{2^{-j}>4Ar} \mu(Q)^{q'} 2^{k(1+\varepsilon)q'} r^{2\varepsilon q'} 2^{-j(\alpha+\varepsilon)q'} 2^{j[(1+2\varepsilon)q'-1]}$$

since $(1+2\varepsilon)q' > 1$

$$\leq c\mu(Q) r^{(\alpha+\varepsilon)q'} 2^{k(1+\varepsilon)q'}\,.$$

These estimates yield

$$\|[D_k(y,\cdot) - D_k(y,x_Q)]\eta_Q(\cdot)\|_{\dot F_{q'}^{-\alpha,q'}} \leq c\mu(Q)^{1/q'} r^{\alpha+\varepsilon} 2^{k(1+\varepsilon)}$$

and, hence,

$$|D_k(a)(y)| \leq \|a\|_{\dot F_q^{\alpha,q}} \|[D_k(y,\cdot) - D_k(y,x_Q)]\eta_Q(\cdot)\|_{\dot F_{q'}^{-\alpha,q'}}$$
$$\leq c\mu(Q)^{1/q-1/p+1/q'} r^{\alpha+\varepsilon} 2^{k(1+\varepsilon)} \leq c\mu(Q)^{1-1/p+\alpha+\varepsilon} 2^{k(1+\varepsilon)}\,,$$

which proves (3.21) and, hence, the "if" part of Theorem B.

Now we prove the "only if" part. Suppose that $-\varepsilon < \alpha < \varepsilon$,

$$\max\Big(\frac{1}{1+\alpha+\varepsilon}, \frac{1}{1+\varepsilon}\Big) < p \leq 1 \leq q < \infty\,;$$

and $f\in(M^{(\beta,\gamma)})'$ with $0<\beta,\gamma<\varepsilon$, and $f\in\dot F_p^{\alpha,q}$. We follow the idea in [CF]. Let $\Omega_k=\{x\in X:\widetilde S^\alpha_{q,C_0A}(f)(x)>2^k\}$, where

$$\widetilde S^\alpha_{q,C_0A}(f)(x)=\Big\{\sum_{k\in\mathbb Z}\int\limits_{\varrho(x,y)\le C_0A2^{-k}}2^k(2^{k\alpha}|\widehat D_k(f)(y)|)^q\,d\mu(y)\Big\}^{1/q},$$

C_0 is a constant which will be chosen later and $\widehat D_k$ is as in Theorem (2.4). Set $B_k=\{Q:Q$ a "dyadic cube" in X such that $\mu(Q\cap\Omega_k)\ge\frac12\mu(Q)$ and $\mu(Q\cap\Omega_{k+1})<\frac12\mu(Q)\}$. It is easy to see that for each dyadic cube Q in X there is a unique $k\in\mathbb Z$ such that $Q\in B_k$. For each dyadic cube $Q\in B_k$ there is a unique maximal dyadic cube $Q'\in B_k$ such that $Q\subseteq Q'$. Denote the collection of all maximal dyadic cubes in B_k by Q_k^i, $i\in I_k$, an index set which depends on k (it is possibly finite). We then have

$$\bigcup_{Q:\text{dyadic cube}}Q=\bigcup_k\bigcup_{i\in I_k}\bigcup_{Q\subseteq Q_k^i,\,Q\in B_k}Q. \tag{3.24}$$

Write $D_Q=D_k$ and $\widehat D_Q=\widehat D_k$ for dyadic cubes Q with $\mu(Q)\sim2^{-k}$. Applying the Calderón-type reproducing formula of Theorem (2.4), we obtain

$$\begin{aligned}f(x)&=\sum_{k\in\mathbb Z}D_k\widehat D_k(f)(x)=\sum_{k\in\mathbb Z}\sum_{\mu(Q)\sim2^{-k}}\int_Q D_k(x,y)\widehat D_k(f)(y)\,d\mu(y)\\&=\sum_Q\int_Q D_Q(x,y)\widehat D_Q(f)(y)\,d\mu(y)\\&=\sum_k\sum_i\sum_{Q\subseteq Q_k^i,\,Q\in B_k}\int_Q D_k(x,y)\widehat D_k(f)(y)\,d\mu(y).\end{aligned}\tag{3.25}$$

To obtain an atomic decomposition of f, we need the following lemma.

LEMMA (3.26). *Suppose that B_k is as above. Then there exists a constant c such that*

$$\sum_{Q\in B_k}\int_Q(\mu(Q)^{-\alpha}|\widehat D_k(f)|)^q\,d\mu(y)\le c2^{kq}\mu(\Omega_k).\tag{3.27}$$

Proof. Let $\widetilde\Omega_k=\{x\in X:M(\chi_{\Omega_k})(x)>1/2\}$, where M is the Hardy–Littlewood maximal function. By the Hardy–Littlewood maximal function theorem, $\mu(\widetilde\Omega_k)\le c\mu(\Omega_k)$. Thus,

$$\int\limits_{\widetilde\Omega_k\setminus\Omega_{k+1}}\widetilde S^\alpha_{q,C_0A}(f)^q(x)\,d\mu(x)\le2^{(k+1)q}\mu(\widetilde\Omega_k)\le c2^{kq}\mu(\Omega_k).\tag{3.28}$$

On the other hand,

$$\begin{aligned}&\int\limits_{\widetilde\Omega_k\setminus\Omega_{k+1}}[\widetilde S^\alpha_{q,C_0A}(f)(x)]^q(x)\,d\mu(x)\\&=\sum_j\int_X(2^{j\alpha}|\widehat D(f)(y)|)^q2^j\mu(\{x\in\widetilde\Omega_k\setminus\Omega_{k+1}:\varrho(x,y)\le C_0A2^{-j}\})\,d\mu(y)\\&\ge c\sum_{Q\in B_k}\int_Q(\mu(Q)^{-\alpha}|\widehat D_Q(f)(y)|)^q\mu(Q)^{-1}\\&\qquad\times\mu(\{x\in\widetilde\Omega_k\setminus\Omega_{k+1}:\varrho(x,y)\le C_0A\mu(Q)\})\,d\mu(y).\end{aligned}\tag{3.29}$$

Since if $y\in Q\in B_k$ then $Q\subset\widetilde\Omega_k$, there exists a constant C_0 such that if $Q\in B_k$ then

$$\begin{aligned}\mu(\{x\in\widetilde\Omega_k\setminus\Omega_{k+1}:\varrho(x,y)\le C_0A\mu(Q)\})&\ge\mu((Q\cap\widetilde\Omega_k)\setminus\Omega_{k+1})\\&=\mu(Q\cap\widetilde\Omega_k)-\mu(Q\cap\Omega_{k+1})\ge\mu(Q)-\frac12\mu(Q)=\frac12\mu(Q).\end{aligned}\tag{3.30}$$

Substituting (3.30) into (3.29) together with (3.28) yields (3.27).

Now set

$$\lambda_{k,i}=c\mu(Q_k^i)^{1/p}\Big(\sum_{Q\subseteq Q_k^i,\,Q\in B_k}\int_Q(\mu(Q)^{-\alpha}|\widehat D_Q(f)(y)|)^q\,d\mu(y)\Big)^{1/q}$$

and

$$\begin{aligned}a_{k,i}&=c^{-1}\mu(Q_k^i)^{1/q}\Big(\sum_{Q\subseteq Q_k^i,\,Q\in B_k}\int_Q(\mu(Q)^{-\alpha}|\widehat D_Q(f)(y)|)^q\,d\mu(y)\Big)^{-1/q}\\&\quad\times\sum_{Q\subseteq Q_k^i,\,Q\in B_k}\int_Q D_Q(x,y)\widehat D_Q(f)(y)\,d\mu(y).\end{aligned}$$

By Lemma (3.26),

$$\begin{aligned}&\sum_{Q\subseteq Q_k^i,\,Q\in B_k}\int_Q(\mu(Q)^{-\alpha}|\widehat D_Q(f)(y)|)^q\,d\mu(y)\\&\le\sum_{Q\in B_k}\int_Q(\mu(Q)^{-\alpha}|\widehat D_Q(f)(y)|)^q\,d\mu(y)\le c2^{qk}|\Omega_k|<\infty\end{aligned}$$

which shows $\lambda_{k,i}$ and $a_{k,i}$ are well defined. Thus, by (3.25),

$$f(x)=\sum_{k,i}\lambda_{k,i}a_{k,i}.$$

It is easy to see that the $a_{k,i}$ satisfy the condition (i) of Definition (1.20). To check (ii), by duality of $\dot F_q^{\alpha,q}$ for $1<q<\infty$,

$$\|a_{k,i}\|_{\dot F_p^{\alpha,q}} = \sup_{\|g\|_{\dot F_{q'}^{-\alpha,q'}}\le 1} |\langle a_{k,i}, g\rangle|$$

$$\le \sup_{\|g\|_{\dot F_{q'}^{-\alpha,q'}}\le 1} c^{-1}\Big\{\sum_{Q\subseteq Q_k^i, Q\in B_k}\int_Q |\widehat D_Q(f)(y)||D_Q^*(g)(y)|\,d\mu(y)\,\mu(Q_k^i)^{1/q}\Big\}$$

$$\times\Big\{\Big(\sum_{Q\subseteq Q_k^i, Q\in B_k}\int_Q (\mu(Q)^{-\alpha}|\widehat D_Q(f)(y)|)^q\,d\mu(y)\Big)^{-1/q}\Big\}$$

$$\le \sup_{\|g\|_{\dot F_{q'}^{-\alpha,q'}}\le 1} c^{-1}\mu(Q_k^i)^{1/q}\Big(\sum_{Q\subseteq Q_k^i}\int_Q (\mu(Q)^{\alpha}|D_Q^*(g)(y)|)^{q'}\,d\mu(y)\Big)^{1/q'}$$

by Hölder's inequality

$$\le \mu(Q_k^i)^{1/q-1/p}$$

since

$$\Big(\sum_{Q\subseteq Q_k^i}\int_Q (\mu(Q)^{\alpha}|\widehat D_Q(g)(y)|)^{q'}\,d\mu(y)\Big)^{1/q'}$$

$$\le \Big(\sum_{Q:\text{dyadic}}\int_Q (\mu(Q)^{\alpha}|\widehat D_Q(g)(y)|)^{q'}\,d\mu(y)\Big)^{1/q'} \le \|g\|_{\dot F_{q'}^{-\alpha,q'}} \le c\,.$$

If $q=1$, then

$$\|a_{k,i}\|_{\dot F_1^{\alpha,1}}$$

$$= \sum_j 2^{j\alpha}\|D_j(a_{k,i})\|_1$$

$$\le c^{-1}\mu(Q_k^i)^{1-1/p}\sum_{Q\subseteq Q_k^i, Q\in B_k}\int_Q \sum_j 2^{j\alpha}\|D_jD_Q\|_1|\widehat D_Q(f)(y)|\,d\mu(y)$$

$$\times\Big(\sum_{Q\subseteq Q_k^i, Q\in B_k}\int_Q \mu(Q)^{-\alpha}|\widehat D_Q(f)(y)|\,d\mu(y)\Big)^{-1}$$

$$\le c^{-1}\mu(Q_k^i)^{1-1/p}\Big\{\sum_{Q\subseteq Q_k^i, Q\in B_k}\int_Q \sum_j 2^{j\alpha}2^{-|j-\log_2\mu(Q)^{-1}|}|\widehat D_Q(f)(y)|\,d\mu(y)\Big\}$$

$$\times\Big\{\sum_{Q\subseteq Q_k^i, Q\in B_k}\int_Q \mu(Q)^{-\alpha}|\widehat D_Q(f)(y)|\,d\mu(y)\Big\}^{-1}$$

$$\le \mu(Q_k^i)^{1-1/p}\sum_{Q\subseteq Q_k^i, Q\in B_k}\int_Q \mu(Q)^{-\alpha}|\widehat D_Q(f)(y)|\,d\mu(y)$$

$$\times\Big(\sum_{Q\subseteq Q_k^i, Q\in B_k}\int_Q \mu(Q)^{-\alpha}|\widehat D_Q(f)(y)|\,d\mu(y)\Big)^{-1}$$

$$= \mu(Q_k^i)^{1-1/p}\,.$$

To see that the $a_{k,i}$ satisfy the cancellation condition (iii) of (1.20), suppose that $\operatorname{supp} a_{k,i}\subseteq B(x_0,r)$ and let g, $g_{\widetilde B}$ and $\eta_{\widetilde B}$ be as in (1.20)(iii). Notice first that

$$\sum_{Q\subseteq Q_k^i, Q\in B_k}\Big|\Big\langle \int_Q D_Q(\cdot,y)\widehat D_Q(f)(y)\,d\mu(y), g\Big\rangle\Big|$$

$$\le \Big(\sum_{Q\subseteq Q_k^i, Q\in B_k}\int_Q (\mu(Q)^{\alpha}|D_Q^*(g)(y)|)^{q'}\,d\mu(y)\Big)^{1/q'}$$

$$\times\Big(\sum_{Q\subseteq Q_k^i, Q\in B_k}\int_Q (\mu(Q)^{-\alpha}|\widehat D_Q(f)(y)|)^{q}\,d\mu(y)\Big)^{1/q} \le c$$

where the last inequality follows from Lemma (3.26) and the fact that g belongs to $\dot F_{q'}^{-\alpha,q'}$ if $g\in M^{(\beta,\gamma)}$ with $\max(0,-\alpha)<\beta<\varepsilon$ and $\max(0,\alpha)<\gamma<\varepsilon$. This shows

$$\Big\langle \sum_{Q\subseteq Q_k^i, Q\in B_k}\int_Q D_Q(\cdot,y)\widehat D_Q(f)(y)\,d\mu(y), g\Big\rangle$$

$$= \sum_{Q\subseteq Q_k^i, Q\in B_k}\Big\langle \int_Q D_Q(\cdot,y)\widehat D_Q(f)(y)\,d\mu(y), g\Big\rangle$$

and, hence,

$$\langle a_{k,i}, g\rangle = c^{-1}\mu(Q_k^i)^{1/q}\Big(\sum_{Q\subseteq Q_k^i, Q\in B_k}\int_Q (\mu(Q)^{-\alpha}|\widehat D_Q(f)(y)|)^{q}\,d\mu(y)\Big)^{-1/q}$$

$$\times\Big(\sum_{Q\subseteq Q_k^i, Q\in B_k}\Big\langle \int_Q D_Q(\cdot,y)\widehat D_Q(f)(y)\,d\mu(y), g\Big\rangle\Big)$$

$$= c^{-1}\mu(Q_k^i)^{1/q}\Big(\sum_{Q\subseteq Q_k^i, Q\in B_k}\int_Q (\mu(Q)^{-\alpha}|\widehat D_Q(f)(y)|)^{q}\,d\mu(y)\Big)^{-1/q}$$

$$\times\Big(\sum_{Q\subseteq Q_k^i, Q\in B_k}\int_Q \langle D_Q(\cdot,y), g\rangle\widehat D_Q(f)(y)\,d\mu(y)\Big)$$

$$= c^{-1}\mu(Q_k^i)^{1/q}\Big(\sum_{Q\subseteq Q_k^i, Q\in B_k}\int_Q (\mu(Q)^{-\alpha}|\widehat D_Q(f)(y)|)^{q}\,d\mu(y)\Big)^{-1/q}$$

$$\times \sum_{Q\subseteq Q_k^i, Q\in B_k} \int_Q \langle D_Q(\cdot,y), [g-g_{\widetilde{B}}]\eta_{\widetilde{B}}\rangle \widehat{D}_Q(f)(y)\,d\mu(y)$$

since supp $D_Q(\cdot,y)\subseteq \widetilde{B}$ and $\int D_Q(\cdot,y)\,d\mu(y)=0$

$$= c^{-1}\mu(Q_k^i)^{1/q}\Big(\sum_{Q\subseteq Q_k^i, Q\in B_k}\int_Q (\mu(Q)^{-\alpha}|\widehat{D}_Q(f)(y)|)^q\,d\mu(y)\Big)^{-1/q}$$

$$\times \sum_{Q\subseteq Q_k^i, Q\in B_k} \Big\langle \int_Q D_Q(x,y)\widehat{D}_Q(f)(y)\,d\mu(y), [g-g_{\widetilde{B}}]\eta_{\widetilde{B}}\Big\rangle$$

$$= \langle a_{k,i}, [g-g_{\widetilde{B}}]\eta_{\widetilde{B}}\rangle$$

since $[g-g_{\widetilde{B}}]\eta_{\widetilde{B}}\in M^{(\beta,\gamma)}$ with $\max(0,-\alpha)<\beta<\varepsilon$ and $\max(0,\alpha)<\gamma<\varepsilon$.

Finally, we check the condition $\sum_{k,i}|\lambda_{k,i}|^p<\infty$:

$$\sum_{k,i}|\lambda_{k,i}|^p$$

$$= \sum_{k,i}\mu(Q_k^i)^{1-p/q}\Big(\sum_{Q\subseteq Q_k^i, Q\in B_k}\int_Q (\mu(Q)^{-\alpha}|\widehat{D}_Q(f)(y)|)^q\,d\mu(y)\Big)^{p/q}$$

$$\le \sum_k \Big(\sum_i \mu(Q_k^i)\Big)^{1-p/q}\Big(\sum_i\sum_{Q\subseteq Q_k^i, Q\in B_k}\int_Q (\mu(Q)^{-\alpha}|\widehat{D}_Q(f)(y)|)^q\,d\mu(y)\Big)^{p/q}$$

by Hölder's inequality

$$\le c\sum_k \mu(\Omega_k)^{1-p/q}\Big(\sum_{Q\in B_k}\int_Q (\mu(Q)^{-\alpha}|\widehat{D}_Q(f)(y)|)^q\,d\mu(y)\Big)^{p/q}$$

since $\mu(\Omega_k\cap Q_k^i)\ge \frac12\mu(Q_k^i)$ and $Q_k^i, i\in I_k$, are disjoint

$$\le c\sum_k \mu(\Omega_k)^{1-p/q}2^{pk}\mu(\Omega_k)^{p/q} \quad \text{by Lemma (3.26)}$$

$$\le c\sum_k 2^{kp}\mu(\Omega_k)\le c\|\widetilde{S}^{\alpha}_{q,C_0A}(f)\|_p$$

$$\le c\|S^{\alpha}_q(f)\|_p \quad \text{by Theorem A,}$$

which shows that f has an atomic decomposition; this completes the proof of Theorem B.

Now we prove Theorem C.

Proof of Theorem C. In [MS2], it was shown that if $f\in H^p$ for $1/(1+\varepsilon)<p\le 1$, then $f=\sum_k\lambda_k a_k$, where the a_k are $(p,\infty,0)$ atoms and $\sum_k|\lambda_k|^p<\infty$. It is easy to see that $(p,\infty,0)$ atoms are $(p,2,0)$ atoms (see [CW]). Thus, by Theorem B and the remark following Definition (1.20), if $f\in H^p$ then $f\in \dot F_p^{0,2}$ for $1/(1+\varepsilon)<p\le 1$. Conversely, if $f\in \dot F_p^{0,2}$ for $1/(1+\varepsilon)<p\le 1$, then $f=\sum_k\lambda_k a_k$, where the a_k are $(p,2,0)$ atoms and $\sum_k|\lambda_k|^p<\infty$ by Theorem B and the remark following Definition (1.20). It can be shown that if $f=\sum_k\lambda_k a_k$, where the a_k are $(p,2,0)$ atoms and $\sum_k|\lambda_k|^p<\infty$, then $f\in H^p$ for $1/(1+\varepsilon)<p\le 1$. See [MS2] for the proof that if $f=\sum_k\lambda_k a_k$, where the a_k are $(p,\infty,0)$ atoms and $\sum_k|\lambda_k|^p<\infty$, then $f\in H^p$ for $1/(1+\varepsilon)<p\le 1$. We leave the details to the reader. This completes the proof of Theorem C.

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