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# On quasi-multipliers

by

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**Abstract.** A quasi-multipliers is a generalization of the notion of a left (right, double) multiplier. The first systematic account of the general theory of quasi-multipliers on a Banach algebra with a bounded approximate identity was given in a paper by McKennon in 1977. Further developments have been made in more recent papers by Vasudevan and Goel, Kassem and Rowlands, and Lin. In this paper we consider the quasi-multipliers of algebras not hitherto considered in the literature. In particular, we study the quasi-multipliers of $A^*$-algebras, of the algebra of compact operators on a Banach space, and of the Pedersen ideal of a $C^*$-algebra. We also consider the strict topology on the quasi-multiplier space $QM(A)$ of a Banach algebra $A$ with a bounded approximate identity. We prove that, if $M_l(A)$ (resp. $M_r(A)$) denotes the algebra of left (right) multipliers on $A$, then $M_l(A)+M_r(A)$ is strictly dense in $QM(A)$, thereby generalizing a theorem due to Lin.

**1. Introduction.** A quasi-multiplier is a generalization of the notion of a left (right, double) multiplier, and was first introduced by Akemann and Pedersen in ([1], §4). The first systematic account of the general theory of quasi-multipliers on a Banach algebra with a bounded approximate identity was given in a paper by McKennon [14] in 1977. Further developments have been made as a result of more recent contributions by Vasudevan and Goel [21], [22], Kassem and Rowlands [11], and Lin [13]. In this paper we study the quasi-multipliers of algebras not hitherto considered in the literature.

We begin by outlining the necessary background results on quasi-multipliers and then proceed to consider the quasi-multipliers $QM(A)$ of an $A^*$-algebra $A$; in particular, we improve a result due to Vasudevan and Goel ([21], Theorem 3.4) on extending a quasi-multiplier from an $A^*$-algebra to its auxiliary norm completion. The result enables us to define an "auxiliary" norm on $QM(A)$ and, in the special case when $QM(A)$ is a Banach algebra, we prove that under certain conditions $QM(A)$ is an $A^*$-algebra. In the literature topologies other than the norm topology have been defined on the quasi-multiplier space and properties established for the resulting locally

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convex spaces (see, for example, [11] and [13]). In §3 we consider the strict topology on the quasi-multiplier space of a Banach algebra $A$ with a bounded approximate identity. In particular, we prove that, if $M_l(A)$ (resp. $M_r(A)$) denotes the algebra of left (right) multipliers on $A$, then $M_l(A)+M_r(A)$ is strictly dense in $QM(A)$, thereby generalizing a theorem due to Lin ([13], Theorem 9.3).

In §4, $A$ is the algebra of approximable operators on a Banach space $X$ (that is, operators that can be approximated, in the operator norm, by operators of finite rank), and our investigations lead to characterizations for $QM(A)$ and $QM(A^{**})$. For example, if $X^*$ has the bounded approximation property then $QM(A)$ is topologically isomorphic to $M_r(A)$ and, if in addition $X^*$ has the Radon–Nikodym property, then $QM(A^{**})$ is topologically isomorphic to $M_l(A^{**})$. In the final section we study the quasi-multipliers of the Pedersen ideal $K_A$ of a $C^*$-algebra $A$. In this case the quasi-multipliers are not necessarily continuous. Nevertheless, the space $\delta(K_A)$ of quasi-multipliers on $K_A$, with the quasi-strict topology $\gamma$, has a number of interesting properties. In particular, we show that $\delta(K_A)$ is $\gamma$-complete, and, for certain $C^*$-algebras, $K_A$ is $\gamma$-dense in $\delta(K_A)$. We also establish a characterization for the dual space $(\delta(K_A),\gamma)^*$.

**2. Preliminaries and quasi-multipliers of $A^*$-algebras.** Let $A$ be a Banach algebra. A mapping $m: A\times A\to A$ is said to be a *quasi-multiplier* on $A$ if

$$m(ab,c)=am(b,c) \quad\text{and}\quad m(a,bc)=m(a,b)c \tag{2.1}$$

for all $a,b,c\in A$. Let $QM(A)$ denote the set of all bilinear jointly continuous quasi-multipliers on $A$. If $A$ is a Banach algebra with a bounded two-sided approximate identity (abbreviated to a.i. in the sequel), then every quasi-multiplier belongs to $QM(A)$ ([14], Theorem 1), and $QM(A)$ is a Banach space with respect to the norm

$$\|m\|=\sup\{\|m(a,b)\|: a,b\in A,\ \|a\|\le 1,\ \|b\|\le 1\}$$

([14], Theorem 2). If the products $a\circ m$ and $m\circ a$ are defined by

$$(a\circ m)(x,y)=m(xa,y),\qquad (m\circ a)(x,y)=m(x,ay)$$

($m\in QM(A)$, $x,y,a\in A$), then $QM(A)$ becomes a Banach $A$-module.

A mapping $T: A\to A$ is called a *left* (resp. *right*) *multiplier* on $A$ if $T(ab)=(Ta)b$ (resp. $T(ab)=a(Tb)$) for all $a,b$ in $A$, and $T$ is called a *multiplier* if it is both a left and right multiplier on $A$. Let $M_0(A)$ (resp. $M_l(A), M_r(A)$) be the set of all continuous linear (left, right) multipliers on $A$. Then both $M_l(A)$ and $M_r(A)$ are closed subalgebras of the Banach algebra $\mathcal{L}(A)$ of all continuous linear operators on $A$ and $M_0(A)$ is a closed commutative subalgebra of $\mathcal{L}(A)$. A pair $(S,T)$ of mappings



$S,T: A\to A$ is said to be a *double multiplier* on $A$ if $aSb=(Ta)b$ for all $a,b\in A$. If $M(A)$ denotes the set of all continuous linear double multipliers on $A$, then, for each $(S,T)\in M(A)$, we have $S\in M_l(A)$, $T\in M_r(A)$, and $\|(S,T)\|=\max(\|S\|,\|T\|)$ defines a norm on $M(A)$ relative to which it is a Banach algebra. For further details on the algebras of left, right and double multipliers on a Banach algebra we refer the reader to ([20], §3).

Each of the linear mappings

$$\Phi: A\to QM(A),\qquad \lambda: M_l(A)\to QM(A),$$
$$\varrho: M_r(A)\to QM(A),\qquad \Psi: M(A)\to QM(A),$$

defined respectively by

$$(\Phi(a))(x,y)=xay,\qquad (\lambda(S))(x,y)=xSy,$$
$$(\varrho(T))(x,y)=(Tx)y,\qquad \Psi(S,T)=\lambda(S),$$

is a norm decreasing embedding; if, in addition, $A$ has a minimal a.i. $\{e_\alpha:\alpha\in I\}$ (that is, $\|e_\alpha\|\le 1$ for all $\alpha\in I$), then the mappings are isometric.

A bounded a.i. $\{e_\alpha:\alpha\in I\}$ in $A$ is said to be an *ultra-approximate identity* if, for all $m\in QM(A)$ and $a\in A$, the nets $\{m(a,e_\alpha)\}$ and $\{m(e_\alpha,a)\}$ are Cauchy ([14], p. 110). In this case $\lambda$ and $\varrho$ are surjective; for, if $m\in QM(A)$, the mappings $S,T$ on $A$ defined by

$$Sa=\lim_\alpha m(e_\alpha,a),\qquad Ta=\lim_\alpha m(a,e_\alpha),$$

belong to $M_l(A)$ and $M_r(A)$ respectively, and $\lambda(S)=m=\varrho(T)$. Under these circumstances we can use either of the isomorphisms $\lambda$ or $\varrho$ to define multiplication in $QM(A)$. Thus, for example, the equation

$$(m_1\odot m_2)(a,b)=m_1(a,\lim_\alpha m_2(e_\alpha,b))$$

defines a product in $QM(A)$, and, if we assume that the ultra-approximate identity is minimal, then $QM(A)$ becomes a Banach algebra, with $\lambda$ (resp. $\varrho$) an isometric algebraic isomorphism of $M_l(A)$ ($M_r(A)$) onto $QM(A)$.

A bilinear mapping $m: A\times A\to A$ can be extended in two natural ways to a bilinear map $A^{**}\times A^{**}\to A^{**}$; we outline the construction in stages, as follows:

(i) $m^*: A^*\times A\to A^*$, $\quad\langle b, m^*(f,a)\rangle=\langle m(a,b),f\rangle$,
$m^{**}: A^{**}\times A^*\to A^*$, $\quad\langle a, m^{**}(F,f)\rangle=\langle m^*(f,a),F\rangle$,
$m^{***}: A^{**}\times A^{**}\to A^{**}$, $\quad\langle f, m^{***}(F,G)\rangle=\langle m^{**}(G,f),F\rangle$;

(ii) $(m')^*: A\times A^*\to A^*$, $\quad\langle b,(m')^*(a,f)\rangle=\langle m(b,a),f\rangle$,
$(m')^{**}: A^*\times A^{**}\to A^*$, $\quad\langle a,(m')^{**}(f,F)\rangle=\langle (m')^*(a,f),F\rangle$,
$(m')^{***}: A^{**}\times A^{**}\to A^{**}$, $\quad\langle f,(m')^{***}(F,G)\rangle=\langle (m')^{**}(f,F),G\rangle$

($a, b \in A, f \in A^*, F, G \in A^{**}$). It is easy to check that when $m$ is continuous then $m^*, m^{**}, m^{***}, (m')^*, (m')^{**}, (m')^{***}$ are continuous. Moreover, by routine calculations we can show that, if $m \in QM(A)$, then $m^{***}$ and $(m')^{***}$ are quasi-multipliers on $(A^{**}, \cdot)$ and $(A^{**}, *)$ respectively, where $\cdot$ (resp. $*$) denotes the first (second) Arens product on $A^{**}$. (For the definitions of the Arens products on $A^{**}$, we refer the reader to ([20], §4); in particular, we note that $(A^{**}, \cdot)$ (resp. $L(A^{**}, *)$) has a right (left) identity if and only if $A$ has a bounded right (left) approximate identity ([3], p. 146, Proposition 9)). It is also straightforward to show that $m^{***}(\widehat{a}, \widehat{b}) = \widehat{m(a, b)}$ for all $a, b \in A$.

The following is a simpler proof of ([21], Theorem 2.1).

THEOREM 2.1. *Let $A$ be a Banach algebra and suppose that $E$ (resp. $I$) is a right (left) identity in $A^{**}$ with respect to the first (second) Arens product. Then the mapping $m \to m^{***}(E, I)$ is a topological linear isomorphism of $QM(A)$ into $A^{**}$, with*

$$\|m\| \leq \|m^{***}(E, I)\| \leq \|m\| \, \|E\| \, \|I\| .$$

Proof. It is clear that the mapping $m \to m^{**}(E, I)$ is linear. For any $a, b \in A$,

$$\|m(a, b)\| = \|\widehat{m(a, b)}\| = \|m^{***}(\widehat{a}, \widehat{b})\| ,$$

and since $m^{***}$ is a quasi-multiplier on $A^{**}$ we have

$$\|m(a, b)\| = \|a \cdot m^{***}(E, I) \cdot b\| \leq \|m^{***}(E, I)\| \, \|a\| \, \|b\| ,$$

which implies that

$$\|m\| \leq \|m^{***}(E, I)\| .$$

The right side inequality follows from the fact that $m^{***} \in QM(A^{**}, \cdot)$ and that $\|m^{***}\| \leq \|m\|$. ∎

COROLLARY 2.2. *If $\|E\| = \|I\| = 1$, then the mapping $m \to m^{***}(E, I)$ is isometric.*

Similarly we can show that $m \to m^{***}(I, E)$ is a topological linear isomorphism of $QM(A)$ into $A^{**}$. The following generalizes a result due to Vasudevan and Goel ([22], Corollary 3.1).

THEOREM 2.3. *Let $A$ be a Banach algebra. The image of $\Phi(A)$ under the mapping $m \to m^{***}(I, E)$ is $\widehat{A}$.*

Proof. We show that, for each $a \in A$, $(\Phi(a))^{***}(I, E) = \widehat{a}$. Since $E$ (resp. $I$) is a right (left) identity in $(A^{**}, \cdot)$ (resp. $(A^{**}, *)$), $E$ (resp. $I$) is the weak*-limit of a bounded right (left) a.i. in $A$. Thus, for any $f$ in $A^*$,



$$\begin{aligned}\langle f, (\Phi(a))^{***}(I, E)\rangle &= \langle(\Phi(a))^{**}(E, f), I\rangle = \lim_\beta \langle \widetilde{e}_\beta, (\Phi(a))^{**}(E, f)\rangle \\ &= \lim_\beta \langle(\Phi(a))^*(f, \widetilde{e}_\beta), E\rangle = \lim_\alpha \lim_\beta \langle e_\alpha, (\Phi(a))^*(f, \widetilde{e}_\beta)\rangle \\ &= \lim_\alpha \lim_\beta \langle \widetilde{e}_\beta a e_\alpha, f\rangle = \langle a, f\rangle ,\end{aligned}$$

which implies that $(\Phi(a))^{***}(I, E) = \widehat{a}$, as required. ∎

Let $A^{**}_\sigma = \{F \in A^{**} : a \cdot F \cdot b \in A \text{ for all } a, b \in A\}$. It is easy to show that $A^{**}_\sigma$ is a closed subspace of $A^{**}$ and the equation

$$\widehat{\sigma(F)(a, b)} = a \cdot F \cdot b \quad (a, b \in A, \; F \in A^{**}_\sigma)$$

defines a norm decreasing linear mapping $\sigma$ of $A^{**}_\sigma$ into $QM(A)$. The kernel of $\sigma$ is $(AA^*A)^\perp$. For the sake of completeness we give the following properties of $\sigma$.

LEMMA 2.4 (cf. [21], Theorem 2.4). *Let $A$ be a Banach algebra and suppose that $E$ (resp. $I$) is a right (left) identity in $A^{**}$ with respect to the first (second) Arens product. Then*

(i) *$F \in A^{**}_\sigma$ if and only if there exist an $m$ in $QM(A)$ and a $G \in \ker \sigma$ such that $F = m^{***}(E, I) + G$, and*

(ii) *$\sigma$ maps $A^{**}_\sigma$ onto $QM(A)$.*

Proof. (i) Suppose that $F \in A^{**}_\sigma$. Then the mapping $(\widehat{a}, \widehat{b}) \to a \cdot F \cdot b$ defines an element of $QM(\widehat{A})$ and so, since $QM(A)$ and $QM(\widehat{A})$ are isomorphic (via the correspondence $m \to m^{***}|_{\widehat{A}}$), there exists an $m \in QM(A)$ such that $m^{***}(\widehat{a}, \widehat{b}) = a \cdot F \cdot b$. Thus, for any $f \in A^*$,

$$\langle f, a \cdot m^{***}(E, I) \cdot b\rangle = \langle f, m^{***}(\widehat{a}, \widehat{b})\rangle = \langle f, a \cdot F \cdot b\rangle ,$$

which implies that $a \cdot (m^{***}(E, I) - F) \cdot b = 0$; that is, $m^{***}(E, I) - F \in \ker \sigma$, and so $F = m^{***}(E, I) + G$ for some $G \in \ker \sigma$, as required.

On the other hand, suppose that $F \in A^{**}$ has the representation $F = m^{***}(E, I) + G$ for some $G \in \ker \sigma$ and $m \in QM(A)$. Then

$$a \cdot F \cdot b = m^{***}(\widehat{a}, \widehat{b}) + a \cdot G \cdot b = m^{***}(\widehat{a}, \widehat{b})$$

since $G \in \ker \sigma$; that is, $F \in A^{**}_\sigma$.

(ii) Let $m$ be any element of $QM(A)$. Then $m^{***}(E, I) \in A^{**}_\sigma$ and it is easy to show that $\sigma(m^{***}(E, I)) = m$; that is, $\sigma$ is a surjection. ∎

We now turn our attention to Banach $*$-algebras. If $A$ is a Banach $*$-algebra, then we can define an involution in $QM(A)$ by setting

$$m^*(a, b) = (m(b^*, a^*))^* .$$

It is clear that, if $a \to a^*$ is continuous in $A$, then $m \to m^*$ is a continuous mapping in $QM(A)$.

THEOREM 2.5. *Let $A$ be a Banach $*$-algebra and suppose that $E$ (resp. $I$) is a right (left) identity for the first (second) Arens product on $A^{**}$. If, in addition, $(AA^*A)^\perp = \{0\}$, then $m \to m^{***}(E,I)$ is a continuous linear $*$-isomorphism of $QM(A)$ into $A^{**}$.*

Proof. In view of Theorem 2.1 it is enough to prove that $m^* \to (m^{***}(E,I))^*$. (In this proof we are using the $*$-notation in different senses but it should not give rise to confusion.) First it is easy to show that the identities $\langle a, f^*\rangle = \overline{\langle a^*, f\rangle}$, $\langle f, F^*\rangle = \overline{\langle f^*, F\rangle}$ ($a \in A, f \in A^*, F \in A^{**}$) define involutions in $A^*$ and $A^{**}$ respectively. A straightforward application of the above identities enables us to prove the following:

$$(b * f \cdot a)^* = a^* * f^* \cdot b^*, \qquad a \cdot F^* \cdot b = (b^* \cdot F \cdot a^*)^* .$$

Thus, since $m^*$ is a quasi-multiplier on $A$, $(m^*)^{***}$ is a quasi-multiplier on $A^{**}$, and so

$$\begin{aligned} a \cdot (m^*)^{***}(E,I) \cdot b &= (m^*)^{***}(\widehat{a}, \widehat{b}) = \widehat{m^*(a,b)} \\ &= (\widehat{m(b^*, a^*)})^* = (m^{***}(\widehat{b^*}, \widehat{a^*}))^* \\ &= (b^* \cdot m^{***}(E,I) \cdot a^*)^* = a \cdot (m^{***}(E,I))^* \cdot b . \end{aligned}$$

It follows that $(m^*)^{***}(E,I) - (m^{***}(E,I))^* \in (AA^*A)^\perp$, and so we have $(m^*)^{***}(E,I) = (m^{***}(E,I))^*$, as required. ■

The following is an improvement of a result given by Vasudevan and Goel in ([21], Theorem 3.4).

THEOREM 2.6. *Let $A$ be an $A^*$-algebra with a bounded a.i. and let $\mathcal{U}$ be its auxiliary norm completion. Then, for each $m \in QM(A)$, there exists a unique $\overline{m}$ in $QM(\mathcal{U})$ such that $m = \overline{m}|_{A\times A}$.*

Proof. Let $|\cdot|$ denote the auxiliary norm in $A$. By ([19], Corollary 4.1.16) there exists a positive number $\beta$ such that $|a| \le \beta\|a\|$ for all $a \in A$. It follows that, for each $g \in \mathcal{U}^*$, the restriction of $g$ to $A$, $g_A$ say, is an element of $(A, \|\cdot\|)^*$ (abbreviated to $A^*$ in the sequel), with $\|g_A\| \le \beta|g|$. Thus, if $F \in A^{**}$, the functional $\overline{F}$ on $\mathcal{U}^*$ defined by $\langle g, \overline{F}\rangle = \langle g_A, F\rangle$ is an element of $\mathcal{U}^{**}$.

Let $m$ be any element of $QM(A)$ and let $G = m^{***}(E,E)$, where $E$ is a right identity for the first Arens product and a left identity for the second. Now $G \in A^{**}_\sigma$. We show that $\overline{G} \in \mathcal{U}^{**}_\sigma$, where $\mathcal{U}^{**}_\sigma$ is defined in an analogous way to $A^{**}_\sigma$. First we prove that, for $a, b \in A$,

$$\pi_{\mathcal{U}}(\sigma(G)(a,b)) = \pi_{\mathcal{U}}(a) \cdot \overline{G} \cdot \pi_{\mathcal{U}}(b) , \tag{2.2}$$

where $\pi_{\mathcal{U}}$ denotes the canonical embedding of $\mathcal{U}$ in $\mathcal{U}^{**}$. For any $g \in \mathcal{U}^*$,

$$\begin{aligned} \langle g, \pi_{\mathcal{U}}(\sigma(G)(a,b))\rangle &= \langle \sigma(G)(a,b), g\rangle = \langle \sigma(G)(a,b), g_A\rangle \\ &= \langle g_A, a \cdot G \cdot b\rangle = \langle b * g_A \cdot a, G\rangle . \end{aligned} \tag{2.3}$$



On the other hand,

$$\langle g, \pi_{\mathcal{U}}(a) \cdot \overline{G} \cdot \pi_{\mathcal{U}}(b)\rangle = \langle (b * g \cdot a)_A, G\rangle ,$$

and since $(b * g \cdot a)_A = b * g_A \cdot a$, we have

$$\langle g, \pi_{\mathcal{U}}(a) \cdot \overline{G} \cdot \pi_{\mathcal{U}}(b)\rangle = \langle b * g_A \cdot a, G\rangle . \tag{2.4}$$

Thus (2.2) follows from (2.3) and (2.4).

Let $u, v \in \mathcal{U}$. Since $A$ is $|\cdot|$-dense in $\mathcal{U}$, there exist sequences $\{a_n\}$, $\{b_m\}$ in $A$ such that $a_n \to u$, $b_m \to v$, with respect to the auxiliary norm on $\mathcal{U}$. Thus

$$\lim_{n,m} \pi_{\mathcal{U}}(a_n) \cdot \overline{G} \cdot \pi_{\mathcal{U}}(b_m) = \pi_{\mathcal{U}}(u) \cdot \overline{G} \cdot \pi_{\mathcal{U}}(v) ,$$

and so by (2.1),

$$\pi_{\mathcal{U}}(u) \cdot \overline{G} \cdot \pi_{\mathcal{U}}(v) = \lim_{n,m} \pi_{\mathcal{U}}(\sigma(G)(a_n, b_m)) .$$

It follows that $\pi_{\mathcal{U}}(u) \cdot \overline{G} \cdot \pi_{\mathcal{U}}(v) \in \pi_{\mathcal{U}}(\mathcal{U})$; that is, $\overline{G} \in \mathcal{U}^{**}_\sigma$, as required.

Let $\overline{m} = \sigma(\overline{G})$. Then $\overline{m} \in QM(\mathcal{U})$, and, for $a, b \in A$,

$$\begin{aligned} \pi_{\mathcal{U}}(\overline{m}(a,b)) &= \pi_{\mathcal{U}}(a) \cdot \overline{G} \cdot \pi_{\mathcal{U}}(b) \\ &= \pi_{\mathcal{U}}(\sigma(G)(a,b)) = \pi_{\mathcal{U}}(m(a,b)) , \end{aligned}$$

which implies that $\overline{m}|_{A\times A} = m$. The uniqueness of $m$ follows immediately from the fact that $A$ is auxiliary norm dense in $\mathcal{U}$. ■

The above result enables us to make the following.

DEFINITION 2.7. For each $m \in QM(A)$, we define the "auxiliary" norm on $QM(A)$ by

$$|m| = |\overline{m}| ;$$

our use of the terminology will be justified later.

Before we make a further study of the space $(QM(A), |\cdot|)$ we require some results on the double multipliers of an $A^*$-algebra; the theorems proved are variants of ([12], Theorems 3.3–3.7).

THEOREM 2.8. *Let $A$ be an $A^*$-algebra with a bounded a.i. and let $\mathcal{U}$ be its auxiliary norm completion. Then each $(S,T) \in M(A)$ has a unique extension to a double multiplier $(S', T') \in M(\mathcal{U})$.*

Proof. We first show that each $S \in M_l(A)$ has a unique extension to an element $S' \in M_l(\mathcal{U})$. With $E$ as in the proof of Theorem 2.6, let $F = S^{**}E$. For $g \in \mathcal{U}^*$, the equation $\langle g, \overline{F}\rangle = \langle g_A, F\rangle$, where $g_A$ denotes the restriction of $g$ to $A$, defines an element $\overline{F} \in \mathcal{U}^{**}$. Moreover, $\overline{F} \cdot a \in \pi_{\mathcal{U}}(A)$ ($a \in A$); for, $\overline{F} \cdot a = \overline{F \cdot a}$ and since $F \cdot a = \widehat{Sa}$, it follows that $\overline{F} \cdot a \in \pi_{\mathcal{U}}(A)$. Since

$\mathcal{U}$ is the $|\cdot|$-closure of $A$ the above implies that $\overline{F}\cdot u \in \pi_{\mathcal{U}}(\mathcal{U})$ for all $u \in \mathcal{U}$. This enables us to define a mapping $S' : \mathcal{U} \to \mathcal{U}$ by

$$\pi_{\mathcal{U}}(S'u) = \overline{F}\cdot \pi_{\mathcal{U}}(u) \quad (u \in \mathcal{U}). \tag{2.5}$$

It is straightforward to show that $S' \in M_l(\mathcal{U})$. Moreover, $S'$ is an extension of $S$; for, if $a \in A$, then

$$\pi_{\mathcal{U}}(S'a) = \overline{F}\cdot \pi_{\mathcal{U}}(a) = \overline{F\cdot a} = \widehat{Sa} = \pi_{\mathcal{U}}(Sa),$$

which implies that $S'a = Sa$; that is, $S'|_A = S$. The extension is unique since $A$ is $|\cdot|$-dense in $\mathcal{U}$.

In a similar way we can prove that each $T \in M_r(A)$ has a unique extension to an element $T' \in M_r(\mathcal{U})$. The equation corresponding to (2.4) is given by

$$\pi_{\mathcal{U}}(T'u) = \pi_{\mathcal{U}}(u)\cdot \overline{F}.$$

It follows that, if $(S,T) \in M(A)$, then $(S',T')$ is its unique extension to $M(\mathcal{U})$. ∎

THEOREM 2.9. *Let $A$ be an $A^*$-algebra with a bounded a.i. Then $M(A)$ is an $A^*$-algebra.*

Proof. We recall that $M(A)$ is a Banach algebra with respect to the norm $\|(S,T)\| = \max(\|S\|,\|T\|)$. In addition, since $A$ is a $*$-algebra, $M(A)$ is a Banach $*$-algebra, the involution being defined by $(S,T)^* = (T^*,S^*)$, where $S^*a = (Sa^*)^*$ and $T^*a = (Ta^*)^*$ $(a \in A)$. Thus it is enough to show that we can define an auxiliary norm on $M(A)$.

By Theorem 2.8, each $(S,T) \in M(A)$ has a unique extension to an element $(S',T') \in M(\mathcal{U})$. Define the auxiliary norm on $M(A)$ by $|(S,T)| = |(S',T')|$. It is easy to check that $(ST^*)' = S'(T')^*$ and $(S^*T)' = (S')^*T'$, and so

$$\begin{aligned}|(S,T)(S,T)^*| &= |((ST^*)',(S^*T)')| = |(S'(T')^*,(S')^*T')|\\ &= |(S',T')(S',T')^*| = |(S',T')|^2\end{aligned}$$

since $M(\mathcal{U})$ is a $B^*$-algebra. Thus $|(S,T)(S,T)^*| = |(S,T)|^2$; that is, $M(A)$ with the norms $\|(\cdot,\cdot)\|$ and $|(\cdot,\cdot)|$ is an $A^*$-algebra. ∎

By using the same arguments as the ones used to prove ([12], Theorem 3.5) (resp. ([12], Theorem 3.7)) we can show that $M(A)$ is algebraically $*$-isomorphic and auxiliary norm isometric to the subalgebra $K = \{(V,W) \in M(\mathcal{U}) : V(A) \subseteq A,\ W(A) \subseteq A\}$ of $M(\mathcal{U})$ (resp. to a subalgebra of $(\mathcal{U}^{**},\cdot)$).

THEOREM 2.10. *Let $A$ be an $A^*$-algebra with a bounded a.i. and let $\mathcal{U}$ be its auxiliary norm completion. Then $QM(A)$ is linearly $*$-isomorphic and auxiliary norm isometric to a subspace of $QM(\mathcal{U})$.*



Proof. Let $\mathcal{P} = \{m \in QM(\mathcal{U}) : m(A\times A) \subseteq A\}$. Clearly $\mathcal{P}$ is a $*$-subspace of $QM(\mathcal{U})$. If $m \in \mathcal{P}$, then $m|_{A\times A}$ is a quasi-multiplier on $A$, and so, since $A$ has a bounded a.i., $m|_{A\times A} \in QM(A)$. Let $\widetilde{\Psi}$ denote the mapping of $QM(A)$ into $\mathcal{P}$ defined by $\widetilde{\Psi}(q) = \overline{q}$, where $\overline{q}$ is the unique extension of $q \in QM(A)$ as given in Theorem 2.6. It follows that $\widetilde{\Psi}$ is a linear surjection of $QM(A)$ onto $\mathcal{P}$. It is routine to show that, for $m \in QM(A)$, $\widetilde{\Psi}(m^*)|_{A\times A} = (\widetilde{\Psi}(m)|_{A\times A})^*$, which implies that $\widetilde{\Psi}(m^*) = (\widetilde{\Psi}(m))^*$; that is, $\widetilde{\Psi}$ is a $*$-isomorphism. Finally, it is clear from Definition 2.7 that $\widetilde{\Psi}$ is auxiliary norm isometric. ∎

If $A$ has a minimal ultra-approximate identity, then $QM(A)$ is a Banach algebra. In this case the above theorem may be strengthened to give the following.

THEOREM 2.11. *Let $A$ be an $A^*$-algebra with a minimal ultra-approximate identity. Then $QM(A)$ is algebraically $*$-isomorphic and auxiliary norm isometric to a subalgebra of $QM(\mathcal{U})$.*

Proof. Let $\{e_\alpha\}$ denote the minimal ultra-approximate identity in $A$. For $m_1, m_2 \in QM(A)$, the product $m_1 \odot m_2$ is given by

$$(m_1 \odot m_2)(a,b) = m_1(a, \lim_\alpha m_2(e_\alpha, b)).$$

We now extend the above definition to define a product in $\mathcal{P}$.

Since $A$ is an $A^*$-algebra there exists a positive number $\beta$ such that $|a| \le \beta\|a\|$ for all $a \in A$. Moreover, since $A$ is auxiliary norm dense in $\mathcal{U}$, it follows that, for $m \in QM(A)$ and $u \in \mathcal{U}$, the nets $\{\overline{m}(e_\alpha,u)\}$ and $\{\overline{m}(u,e_\alpha)\}$ are Cauchy in $\mathcal{U}$ and hence convergent. Thus we can define a product in $\mathcal{P}$ by setting

$$(\overline{m_1} \odot \overline{m_2})(u,v) = \overline{m_1}(u, \lim_\alpha \overline{m_2}(e_\alpha, v)),$$

so that $\mathcal{P}$ is a subalgebra of $QM(\mathcal{U})$. We also note that $\overline{m_1} \odot \overline{m_2}|_{A\times A} = m_1 \odot m_2$, and so

$$\overline{m_1} \odot \overline{m_2} = \overline{m_1 \odot m_2}.$$

It follows from the above and Theorem 2.10 that $\widetilde{\Psi}$ is an algebraic $*$-isomorphism of $QM(A)$ onto $\mathcal{P}$. ∎

With $A$ as in Theorem 2.11, the mappings $\lambda, \varrho$ and $\Psi$ are isometric embeddings of $(M_l(A),\|\cdot\|)$, $(M_r(A),\|\cdot\|)$ and $(M(A),\|\cdot\|)$ respectively into $(QM(A),\|\cdot\|)$. If, instead, we consider the auxiliary norms on $M(A)$ and $QM(A)$, then we have the following

THEOREM 2.12. *Let $A$ be an $A^*$-algebra with a minimal ultra-approximate identity. Then there exists a positive number $\beta$ such that*

(i) $\beta^{-1}|(S,T)| \le |\Psi(S,T)| \le |(S,T)|$ *for all* $(S,T) \in M(A)$,

(ii) $\beta^{-1}|S| \le |\lambda(S)| \le |S|$ $(S \in M_l(A))$,
(iii) $\beta^{-1}|T| \le |\varrho(T)| \le |T|$ $(T \in M_r(A))$.

Proof. (i) Let $\{e_\alpha\}$ denote the minimal ultra-approximate identity in $A$. Since $A$ is an $A^*$-algebra there exists a positive number $\beta$ such that $|a| \le \beta\|a\|$ for all $a \in A$. If $(S,T) \in M(A)$ and $(S',T')$ its unique extension to $M(\mathcal{U})$, then, for any $u \in \mathcal{U}$,

$$|S'u| = \lim_\alpha |e_\alpha S'u| = \lim_\alpha |\Psi(S',T')(e_\alpha,u)| \le \beta|\Psi(S',T')||u|,$$

which implies that $|S'| \le \beta|\Psi(S',T')|$. (We are using the same notation to denote the embeddings of $M(A)$ in $QM(A)$ and $M(\mathcal{U})$ in $QM(\mathcal{U})$ but this should not cause any confusion.) Since $(\Psi(S',T'))|_{A\times A} = \Psi(S,T)$, we have $|\Psi(S,T)| = |\Psi(S',T')|$. Now $|(S,T)| = |(S',T')| = |S'|$, and so it follows that $\beta^{-1}|(S,T)| \le |\Psi(S,T)|$. The right hand side inequality holds since $\Psi : (M(A), |\cdot|) \to (QM(A), |\cdot|)$ is norm decreasing.

(ii) and (iii) may be proved by the same methods. ∎

THEOREM 2.13. *Let $A$ be as in Theorem 2.12. Then, for any $m \in QM(A)$,*

$$\beta^{-1}|m|^2 \le |m \odot m^*| \le \beta^2|m|^2.$$

Proof. Let $m \in QM(A)$. Since $\Psi$ is surjective there exists an $(S,T) \in M(A)$ such that $m = \Psi(S,T)$. We also note that $m^* = \Psi((S,T)^*)$. Thus

$$\begin{aligned}|m \odot m^*| &= |\Psi(S,T) \odot \Psi((S,T)^*)| = |\Psi((S,T)(S,T)^*)| \\ &\le |(S,T)(S,T)^*| = |(S,T)|^2\end{aligned}$$

since $M(A)$ is an $A^*$-algebra by Theorem 2.9. By Theorem 2.12(i),

$$|m \odot m^*| \le \beta^2|m|^2;$$

also

$$\begin{aligned}|m|^2 &= |\Psi(S,T)|^2 \le |(S,T)|^2 = |(S,T)(S,T)^*| \\ &\le \beta|\Psi(S,T) \odot \Psi((S,T)^*)| = \beta|m \odot m^*|,\end{aligned}$$

and so

$$\beta^{-1}|m|^2 \le |m \odot m^*| \le \beta^2|m|^2,$$

as required. ∎

COROLLARY 2.14. *If $\beta = 1$, then $|\cdot|$ satisfies the $B^*$-condition and $QM(A)$ is an $A^*$-algebra.*

The above corollary justifies the use of the term "auxiliary" norm in Definition 2.7.



**3. Quasi-multipliers and the strict topology.** Let $A$ be a Banach algebra with a bounded a.i.

DEFINITION 3.1. The *left strict* $\beta_l$, *right strict* $\beta_r$, *strict* $\beta$ and *quasi-strict* $\gamma$ topologies on $QM(A)$ are defined respectively by the following families of semi-norms:

(i) $m \to \|a \circ m\|$,
(ii) $m \to \|m \circ a\|$,
(iii) $m \to \|a \circ m\|$ and $m \to \|m \circ a\|$,
(iv) $m \to \|a \circ m \circ b\|$

$(a, b \in A,\ m \in QM(A))$.

Clearly $\gamma \subseteq \beta_l$ and $\beta_r \subseteq \beta$. The properties of $(QM(A),\gamma)$ have been studied in some detail in ([11], §3); in this section we turn our attention to $(QM(A),\beta)$.

THEOREM 3.2. *$QM(A)$ is $\beta$-complete.*

Proof. We first note that, for each $a \in A$ and $m \in QM(A)$, the mappings $S_a$ and $T_a$, given by

$$S_a(b) = m(a,b), \quad T_a(b) = m(b,a),$$

define elements in $M_l(A)$ and $M_r(A)$ respectively, and it is easy to show that $\lambda(S_a) = a \circ m$ and $\varrho(T_a) = m \circ a$.

Let $\{m_\alpha : \alpha \in I\}$ be a $\beta$-Cauchy net in $QM(A)$ and let $a \in A$. It follows from the definition of the $\beta$-topology that the nets $\{\lambda(S_a)_\alpha\}$ and $\{\varrho(T_a)_\alpha\}$, where $(S_a)_\alpha b = m_\alpha(a,b)$ and $(T_a)_\alpha b = m_\alpha(b,a)$, are norm-Cauchy in $QM(A)$. Since $\lambda$ and $\varrho$ are topological embeddings, the nets $\{(S_a)_\alpha\}$ and $\{(T_a)_\alpha\}$ are norm-Cauchy in $M_l(A)$ and $M_r(A)$ respectively. Both $M_l(A)$ and $M_r(A)$ are Banach spaces and so there exist $S^{(a)}$ in $M_l(A)$ and $T^{(a)}$ in $M_r(A)$ such that

$$\|(S_a)_\alpha - S^{(a)}\| \to 0, \quad \|(T_a)_\alpha - T^{(a)}\| \to 0.$$

Since $\gamma \subseteq \beta$, the net $\{m_\alpha\}$ is $\gamma$-Cauchy. The space $QM(A)$ is $\gamma$-complete ([14], Theorem 6) and so there exists an element $m_0$ in $QM(A)$ such that

$$\lim_\alpha m_\alpha(x,y) = m_0(x,y)$$

for all $x, y \in A$. For any $b, c \in A$,

$$\begin{aligned}(\lambda(S^{(a)}))(b,c) &= \lim_\alpha(\lambda((S_a)_\alpha))(b,c) = \lim_\alpha bm_\alpha(a,c) \\ &= (a \circ m_0)(b,c),\end{aligned}$$

which implies that $\lambda(S^{(a)}) = a \circ m_0$. Similarly we can prove that $\varrho(T^{(a)}) = m_0 \circ a$. Thus

$$\|a \circ m_\alpha - a \circ m_0\| = \|\lambda(S_a)_\alpha - \lambda(S^{(a)})\| \le \|(S_a)_\alpha - S^{(a)}\| \to 0$$

and

$$\|m_\alpha \circ a - m_0 \circ a\| = \|\varrho(T_a)_\alpha - \varrho(T^{(a)})\| \le \|(T_a)_\alpha - T^{(a)}\| \to 0\,,$$

which implies that $m_0$ is the $\beta$-limit of the net $\{m_\alpha\}$; that is, $QM(A)$ is $\beta$-complete, as required. ■

Since $\gamma \subseteq \beta$, every $\beta$-bounded set is $\gamma$-bounded. But the $\gamma$-bounded and norm bounded subsets of $QM(A)$ coincide ([11], Theorem 3.2), and so every $\beta$-bounded subset of $QM(A)$ is norm bounded. Clearly every norm bounded subset of $QM(A)$ is $\beta$-bounded. We thus have the following

THEOREM 3.3. *$(QM(A),\beta)$, $(QM(A),\gamma)$ and $(QM(A),\|\cdot\|)$ have the same bounded sets.*

Our next aim is to generalize a theorem due to Lin ([13], Theorem 9.3). Let

$$A^{**}_\mu = \{F \in A^{**} : F\cdot a \in A \text{ for all } a \in A\}\,,$$
$$A^{**}_\nu = \{F \in A^{**} : a\cdot F \in A \text{ for all } a \in A\}\,,$$
$$A^{**}_\sigma = \{F \in A^{**} : a\cdot F\cdot b \in A \text{ for all } a, b \in A\}\,.$$

The *strict topology* on $A^{**}$ is defined to be the locally convex topology determined by the semi-norms $F \to \|F\cdot a\|$ and $F \to \|a\cdot F\|$ $(a \in A,\ F \in A^{**})$.

It is clear that $A^{**}_\mu + A^{**}_\nu \subseteq A^{**}_\sigma$, but, in fact, more is true:

LEMMA 3.4. *$A^{**}_\mu + A^{**}_\nu$ is strictly dense in $A^{**}_\sigma$.*

Proof. Let $\{e_\alpha : \alpha \in I\}$ be a bounded a.i. for $A$, with $\|e_\alpha\| \le C$ $(\alpha \in I)$, and suppose that $F \in A^{**}_\sigma$. For each $\alpha \in I$, let $F_\alpha = e_\alpha\cdot F - e_\alpha\cdot F\cdot e_\alpha + F\cdot e_\alpha$. Clearly $F\cdot e_\alpha \in A^{**}_\nu$ and $e_\alpha \cdot F \in A^{**}_\mu$, and so to complete the proof we show that $F_\alpha$ converges strictly to $F$. Let $a \in A$. Then

$$\begin{aligned}\|a\cdot F_\alpha - a\cdot F\| &\le \|a\cdot F\cdot e_\alpha - ae_\alpha\cdot F\cdot e_\alpha\| + \|ae_\alpha\cdot F - a\cdot F\|\\ &\le \|a - ae_\alpha\|\,\|F\|\,C + \|ae_\alpha - a\|\,\|F\| \to 0\,;\end{aligned}$$

similarly we can show that $\|F_\alpha\cdot a - F\cdot a\| \to 0$. ■

THEOREM 3.5. *Let $A$ be a Banach algebra with a bounded a.i. Then $M_l(A) + M_r(A)$ is strictly dense in $QM(A)$.*

Proof. Let $m \in QM(A)$ and suppose that $\{e_\alpha : \alpha \in I\}$ is a bounded a.i. for $A$, with $\|e_\alpha\| \le C$ $(\alpha \in I)$. For each $\alpha \in I$, define mappings $S_\alpha$ and $T_\alpha$ by

$$S_\alpha(a) = m(e_\alpha, a - e_\alpha a),\quad T_\alpha(a) = m(a, e_\alpha) \quad (a \in A)\,.$$

Clearly $S_\alpha \in M_l(A)$ and $T_\alpha \in M_r(A)$.

Let $F = m^{***}(E,E)$, where $E$ is a weak*-cluster point of $\{\widehat{e}_\alpha\}$. For each $\alpha \in I$ we prove the following.

(i) $(\varrho(T_\alpha))^{***}(E,E) = F\cdot e_\alpha$,

(ii) $((\lambda(S_\alpha))')^{***}(E,E) = e_\alpha\cdot F - e_\alpha\cdot F\cdot e_\alpha$.



For (i), $\langle f, (\varrho(T_\alpha))^{***}(E,E)\rangle = \langle(\varrho(T_\alpha))^{**}(E,f), E\rangle$ $(f \in A^*)$; routine calculations show that $(\varrho(T_\alpha))^{**}(E,f) = T^*_\alpha(E\cdot f)$, and so, since $f$ is arbitrary in $A^*$, it follows that

$$(3.1)\qquad (\varrho(T_\alpha))^{***}(E,E) = T^{**}_\alpha E\,.$$

Also, for any $f$ in $A^*$,

$$\begin{aligned}\langle f, T^{**}_\alpha E\rangle &= \langle T^*_\alpha f, E\rangle = \lim_\beta\langle e_\beta, T^*_\alpha f\rangle = \lim_\beta\langle m(e_\beta, e_\alpha), f\rangle\\ &= \lim_\beta\langle e_\beta, m^{**}(\widehat{e}_\alpha, f)\rangle = \langle f, m^{***}(E, \widehat{e}_\alpha)\rangle\\ &= \langle f, m^{***}(E, E * e_\alpha)\rangle = \langle f, m^{***}(E,E)\cdot e_\alpha\rangle\,,\end{aligned}$$

which implies that

$$(3.2)\qquad T^{**}_\alpha E = m^{***}(E,E)\cdot e_\alpha\,.$$

(i) follows from (3.1) and (3.2).

For (ii), we first note that, if $S \in M_l(A)$ then $S^{**} \in M_l(A^{**}, *)$. For each $S \in M_l(A)$, let $\widetilde{\lambda}(S^{**})$ be the element of $QM(A^{**}, *)$ defined by

$$(\widetilde{\lambda}(S^{**}))(F,G) = F * S^{**}G\,.$$

A routine calculation shows that

$$((\lambda(S))')^{***}(F,G) = (\widetilde{\lambda}(S^{**}))(F,G)\,,$$

so that, in particular, for each $\alpha \in I$,

$$(3.3)\qquad ((\lambda(S_\alpha))')^{***}(E,E) = E * S^{**}_\alpha E = S^{**}_\alpha E\,.$$

For each $\alpha \in I$ and $f \in A^*$,

$$\begin{aligned}\langle f, S^{**}_\alpha E\rangle &= \lim_\beta\langle S^*_\alpha f, \widehat{e}_\beta\rangle = \lim_\beta\langle m(e_\alpha, e_\beta - e_\alpha e_\beta), f\rangle\\ &= \lim_\beta\langle e_\alpha, m^*(f, e_\beta - e_\alpha e_\beta)\rangle = \langle m^*(f, e_\alpha), E - e_\alpha\cdot E\rangle\\ &= \langle e_\alpha, m^{**}(E - e_\alpha\cdot E, f)\rangle = \langle f, m^{***}(\widehat{e}_\alpha, E - e_\alpha\cdot E)\rangle\,,\end{aligned}$$

which implies that $S^{**}_\alpha E = m^{***}(\widehat{e}_\alpha, E - \widehat{e}_\alpha)$.

Now $e_\alpha\cdot F - e_\alpha\cdot F\cdot e_\alpha = e_\alpha\cdot m^{***}(E,E) - e_\alpha\cdot m^{***}(E,E)\cdot e_\alpha$ and since $m^{***}$ is an element of $QM(A^{**},\cdot)$, we have

$$e_\alpha\cdot F - e_\alpha\cdot F\cdot e_\alpha = m^{***}(\widehat{e}_\alpha, E - \widehat{e}_\alpha)\,.$$

Thus

$$(3.4)\qquad S^{**}_\alpha E = e_\alpha\cdot F - e_\alpha\cdot F\cdot e_\alpha\,.$$

From (3.3) and (3.4),

$$((\lambda(S_\alpha))')^{***}(E,E) = e_\alpha\cdot F - e_\alpha\cdot F\cdot e_\alpha$$

proving (ii).

Since $a \cdot m^{***}(E,E) \cdot b = m^{***}(\widehat{a},\widehat{b}) = \widehat{m(a,b)}$, we have $F \in A_\sigma^{**}$. Thus, by the proof of Lemma 3.4, $e_\alpha \cdot F - e_\alpha \cdot F \cdot e_\alpha + F \cdot e_\alpha$ converges strictly to $F$; that is, for each $a \in A$,

$$\|(((\lambda(S_\alpha))')^{***}(E,E) + (\varrho(T_\alpha))^{***}(E,E) - m^{***}(E,E)) \cdot a\| \to 0 \tag{3.5}$$

and

$$\|a \cdot (((\lambda(S_\alpha))')^{***}(E,E) + (\varrho(T_\alpha))^{***}(E,E) - m^{***}(E,E))\| \to 0\,. \tag{3.6}$$

Next we require the following identities:

(iii) $(m')^{***}(E,E) \cdot a = m^{***}(E,E) \cdot a$,
(iv) $(m \circ a)^{***} = m^{***} \circ \widehat{a}$,
(v) $(a \circ m)^{***} = \widehat{a} \circ m^{***}$

($m \in QM(A)$, $a \in A$). We prove (iii) below; (iv) and (v) can be proved using routine calculations and the property that $m^{***}$ is a quasi-multiplier on $(A^{**}, \cdot)$. Since $(m')^{***}$ is a quasi-multiplier on $(A^{**}, *)$,

$$\begin{aligned}\langle f, (m')^{***}(E,E) \cdot a\rangle &= \langle f, (m')^{***}(E,\widehat{a})\rangle\\ &= \langle a, (m')^{**}(f,E)\rangle = \langle (m')^{*}(a,f), E\rangle\\ &= \langle m^{**}(\widehat{a},f), E\rangle = \langle f, m^{***}(E,\widehat{a})\rangle\,,\end{aligned}$$

which implies that

$$(m')^{***}(E,E) \cdot a = m^{***}(E,E) \cdot a\,.$$

It follows from (iv) that

$$(m \circ a)^{***}(E,E) = (m^{***} \circ \widehat{a})(E,E) = m^{***}(E,E) \cdot a\,,$$

and from (v) that

$$(a \circ m)^{***}(E,E) = a \cdot m^{***}(E,E)\,.$$

Thus, from (3.5), (iii), and the above we have

$$\lim_\alpha \|((\lambda(S_\alpha) + \varrho(T_\alpha) - m) \circ a)^{***}(E,E)\| = 0\,.$$

It follows from Theorem 2.1 that

$$\lim_\alpha \|(\lambda(S_\alpha) + \varrho(T_\alpha) - m) \circ a\| = 0\,.$$

Similarly we can show that $\lim_\alpha \|a \circ (\lambda(S_\alpha) + \varrho(T_\alpha) - m)\| = 0$. Thus $\lambda(S_\alpha) + \varrho(T_\alpha)$ converges strictly to $m$; that is, $M_l(A) + M_r(A)$ is strictly dense in $QM(A)$.

**4. Quasi-multipliers and the algebra of compact operators.** Let $X$ be a Banach space and let $A = K_0(X)$, the algebra of bounded linear operators on $X$ which can be approximated, in the operator norm, by operators of finite rank. In this section our first aim is to establish a characterization for the quasi-multipliers of $A$. We begin, however, with some definitions and



necessary background results. Full details of the results given may be found in [4]. A Banach space $X$ is said to have the *approximation property* if, for every compact set $K$ and $\varepsilon > 0$, there exists a linear operator $T_{K,\varepsilon}$ of finite rank such that $\|T_{K,\varepsilon}(x) - x\| < \varepsilon$ for all $x \in K$. If, in addition, there exists a positive number $C$, independent of $K$ and $\varepsilon$, such that $\|T_{K,\varepsilon}\| < C$, then $X$ is said to have the *bounded approximation property*. If $C = 1$, then $X$ is said to have the *metric approximation property*. If $X^*$ has the bounded approximation property, then $K_0(X) = K(X)$, the algebra of all compact operators on $X$ ([4], Theorem 3.5), and $K(X)$ has a bounded a.i.; the latter property follows from ([4], Theorems 3.10 and 3.11), ([4], p. 93), and ([3], p. 59, Proposition 6).

The tensor $x \otimes x'$ ($x \in X$, $x' \in X^*$) determines a bounded linear operator on $X$ according to the equation

$$(x \otimes x')y = (x'(y))x \quad (y \in X) \tag{4.1}$$

and so the elements of the tensor product $X \otimes X^*$ are operators on $X$ which are of finite rank. If $u \in X \otimes X^*$ and $u = \sum_{i=1}^n x_i \otimes x_i'$, the right hand side of the equation

$$\|u\|^\vee = \sup\Big\{\Big|\sum_{i=1}^n \langle x_i, f\rangle\langle x_i', F\rangle\Big| : f \in X^*,\ F \in X^{**}\Big\}$$

is independent of the representation of $u$ and defines a norm on $X \otimes X^*$, called the *inductive tensor norm*. The completion of $X \otimes X^*$ with respect to $\|\cdot\|^\vee$ is denoted by $X \check{\otimes} X^*$ and it is straightforward to show that $X \check{\otimes} X^*$ is isometrically isomorphic to $K_0(X)$.

The *projective tensor norm* on $X \otimes X^*$ is defined by

$$\|u\|^\wedge = \inf\Big\{\sum_{i=1}^n \|x_i\|\,\|x_i'\| : u = \sum_{i=1}^n x_i \otimes x_i'\Big\}\,,$$

where the infimum is taken over all representations of $u$. The completion of $X \otimes X^*$ with respect to $\|\cdot\|^\wedge$ is denoted by $X \hat{\otimes} X^*$ and is called the *projective tensor product* of $X$ and $X^*$. The mapping $\phi$ of $X \otimes X^*$ into $\mathcal{L}(X)$ as defined by (4.1) is a norm decreasing mapping on $(X \otimes X^*, \|\cdot\|^\wedge)$ and so induces a contraction from $X \hat{\otimes} X^*$ into $\mathcal{L}(X)$. Clearly its image is linearly isomorphic to the quotient space $X \hat{\otimes} X^* / \ker\phi$; $\phi(X \hat{\otimes} X^*)$, equipped with the quotient norm, is denoted by $N(X)$. The elements of $N(X)$ are called the *nuclear operators* on $X$ and the norm is called the *trace norm*. In fact, if $u \in N(X)$, then the trace norm of $u$ is given by

$$\|u\|_N = \inf\Big\{\sum_{i=1}^\infty \|x_i\|\,\|x_i'\| : u = \sum_{i=1}^\infty x_i \otimes x_i'\Big\}\,,$$

where the infimum is taken over all representations of $u$.

A bounded linear operator $T$ on $X$ is said to be an *integral operator* if there exists a constant $C > 0$ such that

$$\Big|\sum_{i=1}^{n}\langle Tx_i, x_i'\rangle\Big| \le C\Big\|\sum_{i=1}^{n} x_i \otimes x_i'\Big\|$$

for all $\sum_{i=1}^{n} x_i \otimes x_i' \in X \otimes X^*$, where $\|\cdot\|$ denotes the operator norm on $\mathcal{L}(X)$. The infimum over all possible constants $C$ is called the *integral norm* of $T$ and is denoted by $\|T\|_I$. If $I(X)$ denotes the integral operators on $X$, then $(I(X), \|\cdot\|_I)$ is a Banach space ([23], p. 258). Every nuclear operator is integral, its integral norm being dominated by its nuclear norm. The dual of $K_0(X)$ is isometrically isomorphic to $(I(X^*), \|\cdot\|_I)$; the correspondence $F \leftrightarrow f$ between $(K_0(X))^*$ and $I(X^*)$ is described by the relation

$$\langle u, F\rangle = \sum_{i=1}^{n}\langle x_i, f x_i'\rangle \quad (f \in I(X^*)), \tag{4.2}$$

where $u = \sum_{i=1}^{n} x_i \otimes x_i' \in X \otimes X^*$.

For our investigations in this section we require, in addition to the above, the following properties of integral operators.

(i) $I(X)$ is a two-sided ideal of $\mathcal{L}(X)$ and, for $f, g \in I(X)$, $T \in \mathcal{L}(X)$, $\|f \circ T \circ g\|_I \le \|f\|_I \|T\|\ \|g\|_I$; the proof is routine.

(ii) An operator $f$ on $X$ is integral if and only if $f^*$ is integral and $\|f\|_I = \|f^*\|_I$ ([5], p. 236, Corollary 11).

(iii) An integral operator on $X$ is weakly compact ([4], p. 228, Corollary 3.6) and so $f^{**}$ maps $X^{**}$ into $\widehat{X}$ ([4], p. 227).

The bilinear functional $(x', x'') \to \langle x', x''\rangle$ on $X^* \times X^{**}$ induces a unique linear functional $\psi$ on $X^* \otimes X^{**}$ such that

$$\psi\Big(\sum_{i=1}^{n} x_i' \otimes x_i''\Big) = \sum_{i=1}^{n}\langle x_i', x_i''\rangle \tag{4.3}$$

([3], p. 232, Theorem 6). It is clear that $\psi$ is a continuous linear functional on $(X^* \otimes X^{**}, \|\cdot\|^\wedge)$ and so has a unique continuous extension, $\widetilde{\psi}$ say, to $X^*\hat{\otimes}X^{**}$. Since the right hand side of (4.3) is independent of the representation of $u = \sum_{i=1}^{n} x_i' \otimes x_i''$, we refer to it as the *trace* of $u$ and it is written tr.$u$. In particular, if $g \in \mathcal{L}(X^*)$, then $g \circ u = \sum_{i=1}^{n} g x_i' \otimes x_i''$, and so

$$\mathrm{tr.}(g \circ u) = \sum_{i=1}^{n}\langle g x_i', x_i''\rangle = \sum_{i=1}^{n}\langle x_i', g^* x_i''\rangle = \mathrm{tr.}(u \circ g)\,.$$

If $f \in I(X^*)$ and $a = x \otimes x' \in X \otimes X^*$, then $a^* = x' \otimes \widehat{x}$ and so $\mathrm{tr.}(a^* \circ f) = \langle x', f^*\widehat{x}\rangle = \langle x, f x'\rangle = \mathrm{tr.}(f \circ a^*) = \langle a, f\rangle$ (by (4.2)). It follows that, for any $a \in K_0(X)$,

$$\mathrm{tr.}(f \circ a^*) = \mathrm{tr.}(a^* \circ f) = \langle a, f\rangle\,.$$

We let $\pi$ denote the canonical mapping of $X^*\hat{\otimes}X^{**}$ into $I(X^*)$ (in fact, the image of $X^*\hat{\otimes}X^{**}$ under $\pi$ is $N(X^*)$). If $X^*$ has the bounded approximation property, then $\pi$ is injective ([4], p. 80, Theorem 3.4), so that $N(X^*)$ is the projective tensor product $X^*\hat{\otimes}X^{**}$ of $X^*$ and $X^{**}$. In this case tr.$u$ is well defined for every $u \in N(X^*)$. The dual space $(X^*\hat{\otimes}X^{**})^*$ may be identified (isometrically and isomorphically) with $\mathcal{L}(X^{**})$, the correspondence $h \to T_h$ ($h \in (X^*\hat{\otimes}X^{**})^*$, $T_h \in \mathcal{L}(X^{**})$) being defined by the relation

$$\langle x' \otimes x'', h\rangle = \langle x', T_h x''\rangle\,. \tag{4.4}$$

Thus the adjoint $\pi^*$ is a mapping of $(K_0(X))^{**}$ into $\mathcal{L}(X^{**})$.

LEMMA 4.1. *Let $X$ be a Banach space and suppose that $a \in K_0(X)$. Then $\pi^*\widehat{a} = a^{**}$.*

Proof. Let $x' \in X^*$, $x'' \in X^{**}$. It follows from (4.4) that, for any $a \in K_0(X)$,

$$\begin{aligned}\langle x', (\pi^*\widehat{a})x''\rangle &= \langle x' \otimes x'', \pi^*\widehat{a}\rangle = \langle \pi(x' \otimes x''), \widehat{a}\rangle\\ &= \langle a, \pi(x' \otimes x'')\rangle = \mathrm{tr.}(a^* \circ \pi(x' \otimes x''))\\ &= \mathrm{tr.}(\pi(a^*x' \otimes x'')) = \langle a^*x', x''\rangle = \langle x', a^{**}x''\rangle\,,\end{aligned}$$

which implies that $(\pi^*\widehat{a}) = a^{**}$. ∎

For the remainder of this section, unless stated otherwise, we assume that $X^*$ has the bounded approximation property and in the sequel the algebra $K_0(X)$ is denoted by $A$. We also use $\iota_X$ (resp. $\iota_{X^*}$) to denote the canonical embedding of $X$ (resp. $X^*$) into $X^{**}$ (resp. $X^{***}$) and, for each $h \in \mathcal{L}(X^{**})$, the "$h$-flat" mapping is the element of $\mathcal{L}(X^*)$ defined by $h^\flat = \iota_X^* \circ h^* \circ \iota_{X^*}$.

THEOREM 4.2. *Let $X$ be a Banach space and suppose that $X^*$ has the bounded approximation property. Then $\varrho$ is a topological isomorphism of $M_r(A)$ onto $QM(A)$.*

Proof. Since $X^*$ has the bounded approximation property, $A$ has a bounded two-sided approximate identity and so $\varrho$ is a topological isomorphism (isometric if $X^*$ has the metric approximation property). Thus to complete the proof we have to show that $\varrho$ is onto.

Let $m \in QM(A)$ and let $E$ be a right (left) identity with respect to the first (second) Arens product on $A^{**}$. Let $F = m^{***}(E, E)$ and $g = (\pi^*F)^\flat$. The algebra $M_r(A)$ is isometrically isomorphic to $\mathcal{L}(X^*)$ (see, for example, ([10], Theorem 19 and Corollary 1)), and the correspondence $\tau$ which maps $g \in \mathcal{L}(X^*)$ onto $\tau_g \in M_r(A)$ has the property that $\iota_X \circ \tau_g(a) = a^{**} \circ g^* \circ \iota_X$

for all $a \in A$. Thus, for any $a, b \in A$, $x \in X$, $x' \in X^*$, we have

$$\begin{aligned}\langle x', \iota_X((\tau_g(a)b)x)\rangle &= \langle x', (a^{**} \circ g^*)\widehat{bx}\rangle \\ &= \langle x', a^{**} \circ (\pi^* F)^{\flat *}(\widehat{bx})\rangle,\end{aligned}$$

and since $(h^\flat)^*|_{\widehat{X}} = h$ for every $h \in \mathcal{L}(X^{**})$, it follows that

$$\begin{aligned}\langle x', \iota_X((\tau_g(a)b)x)\rangle &= \langle x', (a^{**} \circ \pi^* F \circ b^{**} \circ \iota_X)x\rangle \\ &= \langle x', (\pi^*\widehat{a} \circ \pi^* F \circ \pi^*\widehat{b} \circ \iota_X)x\rangle\end{aligned}$$

by Lemma 4.1.

The mapping $\pi^*$ satisfies $\pi^*(F \cdot G) = \pi^* F \circ \pi^* G$ for all $F, G \in A^{**}$ ([9], Proposition 3.2(iii)) and so

$$\begin{aligned}\langle x', \iota_X((\tau_g(a)b)x)\rangle &= \langle x', \pi^*(a \cdot m^{***}(E,E) \cdot b)\widehat{x}\rangle \\ &= \langle x', (\pi^*(m^{***}(\widehat{a},\widehat{b})))\widehat{x}\rangle = \langle x', (\pi^*(\widehat{m(a,b)}))\widehat{x}\rangle \\ &= \langle x', (m(a,b))^{**}\widehat{x}\rangle = \langle x', (\widehat{m(a,b)})x\rangle,\end{aligned}$$

which implies that $\tau_g(a)b = m(a,b)$; that is, $\varrho(\tau_g) = m$, proving that $\varrho$ is surjective, as required. ■

COROLLARY 4.3. *Let $X$ be a Banach space and suppose that $X^*$ has the bounded approximation property. Then $QM(A)$ is topologically isomorphic to $\mathcal{L}(X^*)$; the topological isomorphism is an isometric one if $X^*$ has the metric approximation property.*

COROLLARY 4.4 (Vasudevan and Goel ([22], Lemma 3.2)). *If $H$ is a Hilbert space and $A = K(H)$, then $\mathcal{L}(H)$ and $QM(A)$ are isometrically isomorphic.*

Proof. The result follows immediately from Corollary 4.3 since every Hilbert space has the metric approximation property. ■

Before our next result we explain how we consider left, right and double multipliers of $A^{**}$; our approach is due to Grosser ([9], p. 547). Suppose that $\mathcal{V}$ is a left module over a Banach algebra $\mathcal{A}$; that is, $\mathcal{V}$ is a Banach space and there is a continuous bilinear mapping $\mathcal{A} \times \mathcal{V} \to \mathcal{V}$ such that $a(bv) = (ab)v$ $(a, b \in \mathcal{A}, v \in \mathcal{V})$. A *right multiplier* of $\mathcal{V}$ is defined to be a mapping $T : \mathcal{A} \to \mathcal{V}$ such that $T(ab) = aTb$ for all $a, b \in \mathcal{A}$. If $\mathcal{A}$ has a bounded right a.i., then every right multiplier is linear and continuous. Following the notation introduced in §2 we denote the set of all linear and continuous right multipliers of $\mathcal{V}$ by $M_r(\mathcal{V})$. The space $M_l(\mathcal{W})$ of all linear and continuous *left multipliers* of a right $\mathcal{A}$-module $\mathcal{W}$ is defined analogously. If $\mathcal{V}$ is an $\mathcal{A}$-bi-module (that is, a left and a right $\mathcal{A}$-module simultaneously satisfying $a(vb) = (av)b$ for all $a, b \in \mathcal{A}$ and $v \in \mathcal{V}$), then the space $M(\mathcal{V})$ of continuous linear *double multipliers* of $\mathcal{V}$ consists of all pairs $(S, T)$, $S \in M_l(\mathcal{V})$ and $T \in M_r(\mathcal{V})$, satisfying $aSb = (Ta)b$ $(a, b \in \mathcal{A})$. In particular, $\mathcal{A}^{**}$ is an $\mathcal{A}$-bi-module and so the right, left and double multipliers of $\mathcal{A}^{**}$ are defined in the above sense. If $S$ is a linear continuous mapping of $\mathcal{A}$ into $\mathcal{A}^{**}$ and $S^t$ denotes the restriction of its adjoint to $\mathcal{A}^*$, then it is not difficult to show that $S$ is a left multiplier of $\mathcal{A}^{**}$ if and only if $S^t(a * f) = a * S^t f$ for all $a \in \mathcal{A}$, $f \in \mathcal{A}^*$. Similarly, a continuous linear operator $T : \mathcal{A} \to \mathcal{A}^{**}$ is a right multiplier of $\mathcal{A}^{**}$ if and only if $T^t(f \cdot a) = T^t f \cdot a$ for all $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$, and the pair $(S, T)$ (where $S, T : \mathcal{A} \to \mathcal{A}^{**}$ are continuous linear mappings) is a double multiplier of $\mathcal{A}^{**}$ if and only if $\langle b, S^t(f \cdot a)\rangle = \langle a, T^t(b * f)\rangle$ for all $a, b \in A$, and $f \in \mathcal{A}^*$. Thus we may regard $M_l(\mathcal{A}^{**})$ and $M_r(\mathcal{A}^{**})$ as subspaces of $\mathcal{L}(\mathcal{A}^*)$ and $M(\mathcal{A}^{**})$ as a subspace of $\mathcal{L}(\mathcal{A}^*) \times \mathcal{L}(\mathcal{A}^*)$.

We now return to the algebra $A$ and to the spaces $M_r(A^{**})$ and $M_l(A^{**})$. In ([8], Theorem 1) Grosser proved that $M_r(A^{**})$ is isometrically isomorphic to $\mathcal{L}(X^{**})$ and that $M_l(A^{**})$ is isometrically isomorphic to $\mathcal{L}(X^*)$. We note that the characterizations do not require $X^*$ to have the bounded approximation property and are therefore valid for any Banach space $X$ and $A = K_0(X)$. For the sake of completeness we give the results in the following theorem using the notation and terminology developed in this paper.

THEOREM (Grosser [8], Theorem 1). *Let $X$ be a Banach space and $A = K_0(X)$.*

(a) *The mapping $\sigma' : \mathcal{L}(X^*) \to M_l(A^{**})$, which acts on $A^*$ according to the equation*

$$\sigma'_g(f) = g \circ f \quad (g \in \mathcal{L}(X^*),\ f \in A^*),$$

*is an isometric isomorphism of $\mathcal{L}(X^*)$ onto $M_l(A^{**})$.*

(b) *For each $h \in \mathcal{L}(X^{**})$ and $f \in A^*$, $(h \circ f^*)^\flat \in A^*$. Consequently, the mapping $\tau' : \mathcal{L}(X^{**}) \to M_r(A^{**})$, whose action on $A^*$ is given by the equation*

$$\tau'_h(f) = (h \circ f^*)^\flat,$$

*is an isometric isomorphism of $\mathcal{L}(X^{**})$ onto $M_r(A^{**})$.*

The isometric isomorphism between $A^*$ and the Banach algebra $(I(X^*), \|\cdot\|_I)$ of integral operators on $X^*$ enables us to define a product on $A^*$ which makes it a Banach algebra. Consequently, we can consider mappings $q : A^* \times A^* \to A^*$ which satisfy the quasi-multiplier condition (2.1). As in §2, we let $QM(A^*)$ denote the space of all jointly continuous quasi-multipliers on $A^*$.

THEOREM 4.5. *Let $X$ be a Banach space and $A = K_0(X)$. Then the equation*

$$(\theta(h))(f, g) = (\tau'_h f) \circ g \quad (h \in \mathcal{L}(X^{**}),\ f, g \in A^*)$$

*defines a norm decreasing linear isomorphism between $\mathcal{L}(X^{**})$ and a subspace of $QM(A^*)$.*

Proof. We recall that $A^* \simeq (I(X^*), \|\cdot\|_I)$, the integral operators on $X^*$, and $I(X^*)$ is a Banach algebra with respect to the integral norm. We first show that $\theta$ maps $\mathcal{L}(X^{**})$ into $QM(A^*)$. Let $h \in \mathcal{L}(X^{**})$ and $f \in A^*$. We first note that $(\tau'_h f)^* = h \circ f^*$; for if $x' \in X^*$, $x'' \in X^{**}$, then

$$\begin{aligned} \langle x', (\tau'_h f)^* x''\rangle &= \langle (\tau'_h f)x', x''\rangle \\ &= \langle x'', (\iota_{X^*} \circ \iota_X^* \circ f^{**} \circ h^* \circ \iota_{X^*})x'\rangle \\ &= \langle x'', (h \circ f^*)^* \widehat{x'}\rangle \quad (\text{since } \iota_{X^*} \circ \iota_X^* = \text{id}.\widehat{X}^*) \\ &= \langle x', (h \circ f^*)x''\rangle, \end{aligned}$$

which implies that $(\tau'_h f)^* = h \circ f^*$. Thus, for $f, g \in A^*$,

$$(\tau'_h(f \circ g))^* = h \circ g^* \circ f^* = (f \circ \tau'_h(g))^*,$$

and so $\tau'_h(f \circ g) = f \circ \tau'_h g$. It follows that, for $f, g, l \in A^*$,

$$\theta(h)(f \circ g, l) = \tau'_h(f \circ g) \circ l = f \circ \tau'_h g \circ l = f \circ \theta(h)(g, l).$$

Similarly, we can show that $\theta(h)(f, g \circ l) = \theta(h)(f, g) \circ l$, so that $\theta(h) \in QM(A^*)$. The linearity of $\theta$ follows immediately from the linearity of $\tau'$. We show that $\theta$ is continuous, as follows:

$$\begin{aligned} \|\theta(h)(f, g)\|_I &= \|(\theta(h)(f, g))^*\|_I = \|(\tau'_h f \circ g)^*\|_I \\ &= \|g^* \circ h \circ f^*\|_I \le \|g^*\|_I \|h\| \, \|f^*\|_I = \|g\|_I \|f\|_I \|h\|, \end{aligned}$$

which implies that $\theta$ is continuous, with $\|\theta\| \le 1$.

Finally, $\theta$ is injective. Suppose that $\theta(h) = 0$. Then $\tau'_h f \circ g = 0$ for all $f, g \in A^*$. In particular, for any $x' \in X$ and $x'' \in X^{**}$, $\tau'_h f \circ \pi(x' \otimes x'') = 0$. Thus, for any $y' \in X^*$,

$$0 = (\tau'_h f \circ \pi(x' \otimes x''))y' = \langle y', x''\rangle (\tau'_h f)x',$$

which implies that, since $x', y'$ and $x''$ are arbitrary, $\tau'_h = 0$. Since $\tau'$ is an isomorphism, $h = 0$, as required. ∎

DEFINITION 4.6. Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity. A mapping $m : \mathcal{A} \times \mathcal{A} \to \mathcal{A}^{**}$ is said to be a *quasi-multiplier* of $\mathcal{A}^{**}$ if

$$m(ab, c) = \widehat{a} \cdot m(b, c) \quad \text{and} \quad m(a, bc) = m(a, b) \cdot \widehat{c} \quad \text{for all } a, b, c \in \mathcal{A}.$$

Let $QM(\mathcal{A}^{**})$ denote the set of all bilinear and jointly continuous quasi-multipliers of $\mathcal{A}^{**}$. Then, as in the case of quasi-multipliers on $\mathcal{A}$, every quasi-multiplier of $\mathcal{A}^{**}$ belongs to $QM(\mathcal{A}^{**})$ and $QM(\mathcal{A}^{**})$ is a Banach space. We note that $QM(\mathcal{A})$ is the subspace of $QM(\mathcal{A}^{**})$ which consists of those $m \in QM(\mathcal{A}^{**})$ such that $m(\mathcal{A}, \mathcal{A}) \subseteq \widehat{\mathcal{A}}$.



We extend $m \in QM(\mathcal{A}^{**})$ to a linear map on $\mathcal{A}^{**} \times \mathcal{A}^{**}$ in the following way:

$$\begin{aligned} m^* &: \mathcal{A}^{***} \times \mathcal{A} \to \mathcal{A}^*, \text{ defined by } \langle b, m^*(\mathcal{F}, a)\rangle = \langle m(a, b), \mathcal{F}\rangle, \\ m^{**} &: \mathcal{A}^{**} \times \mathcal{A}^{***} \to \mathcal{A}^*, \text{ defined by } \langle a, m^{**}(F, \mathcal{F})\rangle = \langle m^*(\mathcal{F}, a), F\rangle, \\ m^{***} &: \mathcal{A}^{**} \times \mathcal{A}^{**} \to \mathcal{A}^{****}, \text{ defined by } \langle \mathcal{F}, m^{***}(F, G)\rangle = \langle m^{**}(G, \mathcal{F}), F\rangle \end{aligned}$$

$(a, b \in \mathcal{A}, F, G \in \mathcal{A}^{**}, \mathcal{F} \in \mathcal{A}^{***})$.

We use the notation $\cdot$ to denote the first Arens product on the algebra $\mathcal{A}^{**}$ and for convenience we also use $\cdot$ to denote the corresponding first Arens product on the algebra $\mathcal{A}^{****}$.

LEMMA 4.7. *For $a, b \in \mathcal{A}$, $\widehat{\widehat{a}} \cdot m^{***}(E, E) \cdot \widehat{\widehat{b}} \in \widehat{\mathcal{A}^{**}}$.*

Proof. Let $\mathcal{F}$ be any element of $\mathcal{A}^{***}$. Then

$$\begin{aligned} \langle \mathcal{F}, \widehat{\widehat{a}} \cdot m^{***}(E, E) \cdot \widehat{\widehat{b}}\rangle &= \langle \widehat{\widehat{b}} \cdot \mathcal{F} \cdot \widehat{a}, m^{***}(E, E)\rangle \\ &= \langle m^{**}(E, \widehat{\widehat{b}} \cdot \mathcal{F} \cdot \widehat{a}), E\rangle = \lim_\alpha \langle e_\alpha, m^{**}(E, \widehat{\widehat{b}} \cdot \mathcal{F} \cdot \widehat{a})\rangle \\ &= \lim_\alpha \langle m^*(\widehat{\widehat{b}} \cdot \mathcal{F} \cdot \widehat{a}, e_\alpha), E\rangle = \lim_\alpha \lim_\beta \langle e_\beta, m^*(\widehat{\widehat{b}} \cdot \mathcal{F} \cdot \widehat{a}, e_\alpha)\rangle \\ &= \lim_\alpha \lim_\beta \langle m(e_\alpha, e_\beta), \widehat{\widehat{b}} \cdot \mathcal{F} \cdot \widehat{a}\rangle = \lim_\alpha \lim_\beta \langle \widehat{\widehat{b}} \cdot \mathcal{F}, \widehat{\widehat{a}} \cdot m\widehat{(e_\alpha, e_\beta)}\rangle \\ &= \lim_\alpha \lim_\beta \langle \mathcal{F}, \widehat{\widehat{a}} \cdot m\widehat{(e_\alpha, e_\beta)} \cdot \widehat{\widehat{b}}\rangle = \lim_\alpha \lim_\beta \langle \mathcal{F}, \iota_{\mathcal{A}^{**}}(\widehat{a} \cdot m(e_\alpha, e_\beta) \cdot \widehat{b})\rangle \\ &= \lim_\alpha \lim_\beta \langle \mathcal{F}, \iota_{\mathcal{A}^{**}}(m(ae_\alpha, e_\beta b))\rangle = \langle \mathcal{F}, m\widehat{(a, b)}\rangle, \end{aligned}$$

which implies that $\widehat{\widehat{a}} \cdot m^{***}(E, E) \cdot \widehat{\widehat{b}} = m\widehat{(a, b)} \in \widehat{\mathcal{A}^{**}}$, as required. ∎

THEOREM 4.8. *Let $X$ be a Banach space and suppose that $X^*$ has the bounded approximation property. Then the mapping $\lambda : M_l(A^{**}) \to QM(A^{**})$, defined by*

$$\lambda(S)(a, b) = \widehat{a} \cdot Sb \quad (S \in M_l(A^{**}),\ a, b \in A),$$

*is a norm decreasing linear mapping of $M_l(A^{**})$ onto $QM(A^{**})$.*

Proof. It is clear that $\lambda$ is linear and norm decreasing. Therefore to complete the proof it is enough to show that $\lambda$ is a surjection.

Let $F = (m^{***}(E, E))|_{\widehat{A^*}}$. In the sequel we regard $F$ as an element of $A^{**}$. Let $g = (\pi^* F)^\flat$. Then $g \in \mathcal{L}(X^*)$; we recall that $\mathcal{L}(X^*)$ is isometrically isomorphic to $M_l(A^{**})$, the isomorphism $\sigma'$ being given by $\sigma'_g(f) = g \circ f$ $(g \in \mathcal{L}(X^*),\ f \in A^*)$. It follows that the identity

$$\langle f, Sa\rangle = \langle a, \sigma'_g(f)\rangle \quad (a \in A)$$

defines an element $S$ of $M_l(A^{**})$; for,

$$\langle f, S(ab)\rangle = \langle ab, \sigma_g'(f)\rangle = \langle a, b * \sigma_g'(f)\rangle$$
$$= \langle a, \sigma_g'(b * f)\rangle = \langle b * f, Sa\rangle = \langle b, f * Sa\rangle\,,$$

which implies that $S(ab) = Sa * \widehat{b}$.

Thus, for all $a, b \in A$, $f \in A^*$,

$$\langle f, \widehat{a}\cdot Sb\rangle = \langle f\cdot a, Sb\rangle = \langle b, \sigma_g'(f\cdot a)\rangle$$
$$= \langle b, (\pi^*F)^b \circ (f\cdot a)\rangle$$
$$= \langle b, (f\cdot a) * F\rangle \quad ([9], \text{Proposition 3.2(ii)})$$
$$= \langle b * (f\cdot a), F\rangle = \langle \widehat{b}\cdot(f\cdot a), F\rangle$$
$$= \langle f, \widehat{a}\cdot F\cdot \widehat{b}\rangle = \langle \widehat{f}, \widehat{\widehat{a}}\cdot m^{***}(E,E)\cdot \widehat{\widehat{b}}\rangle$$
$$= \langle \widehat{f}, \widehat{m(a,b)}\rangle \quad \text{by Lemma 4.7}$$
$$= \langle f, m(a,b)\rangle\,,$$

which implies that $a\cdot Sb = m(a,b)$. It follows that $\lambda(S) = m$, as required. ■

Remark. If $E$ is an identity for the first Arens product (this is the case if, for example, $N(X^*) = I(X^*)$; see ([9], p. 560)), then $\lambda$ is a topological isomorphism. For, in this case, $\lambda(S)(e_\alpha, a) = \widehat{e}_\alpha \cdot S_a \underset{\alpha}{\to} E\cdot S_a = S_a$ for all $a \in A$, which implies that $\lambda$ is injective and $\lambda^{-1}$ is continuous. If, in addition, $X^*$ has the metric approximation property, then $\lambda$ is an isometric isomorphism. Moreover, since $\mathcal{L}(X^{**}) \cong M_l(A^{**})$ we have $\mathcal{L}(X^{**}) \cong QM(A^{**})$ in this case.

**5. Quasi-multipliers of the Pedersen ideal.** Let $A$ be a $C^*$-algebra and $\widetilde{A}$ denote the $C^*$-algebra obtained by adjoining the identity 1 to $A$. An element $a \in A$ is said to be *positive* if it is self-adjoint and $\mathrm{Sp}(a) \subseteq \mathbb{R}_+$ ($\mathrm{Sp}(a) = \{\lambda \in \mathbb{C} : a - \lambda 1$ is singular in $\widetilde{A}\}$); the set of positive elements of $A$ is denoted by $A^+$. A subcone $J$ of $A^+$ is said to be an *order ideal* if the condition $x \le y$ for $y \in J$ and $x \in A^+$ implies that $x \in J$. An order ideal $J$ of $A^+$ is said to be *invariant* if $a^*Ja \subseteq J$ for all $a \in A$. A $*$-subalgebra $B$ of $A$ is said to be *order-related* if $B^+$ is an order ideal in $A^+$ and $B$ is the linear span of $B^+$. In ([16], Theorem 1.3), Pedersen proved that every $C^*$-algebra contains a minimal, dense, order-related, two-sided ideal $K_A$. This ideal has subsequently become known as the *Pedersen ideal*. If $X$ is a locally compact Hausdorff space and $A = C_0(X)$, the complex-valued functions on $X$ which vanish at infinity, then $K_A = C_{00}(X)$, the functions in $C_0(X)$ which have compact supports ([7], p. 109, 7E and 7F). If $A = \mathcal{B}_0(\mathcal{H})$, the $C^*$-algebra of all compact operators on a Hilbert space $\mathcal{H}$, then $K_A = \mathcal{B}_{00}(\mathcal{H})$, the operators on $\mathcal{H}$ of finite rank ([15], Theorem 2.4.7 and Theorem 3.3.3).



LEMMA 5.1. *Let $A$ be a $C^*$-algebra and let $K_A$ denote its Pedersen ideal. If $\{x_i\}$ is a finite set of elements in $K_A$, then the order-related $C^*$-algebra generated by them is contained in $K_A$.*

Proof. See ([16], Proposition 4). ■

We shall require the following version of the Cohen–Hewitt factorization theorem.

THEOREM 5.2. *Let $A$ be a $C^*$-algebra and let $\{x_i : i = 1,\dots,n\}$ be a finite set of elements in $K_A$. Then, for each $\varepsilon > 0$, there exist elements $y_1,\dots,y_n$, $z_1,\dots,z_n$ in $K_A$ and $a, b \in K_A^+$ such that*

$$\|x_i - y_i\| < \varepsilon, \quad \|x_i - z_i\| < \varepsilon\,,$$

*and $x_i = ay_i = z_ib$ for $i = 1,\dots,n$.*

Proof. Let $B$ be the order-related $C^*$-algebra generated by $\{x_i : i = 1,\dots,n\}$. By Lemma 5.1, $B \subseteq K_A$. Every $C^*$-algebra has a bounded a.i. consisting of positive elements (see, for example, the proof of ([18], p. 11, Theorem 1.4.2) and so, without loss of generality, we may assume that $B$ contains an a.i. $\{e_\alpha : \alpha \in I\}$ consisting of elements in $B^+$. By the Cohen–Hewitt factorization theorem there exist elements $y_1,\dots,y_n$, $z_1,\dots,z_n$, and $a, b$ in $B$ such that $x_i = ay_i = z_ib$ and

$$\|x_i - y_i\| < \varepsilon, \quad \|x_i - z_i\| < \varepsilon \quad (i = 1,\dots,n)\,.$$

A close examination of the proof of the Cohen–Hewitt factorization (as given, for example, in ([6], Theorem 16.1, p. 93 et seq.)) shows that $a$ (resp. $b$) is the limit of a sequence of elements in $B^+$, and so, since $B^+$ is closed in $B$, $a$ (resp. $b$) $\in B^+$. Since $B^+ \subseteq K_A^+$, the proof is complete. ■

A quasi-multiplier on $K_A$ is a mapping of $K_A \times K_A \to K_A$ which satisfies conditions (2.1). The Pedersen ideal is not in general a Banach algebra and we cannot therefore make a direct appeal to ([14], Theorem 1) to deduce that the quasi-multipliers on $K_A$ are (i) bilinear and (ii) jointly continuous. However, an application of Theorem 5.2 enables us to establish (i) as follows.

Let $m$ be a quasi-multiplier on $K_A$ and let $w, x, y \in K_A$ and $\alpha \in \mathbb{C}$. By Theorem 5.2 there exist elements $z \in K_A^+$ and $u, v \in K_A$ such that $x = uz$ and $y = vz$. Then

$$m(\alpha x + y, w) = m((\alpha u + v)z, w) = (\alpha u + v)m(z, w)$$
$$= \alpha m(uz, w) + m(vz, w) = \alpha m(x, w) + m(y, w)\,.$$

Similarly we can prove that $m(w, \alpha x + y) = \alpha m(w, x) + m(w, y)$; that is, $m$ is bilinear.

Let $\delta(K_A)$ denote the space of all quasi-multipliers on $K_A$. The members of $\delta(K_A)$ are not, in general, continuous. However, for any $m$ in $\delta(K_A)$, it is easy to show that $a \circ m \circ b \in QM(K_A)$ for $a, b \in K_A$, and that $\|a \circ m \circ b\| \leq \|m(a,b)\|$. Thus we can define the quasi-strict topology $\gamma$ on $\delta(K_A)$; that is, $\gamma$ is determined by the family of semi-norms $\{\gamma_{a,b} : a, b \in K_A\}$, where $\gamma_{a,b}(m) = \|a \circ m \circ b\|$ $(m \in \delta(K_A))$. Since every $C^*$-algebra has a bounded a.i., it follows that $K_A$ has a bounded a.i. Suppose that $\{e_\alpha : \alpha \in I\}$ is a bounded a.i. for $K_A$, with $\|e_\alpha\| \leq C$ for all $\alpha \in I$. Then, for $m \in \delta(K_A)$, $a, b \in K_A$,

$$\begin{aligned}\|a \circ m \circ b\| &= \|\Phi(m(a,b))\| \geq \limsup_\alpha \left\|\Phi(m(a,b))\left(\frac{e_\alpha}{C}, \frac{e_\alpha}{C}\right)\right\| \\ &= \frac{1}{C^2}\lim_\alpha \|e_\alpha m(a,b) e_\alpha\| = \frac{1}{C^2}\|m(a,b)\|.\end{aligned}$$

Thus

$$C^2\|a \circ m \circ b\| \geq \|m(a,b)\| \geq \|a \circ m \circ b\|,$$

which implies that the $\gamma$-topology on $\delta(K_A)$ may also be defined by the semi-norms $m \to \|m(a,b)\|$. In the sequel we find it more convenient to work with this family of semi-norms to establish properties of the locally convex space $(\delta(K_A), \gamma)$.

THEOREM 5.3. *$\delta(K_A)$ is $\gamma$-complete.*

Proof. Let $\{m_\alpha\}$ be a $\gamma$-Cauchy net in $\delta(K_A)$. Then, for $a, b \in K_A$, $\{m_\alpha(a,b)\}$ is a Cauchy net in $K_A$ and so $\lim_\alpha m_\alpha(a,b)$ exists in $A$. Define $m(a,b) = \lim_\alpha m_\alpha(a,b)$. It is clear that, for all $c, d \in K_A$, $m(ca, bd) = cm(a,b)d$. By Theorem 5.2 there exist elements $u, v, w, w'$ in $K_A$ such that $a = wu$ and $b = vw'$, and so

$$m(a,b) = \lim_\alpha w m_\alpha(u,v) w' \in K_A.$$

Thus $m \in \delta(K_A)$, and since $\gamma$-$\lim_\alpha m_\alpha = m$ it follows that $\delta(K_A)$ is $\gamma$-complete. ■

The Pedersen ideal is a $*$-ideal ([17], Lemma 1.1), and so the equation

$$m^*(a,b) = (m(b^*, a^*))^*$$

defines an element of $\delta(K_A)$. The mapping $m \to m^*$ defines an involution on $\delta(K_A)$ and is continuous with respect to the $\gamma$-topology; for, if $m = \gamma$-$\lim_\alpha m_\alpha$, then, for any $x, y \in K_A$,

$$\begin{aligned}\lim_\alpha m_\alpha^*(x,y) &= \lim_\alpha (m_\alpha(y^*, x^*))^* = (\lim_\alpha m_\alpha(y^*, x^*))^* \\ &= (m(y^*, x^*))^* = m^*(x,y),\end{aligned}$$

which implies that $m^* = \gamma$-$\lim_\alpha m_\alpha^*$.

For the next two results we assume that $K_A$ contains a bounded central a.i. This is the case if, for example, $A$ is a quasi-central $C^*$-algebra. The notion of a quasi-central $C^*$-algebra was considered by Archbold in [2], where a $C^*$-algebra is defined to be *quasi-central* if no primitive ideal contains the centre. A $C^*$-algebra is quasi-central if and only if it has a bounded a.i. which belongs to the centre $Z(A)$ of $A$ ([2], Proposition 1), and Archbold also proved ([2], Theorem 3) that, if $I$ is an ideal of $A$, then $\overline{I \cap Z(A)} = \bar{I} \cap Z(A)$. Thus, in the special case when $I = K_A$, $K_A \cap Z(A)$ is dense in $Z(A)$, and so, if $Z(A)$ contains a bounded a.i., then $K_A$ contains a bounded a.i. which is central.



THEOREM 5.4. *Let $A$ be a $C^*$-algebra and suppose that $K_A$ contains a bounded central a.i. Then $K_A$ is $\gamma$-dense in $\delta(K_A)$.*

Proof. Let $\{e_\alpha\}$ be an a.i. in $K_A$ with the required property and suppose that $m \in \delta(K_A)$. Then, for any $a, b \in K_A$,

$$\begin{aligned}m(a,b) &= \lim_\alpha e_\alpha m(a,b) e_\alpha = \lim_\alpha m(e_\alpha a, b e_\alpha) \\ &= \lim_\alpha a m(e_\alpha, e_\alpha) b = \lim_\alpha \Phi(m(e_\alpha, e_\alpha))(a,b),\end{aligned}$$

which implies that $m = \gamma$-$\lim_\alpha \Phi(m(e_\alpha, e_\alpha))$; that is, $K_A$ is $\gamma$-dense in $\delta(K_A)$. ■

THEOREM 5.5. *Let $A$ and $B$ be $C^*$-algebras and suppose that $K_A$ has a bounded central a.i. If $\phi$ is a $*$-homomorphism of $A$ onto $B$, then*

(i) *$\phi$ can be extended to a $*$-linear mapping $\widetilde{\phi}$ of $\delta(K_A)$ into $\delta(K_B)$, and*

(ii) *the mapping $\widetilde{\phi}$ is $\gamma$-continuous.*

Proof. We first note that $\phi$ is norm decreasing ([18], p. 16, Theorem 1.5.7) and that $\phi(K_A) = K_B$ ([17], Corollary 6). Let $b_1, b_2 \in K_B$ and suppose that $x_1, x_2, y_1, y_2$ are elements of $K_A$ such that $\phi(x_1) = \phi(y_1) = b_1$ and $\phi(x_2) = \phi(y_2) = b_2$. If $\{e_\alpha\}$ is a central bounded a.i. in $K_A$, then, for any $m \in \delta(K_A)$,

$$\begin{aligned}\phi(m(x_1, x_2)) &= \lim_\alpha \phi(e_\alpha m(x_1, x_2) e_\alpha) = \lim_\alpha \phi(m(x_1 e_\alpha, e_\alpha x_2)) \\ &= \lim_\alpha \phi(x_1)\phi(m(e_\alpha, e_\alpha))\phi(x_2) \\ &= \lim_\alpha \phi(y_1)\phi(m(e_\alpha, e_\alpha))\phi(y_2) = \phi(m(y_1, y_2)).\end{aligned}$$

Thus the equation $\widetilde{\phi}(m)(b_1, b_2) = \phi(m(x_1, x_2))$ defines a mapping $\widetilde{\phi}(m) : K_B \times K_B \to K_B$. It is routine to show that $\widetilde{\phi}(m) \in \delta(K_B)$ and that the mapping $m \to \widetilde{\phi}(m)$ of $\delta(K_A)$ into $\delta(K_B)$ is linear and is an extension of $\phi$. To complete the proof of (i) we show that $m^* \to (\widetilde{\phi}(m))^*$. Let $a, b \in K_B$ and suppose that $a = \phi(x)$, $b = \phi(y)$, where $x$ and $y$ are in $K_A$. Then, since

$\phi$ is a $*$-homomorphism, $a^* = \phi(x^*)$ and $b^* = \phi(y^*)$. Thus

$$(\widetilde{\phi}(m^*))(a,b) = \phi(m^*(x,y)) = \phi((m(y^*,x^*))^*) = (\phi(m(y^*,x^*)))^* = ((\widetilde{\phi}(m))(b^*,a^*))^* = (\widetilde{\phi}(m))^*(a,b);$$

that is, $\widetilde{\phi}(m^*) = (\widetilde{\phi}(m))^*$, as required.

To prove (ii), let $b_1, b_2$ be any elements of $K_B$ and suppose that $x_1, x_2 \in K_A$ are such that $\phi(x_1) = b_1$ and $\phi(x_2) = b_2$. Then

$$\|\widetilde{\phi}(m)(b_1,b_2)\| = \|\phi(m(x_1,x_2))\| \le \|m(x_1,x_2)\|,$$

which implies that $\widetilde{\phi} : (\delta(K_A),\gamma) \to (\delta(K_B),\gamma)$ is continuous. ∎

LEMMA 5.6. *The sets*

$$V_{a,b} = \{m \in \delta(K_A) : \|m(a,b)\| \le 1,\ a,b \in K_A^+\}$$

*form a neighbourhood base at 0 for the $\gamma$-topology on $\delta(K_A)$.*

Proof. Clearly $V_{a,b}$ is a $\gamma$-neighbourhood of 0 in $\delta(K_A)$ for each $a, b \in K_A^+$. On the other hand, let $U$ be any $\gamma$-neighbourhood of 0. Then there exist elements $x_1,\ldots,x_n,\ y_1,\ldots,y_m$ in $K_A$ such that

$$\{m \in \delta(K_A) : \|m(x_i,y_j)\| \le 1\ (i = 1,\ldots,n,\ j = 1,\ldots,m)\} \subseteq U.$$

By Theorem 5.2 there exist $a_1, b_1 \in K_A^+$ and $u_1,\ldots,u_n,\ v_1,\ldots,v_m$ in $K_A$ such that

$$x_i = u_i a_1\ (i = 1,\ldots,n) \quad \text{and} \quad y_j = b_1 v_j\ (j = 1,\ldots,m).$$

Let $M = \max\{\|u_i\|\ \|v_j\| : 1 \le i \le n,\ 1 \le j \le m\}$ and let $a = \sqrt{M} a_1$, $b = \sqrt{M} b_1$. Then $a, b \in K_A^+$ and, for any $m \in V_{a,b}$, $i = 1,\ldots,n$, $j = 1,\ldots,m$,

$$\|m(x_i,y_j)\| = \|m(u_i a_1, b_1 v_j)\| \le \|u_i\|\ \|m(a_1,b_1)\|\ \|v_j\| \le \|m(a,b)\| \le 1,$$

which implies that $V_{a,b} \subseteq U$, as required. ∎

We now establish a characterization for the $\gamma$-dual of $\delta(K_A)$.

THEOREM 5.7. *Let $A$ be a $C^*$-algebra and $K_A$ its Pedersen ideal. Then*

$$(\delta(K_A),\gamma)^* = \{a \cdot g \cdot b : a,b \in K_A^+,\ g \in A^*\},$$

*where $a \cdot g \cdot b$ is the functional on $\delta(K_A)$ defined by $(a \cdot g \cdot b)(m) = g(m(b,a))$.*

Proof. We note that when $a \cdot g \cdot b$ is restricted to $A$ it agrees with the usual Arens product of $a, b$ and $g$; this justifies our use of the notation to define the functional $a \cdot g \cdot b$. Since

$$|(a \cdot g \cdot b)(m)| \le \|g\|\ \|m(b,a)\| \quad (m \in \delta(K_A))$$

it follows that $a \cdot g \cdot b \in (\delta(K_A),\gamma)^*$ for each $a, b \in K_A^+$ and $g \in A^*$.



On the other hand, suppose that $f \in (\delta(K_A),\gamma)^*$. By Lemma 5.6 there exist $a, b \in K_A^+$ such that $|f(m)| \le 1$ whenever $\|m(b,a)\| \le 1$. This implies that, for any $m \in \delta(K_A)$, $|f(m)| \le \|m(b,a)\|$. On the subspace $W_{a,b} = \{m(b,a) : m \in \delta(K_A)\}$ we define the functional $g$ by

$$g(m(b,a)) = f(m).$$

It is clear that $g$ is well defined and that $g \in (W_{a,b}, \|\cdot\|)^*$, with $\|g\| \le 1$. By the Hahn–Banach theorem $g$ has a continuous extension to all of $A$; we retain the notation $g$ to denote the extension so that $g \in A^*$ and $\|g\| \le 1$. By the first part of the proof $a \cdot g \cdot b \in (\delta(K_A),\gamma)'$. Moreover, for any $m \in \delta(K_A)$, $(a \cdot g \cdot b)(m) = g(m(b,a)) = f(m)$; that is, $f = a \cdot g \cdot b$. ∎

Let $\{A_i : i \in I\}$ be a family of $C^*$-algebras and let $A = (\sum_{i\in I} A_i)_0$; that is, $A$ consists of those elements $a = (a_i)$ such that, for each $\varepsilon > 0$, the set $\{i \in I : \|a_i\| \ge \varepsilon\}$ is finite. With the usual operations of addition and multiplication, $A$ is a $C^*$-algebra, the norm being given by $\|a\| = \sup_i \|a_i\|$ and the involution by $a^* = (a_i^*)$. If $K_{A_i}$ denotes the Pedersen ideal of $A_i$, then the Pedersen ideal $K_A$ of $A$ consists of those elements $(a_i)$ such that $a_i \in K_{A_i}$ and $a_i = 0$ except for a finite number of the $i$'s. We let $\gamma$ and $\gamma_i$ denote the quasi-strict topologies on $\delta(K_A)$ and $\delta(K_{A_i})$ respectively.

THEOREM 5.8. *Let $\{A_i : i \in I\}$ be a family of $C^*$-algebras and suppose that each $A_i$ has a minimal central a.i. If $A = (\sum_{i\in I} A_i)_0$, then $\delta(K_A)$ is topologically $*$-isomorphic to $\prod_{i\in I} \delta(K_{A_i})$, the topology on $\delta(K_A)$ being $\gamma$ and that of $\prod_{i\in I} \delta(K_{A_i})$ being the product of the spaces $(\delta(K_{A_i}),\gamma_i)$.*

Proof. For each $i \in I$, $\phi_i(a) = a_i$ $(a = (a_i) \in A)$ defines a natural $*$-homomorphism $\phi_i$ of $A$ onto $A_i$. The hypothesis ensures that $A$ has a bounded central a.i. and so by Theorem 5.5 each $\phi_i$ has an extension to a $*$-linear mapping, $\widetilde{\phi}_i$ say, of $\delta(K_A)$ into $\delta(K_{A_i})$. Each $\widetilde{\phi}_i$ is given by the equation

$$(\widetilde{\phi}_i(m))(a_i,b_i) = \phi_i(m(\widetilde{a}_i,\widetilde{b}_i)) \quad (m \in \delta(K_A)),$$

where $a_i \to \widetilde{a}_i$ is the natural embedding of $K_{A_i}$ into $K_A$.

We define a mapping $\xi : \delta(K_A) \to \prod_{i\in I} \delta(K_{A_i})$ by

$$\xi(m) = (\widetilde{\phi}_i(m)).$$

Clearly $\xi$ is a $*$-linear mapping. It is also surjective, as follows. Let $(m_i)$ be any element of $\prod_{i\in I} \delta(K_{A_i})$. If $a, b$ are any elements of $K_A$, then $a_i = 0$ and $b_j = 0$ except for a finite number of $i$'s and $j$'s and so $m_i(a_i,b_i) = 0$ except for a finite number of $i$'s. Thus the equation

$$(m(a,b))_i = m_i(a_i,b_i) \quad (i \in I)$$

defines a mapping $m : K_A \times K_A \to K_A$ and it is easy to see that $m \in \delta(K_A)$.

Moreover, $\xi(m) = (m_i)$; for, if $j$ is any index in $I$, then

$$(\widetilde{\phi}_j(m))(a_j, b_j) = \phi_j(m(\widetilde{a}_j, \widetilde{b}_j)) = (m(\widetilde{a}_j, \widetilde{b}_j))_j = m_j(a_j, b_j)\,,$$

which implies that $\widetilde{\phi}_j(m) = m_j$; that is, $\xi(m) = (m_i)$, as required.

It is clear that $\xi$ is injective. Finally, we show that $\xi$ is topological. Let $U$ be any neighbourhood of 0 in $\prod_{i\in I}\delta(K_{A_i})$. Then there exist elements $a_{i_j}, b_{i_j} \in K_{A_{i_j}}$ $(1 \le j \le n)$ such that

$$\{(m_i) : \|m_{i_j}(a_{i_j}, b_{i_j})\| \le 1,\ 1 \le j \le n\} \subseteq U\,.$$

Let $a = \sum_{j=1}^n \widetilde{a}_{i_j}$, $b = \sum_{j=1}^n \widetilde{b}_{i_j}$ and $V = \{m \in \delta(K_A) : \|m(a,b)\| \le 1\}$. If $m \in V$, then, for $1 \le j \le n$,

$$\|(\widetilde{\phi}_{i_j}(m))(a_{i_j}, b_{i_j})\| = \|\phi_{i_j}(m(\widetilde{a}_{i_j}, \widetilde{b}_{i_j}))\|\,.$$

Since $\phi_{i_j}(\widetilde{a}_{i_j}) = \phi_{i_j}(a)$ and $\phi_{i_j}(\widetilde{b}_{i_j}) = \phi_{i_j}(b)$, it follows from the proof of Theorem 5.5 that

$$\|\phi_{i_j}(m(\widetilde{a}_{i_j}, \widetilde{b}_{i_j}))\| = \|\phi_{i_j}(m(a,b))\|\,.$$

Thus

$$\|(\widetilde{\phi}_{i_j}(m))(a_{i_j}, b_{i_j})\| = \|(m(a,b))_{i_j}\| = \|m_{i_j}(a_{i_j}, b_{i_j})\| \le 1\,;$$

that is, $\xi(V) \subseteq U$, and so $\xi$ is continuous.

On the other hand, suppose that $a, b \in K_A$. Then $a_i = 0$ (resp. $b_j = 0$) except for $i$ (resp. $j$) in a finite set of indices, say $I_1$ (resp. $I_2$). Suppose that $(m_i)$ satisfies $\|m_i(a_i, b_i)\| \le 1$ for $i \in I_1 \cup I_2$. Then

$$\|\xi^{-1}(m_i)(a,b)\| = \sup_i \|m_i(a_i, b_i)\| \le 1\,,$$

which implies that $\xi^{-1}$ is continuous, as required. ∎

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