Walsh–Dirichlet kernels $D_j(t)$ (chosen for $\Psi_j(t)$) have the properties (7.2)–(7.4) (cf. [8] or [15]). Hence, the conclusion of Theorem 1* holds for double Walsh series, or more generally, for n-dimensional Walsh series. Its corollaries generalize the corresponding results in [1] and [14]. Other generalizations are left to the reader.

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Nonaccessible filters in measure algebras and functionals on $L^\infty(\lambda)^*$

by

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Abstract. In a nonatomic measure algebra, we construct a nonprincipal filter with the inaccessibility property considered by Kunen [7]. Using that filter we define two "pathological" functionals on $L^\infty(\lambda)^*$. It follows that the Banach space $L^\infty(\lambda)$ is not realcompact whenever the measure λ is not separable.

The main aim of this paper is to prove that a Banach space $L^\infty(\lambda)$ is not realcompact in its weak topology whenever the measure λ is not separable. According to a characterization of realcompact Banach spaces due to Corson (see the next section), it suffices to find a functional from $L^\infty(\lambda)^{**}\setminus L^\infty(\lambda)$ which, roughly speaking, behaves like an element of $L^\infty(\lambda)$, when considered on countable subsets of $L^\infty(\lambda)^*$.

Actually, we shall be dealing with the usual measure λ on the Cantor cube 2^{ω_1}. We find it convenient to treat $C(S)$, the space of continuous functions on the Stone space of λ, rather than the space $L^\infty(\lambda)$ itself. We shall show that one may define a functional with the required properties putting $\mu \to \mu(F)$ for $\mu \in C(S)^*$, where F is a certain closed subset of S. In fact, F will be defined as the set of all ultrafilters from S that extend a suitably chosen filter in $\mathbf{B}$. To make the idea work, we consider a property of filters in measure algebras that has been invented by Kunen [7] for another purpose. It is rather technical; we call it Kunen's property (see Section 2).

It is shown in [7] that in measure algebras of cardinality $\mathfrak{c}$ there are ultrafilters with Kunen's property provided Martin's Axiom holds. However, as explained in Section 2, given such an ultrafilter one may construct a p-point in $\beta\omega\setminus\omega$, which indicates that the existence of ultrafilters with Kunen's property is independent of the usual axioms. Therefore we present a

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construction of a filter with Kunen's property which, although not maximal, is sufficiently large for the purpose mentioned above.

That filter is also used in the last section, where we show by similar methods that there is a functional from $C(S)^{**}$ which lies in the realcompactification of $C(S)$ but is not defined by a single measurable function on S.

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1. Preliminaries. A Banach space E is said to be *realcompact* if (E, weak) has this property in the usual topological sense, that is, if every 0-1 weak Baire measure on E is concentrated at some point (cf. [15]). This is equivalent to saying that (E, weak) is homeomorphic to a closed subset of some product of the real lines. As proved by Corson [1], every Banach space has a Hewitt realcompactification in (E^{**}, weak^*), consisting of all functionals that are weak* continuous on weak* separable subspaces of E^*. That result yields the following useful characterization of realcompactness in terms of functionals on E^* (cf. [14], 2-4-2).

THEOREM 1. *For a Banach space E the following are equivalent:*

(a) *E is realcompact;*

(b) *every $z \in E^{**}$ which is weak* continuous on weak* separable subspaces of E^* is weak* continuous (i.e. $z \in E$);*

(c) *given $z \in E^{**}$, if for every countable subset $D \subseteq E^*$ there is $x \in E$ that agrees with z on D, then z is weak* continuous (i.e. $z \in E$).*

For the basic properties of realcompact Banach spaces and connections of this notion with several others, the reader is referred to Edgar's survey papers [2, 3]. We only recall that realcompact Banach spaces can be quite "big". For instance, it is not difficult to prove that $l^1(\kappa)$ is realcompact provided there is no measurable cardinal less than or equal to κ (see e.g. [14], 16-2-6). A more involved argument shows that under the same cardinal restriction $C(2^\kappa)$, the Banach space of continuous functions on the Cantor cube 2^κ is realcompact (see Talagrand [13], 16-3; cf. Plebanek [9]). Banach spaces $C(K)$, for K compact are also discussed in this context in Plebanek [10].

Given a finite measure λ, $L^\infty(\lambda)$ denotes the usual Banach space of real-valued essentially bounded functions. The Banach space $L^\infty(\lambda)$ is isometric to $C(S)$, the space of continuous real-valued functions on S, the Stone space of the measure algebra. As usual, $C(S)^*$ will be identified with the space $M(S)$ of all signed Radon measures on S of bounded variation ($M^+(S)$ and $M_1^+(S)$ denote the families of nonnegative measures and probability measures, respectively).

It is clear from Theorem 1 that a Banach space E is realcompact whenever E^* is weak* separable. Thus the space $L^\infty(\lambda)$ is realcompact for every separable finite measure λ (since $L^1(\lambda)$ is then separable and weak* dense in $L^\infty(\lambda)^*$).

Now we introduce the notation that will be in constant use. Let $\mathcal{B}(2^{\omega_1})$ be the σ-algebra of Borel subsets of the Cantor cube 2^{ω_1}. Unless stated otherwise, λ stands for the usual (product) measure on $\mathcal{B}(2^{\omega_1})$ and $\mathbf{B} = \mathcal{B}(2^{\omega_1})/\lambda$ is the measure algebra of λ. Concerning measure algebras, we essentially follow the notation of Fremlin's article [6].

The canonical epimorphism from $\mathcal{B}(2^{\omega_1})$ onto $\mathbf{B}$ is denoted by $\widehat{\ }$, that is, $\widehat{B}$ is an element of $\mathbf{B}$ obtained from $B \in \mathcal{B}(2^{\omega_1})$. The Stone space of all ultrafilters in $\mathbf{B}$ is always denoted by S. Further, we shall denote by V the canonical isomorphism between $\mathbf{B}$ and the algebra of closed and open subsets of S, i.e.

$$V(a) = \{p \in S : a \in p\}.$$

We use some standard results on the measure λ. For instance, for every $a \in \mathbf{B}$ there is $B \in \mathcal{B}(2^{\omega_1})$ depending on a countable set of coordinates and such that $a = \widehat{B}$ (see e.g. [6], 1.15). We write $B \sim I$ whenever B depends on a set $I \subseteq \omega_1$, that is, $B = B_0 \times 2^{\omega_1 \setminus I}$ for some $B_0 \in \mathcal{B}(2^I)$. Similarly, every λ-measurable function on 2^{ω_1} is λ-almost everywhere equal to a function depending on a countable set of coordinates.

There are natural correspondences between Radon measures on S, finitely additive measures on $\mathbf{B}$, and measures on $\mathcal{B}(2^{\omega_1})$ that are absolutely continuous with respect to λ. Therefore, we may treat a given $\mu \in M^+(S)$ as either a finitely additive function on $\mathbf{B}$ or as a measure on $\mathcal{B}(2^{\omega_1})$ (denoting them by the same letter).

2. On filters with Kunen's property. Let $(\mathbf{A}, \mu)$ be an arbitrary measure algebra. We say that a filter $\mathbf{F} \subseteq \mathbf{A}$ has *Kunen's property* if for every double sequence $(a_{nk})_{n,k\in\omega} \subseteq \mathbf{A}$ having for every n the properties:

(i) $-a_{n0} \in \mathbf{F}$,

(ii) $a_{n0} \geq a_{n1} \geq a_{n2} \geq \ldots$,

(iii) $\lim_{k\to\infty} \mu(a_{nk}) = 0$,

there is $d \in \mathbf{F}$ such that for every n there is k with $d \cdot a_{nk} = \mathbf{0}$.

The following result is due to Kunen [7].

THEOREM 2. *Under Martin's Axiom, in every nonatomic measure algebra of cardinality* $\mathfrak{c}$ *there exists an ultrafilter with Kunen's property.*

It is easy to check that an ultrafilter with Kunen's property is a weak p-point in the Stone space, that is, a point which is not a cluster point of any countable subset. Theorem 2 was used in [7] as a tool for constructing ultrafilters in $\beta\omega \setminus \omega$ with certain inaccessibility properties (cf. [4]). The existence of weak p-points in the former space was obtained by Kunen in [8] by a different approach that does not require Martin's Axiom. Let us note, however, that ultrafilters with Kunen's property may not exist.

Suppose that $\mathbf{F}$ is an ultrafilter in $\mathbf{B}$ with Kunen's property. Choose a partition $(a_n)_{n\in\omega} \subseteq \mathbf{B}\setminus\mathbf{F}$ and consider the family $\mathcal{P}$ of subsets of ω given by

$$\mathcal{P} = \Big\{N \subseteq \omega : \sum_{n\in N} a_n \in \mathbf{F}\Big\}.$$

Clearly $\mathcal{P}$ is a nonprincipal ultrafilter on ω.

Actually, $\mathcal{P}$ is a p-point in $\beta\omega\setminus\omega$, that is, every $\mathcal{G}_\delta$ set containing $\mathcal{P}$ is a neighbourhood of $\mathcal{P}$. To check this take a partition (N_m) of ω such that no N_m is in $\mathcal{P}$. Putting, for every k, $b_{mk} = \sum\{a_n : n \in N_m,\ n \geq k\}$ we find that b_{mk}'s are as in the definition of Kunen's property. Thus there is $d \in \mathbf{F}$ such that, for every m, $d\cdot b_{mk} = \mathbf{0}$ if k is large enough. Let $M = \{n \in \omega : d\cdot a_n \neq \mathbf{0}\}$. Such a set M belongs to $\mathcal{P}$ and is almost disjoint from every N_m. The property of $\mathcal{P}$ we have checked is one of the combinatorial characterizations of p-pointedness. Since the existence of a p-point is not provable in ZFC (see Shelah [12], VI.4.4.8, cf. [5]), neither is the existence of ultrafilters with Kunen's property.

Now we describe a construction of a certain filter with Kunen's property; the essential part of our construction is based on the following simple trick. Consider a sequence $(a_n)_{n\in\omega}$ of nonzero elements of $\mathbf{B}$. Let $\mathbf{A}$ be a complete subalgebra of $\mathbf{B}$ containing a_n's. It is easy to see that the family

$$(*)\qquad \mathbf{F} = \{x \in \mathbf{A} : \lim_{n\to\infty} \lambda(a_n - x)/\lambda(a_n) = 0\}$$

is a filter in $\mathbf{A}$. Moreover, $\mathbf{F}$ is nonprincipal whenever $\sum_n \lambda(a_n) < \infty$.

LEMMA 2.1. *Every filter $\mathbf{F} \subseteq \mathbf{A}$ defined by $(*)$ has Kunen's property.*

Proof. Let $\mathbf{F}$ be built from the sequence $(a_n)_{n\in\omega}$. Take a double sequence (b_{mk}) from $\mathbf{A}$ with the properties (i)–(iii). For any function $\varphi \in \omega^\omega$ we put $b(\varphi) = \sum_{n\in\omega} b_{m\varphi(m)}$. We have to find a function φ such that $-b(\varphi) \in \mathbf{F}$.

For every $\varepsilon > 0$ there is $\varphi \in \omega^\omega$ such that $\lambda(a_n\cdot b(\varphi)) \leq \varepsilon\lambda(a_n)$ for every n. Indeed, given $m \in \omega$, $-b_{m0} \in \mathbf{F}$ so $\lim_{n\to\infty} \lambda(a_n\cdot b_{m0})/\lambda(a_n) = 0$. Hence there is $N \in \omega$ such that $\lambda(a_n\cdot b_{m0})/\lambda(a_n) \leq \varepsilon 2^{-m-1}$ for every $n > N$. There is $\varphi(m) \in \omega$ such that $\lambda(a_n\cdot b_{m\varphi(m)})/\lambda(a_n) \leq \varepsilon 2^{-m-1}$ for $n < N$. For φ so defined we have

$$\frac{\lambda(a_n\cdot b(\varphi))}{\lambda(a_n)} \leq \sum_{m\in\omega} \frac{\lambda(a_n\cdot b_{m\varphi(m)})}{\lambda(a_n)} \leq \varepsilon,$$

as required.

Let φ_r be a function having the above property with respect to $\varepsilon = 1/r$, $r \geq 1$. Take a function φ that eventually dominates every φ_r, say $\varphi(m) \geq \varphi_r(m)$ for $m \geq r$. Now we have

$$\frac{\lambda(a_n\cdot b(\varphi))}{\lambda(a_n)} \leq \frac{\lambda(a_n\cdot \sum_{m<r} b_{m\varphi(m)})}{\lambda(a_n)} + \frac{\lambda(a_n\cdot \sum_{m\geq r} b_{m\varphi_r(m)})}{\lambda(a_n)} \leq 2/r\,,$$

for all r and n sufficiently large. Hence $\lim_{n\to\infty} \lambda(a_n\cdot b(\varphi))/\lambda(a_n) = 0$, so $-b(\varphi) \in \mathbf{F}$. The proof is complete.

We shall use the following combinatorial lemma.

LEMMA 2.2. *There exists a family $(\Gamma_\alpha)_{\alpha<\omega_1}$ of functions such that every Γ_α maps ω into the family of finite subsets of α and the following are satisfied for every α:*

(a) *$\Gamma_\alpha(n) \subseteq \Gamma_\alpha(n+1)$ for all n;*

(b) *$\bigcup_{n<\omega} \Gamma_\alpha(n) = \alpha$;*

(c) *if $\beta < \alpha$ then there is m such that $\Gamma_\alpha(n)\cap\beta = \Gamma_\beta(n)$ for every $n \geq m$.*

Proof. We put $\Gamma_0(n) = \emptyset$ and proceed by induction. It is clear that, having defined Γ_α, we may put $\Gamma_{\alpha+1}(n) = \Gamma_\alpha(n)\cup\{\alpha\}$.

Now take a limit ordinal $\gamma < \omega_1$ and suppose that Γ_α satisfy (a)–(c) for every $\alpha < \gamma$. Let $(\alpha_k)_{k<\omega}$ be a fixed increasing sequence that is cofinal in γ. We choose for every k a natural number $r_k \geq k$ so that $\Gamma_{\alpha_k}(n)\cap\alpha_i = \Gamma_{\alpha_i}(n)$ whenever $n \geq r_k$ and $i < k$. We shall check that Γ_γ given by

$$\Gamma_\gamma(n) = \bigcup\{\Gamma_{\alpha_k}(n) : r_k \leq n\}$$

has the desired properties.

It is clear that $\Gamma_\gamma(n)$ is a finite subset of γ and (a) holds. If $\xi \in \gamma$ then there is k such that $\alpha_k > \xi$ and so $\xi \in \Gamma_{\alpha_k}(m)$ for some m. Then $\xi \in \Gamma_\gamma(n)$ provided $n \geq \max\{m, r_k\}$. This shows that Γ_γ satisfies (b).

To check (c) we first note that $\Gamma_\gamma(n)\cap\alpha_i = \Gamma_{\alpha_i}(n)$ for every $n \geq r_i$. Indeed, $\Gamma_\gamma(n)\cap\alpha_i$ is the union of the sets $\Gamma_{\alpha_k}(n)\cap\alpha_i$, where $r_k \leq n$. If $k \geq i$ then $\Gamma_{\alpha_k}(n)\cap\alpha_i = \Gamma_{\alpha_i}(n)$ by our choice of r_k. If $k < i$ then $\Gamma_{\alpha_k}(n)\cap\alpha_i = \Gamma_{\alpha_k}(n) \subseteq \Gamma_{\alpha_i}(n)$, since $n \geq r_i$.

Now we prove that (c) holds for Γ_γ. Take any $\alpha < \gamma$ and $\alpha_i > \alpha$. As we have checked, $\Gamma_\gamma(n)\cap\alpha_i = \Gamma_{\alpha_i}(n)$ for n sufficiently large; on the other hand, $\Gamma_{\alpha_i}(n)\cap\alpha = \Gamma_\alpha(n)$ eventually holds by the inductive assumption. This gives $\Gamma_\gamma(n)\cap\alpha = \Gamma_\alpha(n)$, and the proof is complete.

For a given $I \subseteq \omega_1$ we denote by $\mathbf{B}(I)$ the complete subalgebra of $\mathbf{B}$ consisting of all elements depending on I, i.e.

$$\mathbf{B}(I) = \{\widehat{X} : X \in \mathcal{B}(2^{\omega_1}),\ X \sim I\}.$$

PROPOSITION 2.3. *There exists a filter* $\mathbf{F}$ *in* $\mathbf{B}$ *with Kunen's property and such that for every countable subset* I *of* ω_1 *there is* $B \in \mathcal{B}(2^{\omega_1})$ *depending on* $\omega_1 \setminus I$ *with* $\lambda(B) = 1/2$ *and* $\widehat{B} \in \mathbf{F}$.

Proof. Write ω_1 as the union of a pairwise disjoint family of infinite countable sets $(I_\alpha)_{\alpha<\omega_1}$ and put, for every $\alpha < \omega_1$, $J_\alpha = \bigcup_{\beta<\alpha} I_\beta$. Let $\mathbf{F}_\alpha$ be a filter in $\mathbf{B}(I_\alpha)$ defined as in $(*)$ for a sequence $(a_n^\alpha)_n \subseteq \mathbf{B}(I_\alpha)$ (which is assumed to satisfy $\sum_n \lambda(a_n^\alpha) < \infty$).

We define an increasing family $(\mathbf{H}_\alpha)_{0<\alpha<\omega_1}$, where $\mathbf{H}_\alpha$ is a filter in the algebra $\mathbf{B}(J_\alpha)$, as follows. For every $\alpha < \omega_1$ and every n we put

$$c_n^\alpha = \prod_{\xi\in\Gamma_\alpha(n)} a_n^\xi ,$$

where Γ_α's are functions as in Lemma 2.2. It is clear that c_n^α's form a sequence of nonzero elements of the algebra $\mathbf{B}(J_\alpha)$. We let $\mathbf{H}_\alpha$ be the filter defined as in $(*)$, but with respect to c_n^α's, i.e.

$$(*) \qquad \mathbf{H}_\alpha = \{x \in \mathbf{B}(J_\alpha) : \lim_{n\to\infty} \lambda(c_n^\alpha - x)/\lambda(c_n^\alpha) = 0\} .$$

Then $\mathbf{H}_\alpha$ is a nonprincipal filter with Kunen's property (by Lemma 2.1).

We check that $\mathbf{H}_\beta \subseteq \mathbf{H}_\alpha$ whenever $\beta < \alpha$. Take $x \in \mathbf{H}_\beta$; since $\Gamma_\alpha(n) \cap \beta = \Gamma_\beta(n)$ for n large enough we have

$$\lambda(x\cdot c_n^\alpha) = \lambda\Big(x\cdot \prod_{\xi\in\Gamma_\alpha(n)} a_n^\xi\Big) = \lambda\Big(x\cdot \prod_{\xi\in\Gamma_\beta(n)} a_n^\xi\Big)\lambda\Big(\prod_{\xi\in\Gamma_\alpha(n)\setminus\beta} a_n^\xi\Big),$$

since the factors of the latter product are stochastically independent of those from the former. Accordingly,

$$\lambda(c_n^\alpha) = \lambda\Big(\prod_{\xi\in\Gamma_\alpha(n)} a_n^\xi\Big) = \lambda\Big(\prod_{\xi\in\Gamma_\beta(n)} a_n^\xi\Big)\lambda\Big(\prod_{\xi\in\Gamma_\alpha(n)\setminus\beta} a_n^\xi\Big).$$

This gives

$$\frac{\lambda(x\cdot c_n^\alpha)}{\lambda(c_n^\alpha)} = \frac{\lambda(x\cdot c_n^\beta)}{\lambda(c_n^\beta)} \to 1 ,$$

so the inclusion in question is verified.

An analogous argument shows that $\mathbf{F}_\beta \subseteq \mathbf{H}_\alpha$ for $\beta < \alpha$.

Finally, we put

$$\mathbf{F} = \bigcup_{\alpha<\omega_1} \mathbf{H}_\alpha .$$

Being the increasing union of an uncountable family of filters with Kunen's property, $\mathbf{F}$ is a filter with Kunen's property. The rest follows from the fact that $\mathbf{F}$ contains every $\mathbf{F}_\alpha$.

3. A pathological functional. Let $\mathbf{F} \subseteq \mathbf{B}$ be a filter as in Proposition 2.3. We denote by F the closed subset associated with $\mathbf{F}$, i.e.

$$F = \{p \in S : \mathbf{F} \subseteq p\} .$$

The formula $z(\mu) = \mu(F)$ defines a continuous functional on $M(S)$; in fact, $z \in C(S)^{**} \setminus C(S)$ since F is not open in S. We shall check that z has the property as in part (c) of Theorem 1, that is, for every countable family of measures $M \subseteq M(S)$ there exists a function from $C(S)$ that agrees with z on M.

LEMMA 3.1. *Let* $\mu \in M_1^+(S)$ *be a measure singular with respect to* λ *and such that* $\mu(F) = 0$. *Then* F *is disjoint from the support of* μ, *i.e.* μ *vanishes on some neighbourhood of* F.

Proof. Since $\mu(F) = 0$ and μ is outer-regular we can find open neighbourhoods of F of arbitrarily small measure μ. If we consider μ as a finitely additive measure on $\mathbf{B}$ this means that for every $m \in \omega$ there is $d_m \in \mathbf{F}$ with $\mu(d_m) \le 1/(m+1)$. We put $a_{m0} = -d_m$. By singularity, for every m there is a sequence $a_{m0} \ge a_{m1} \ge \ldots$ with $\mu(a_{mk}) \ge 1-2/(m+1)$ and $\lambda(a_{nk}) \le 1/k$.

Now, as $\mathbf{F}$ has Kunen's property, there is $d \in \mathbf{F}$ such that for every m there is k with $d\cdot a_{mk} = 0$. This means $\mu(d) \le 2/(m+1)$ for every m, so $\mu(d) = 0$, and we are done.

LEMMA 3.2. *If* $(\mu_n)_{n\in\omega} \subseteq M_1^+(S)$ *is a sequence of measures that vanishes on* F *then there exists* $g \in C(S)$ *such that* $g = 1$ *on* F *and* $\mu_n(g) = 0$ *for all* n.

Proof. For every n, write $\mu_n = \mu_n^1 + \mu_n^2$, where μ_n^1 is absolutely continuous with respect to λ and μ_n^2 is singular.

Applying the lemma above to the measure $\mu = \sum_n 2^{-n}\mu_n^2$, we infer that there is $d \in \mathbf{F}$ such that $\mu_n^2(d) = 0$ for all n. Every measure μ_n^1, when considered as a measure on $\mathcal{B}(2^{\omega_1})$, is given by $\mu_n^1(B) = \int_B \varphi_n \, d\lambda$, where φ_n is the Radon–Nikodym derivative. Since every φ_n is λ-almost everywhere equal to a function depending on a countable set of coordinates, we may assume that all φ_n's depend on a countable set $I \subseteq \omega_1$. Moreover, we may take I so that there is $D \in \mathcal{B}(2^{\omega_1})$ with $D \sim I$ and $\widehat{D} = d$.

Now we make use of another property of $\mathbf{F}$—there is $B \in \mathcal{B}(2^{\omega_1})$ such that $B \sim \omega_1 \setminus I$, $\widehat{B} \in \mathbf{F}$ and $\lambda(B) = 1/2$. Take $g \in C(S)$ to be the function corresponding to $\chi(B\cap D) - \chi(D\setminus B) \in L^\infty(\lambda)$ (here $\chi(\cdot)$ stands for a characteristic function). We check that g has the required properties.

Indeed, $F \subseteq V(\widehat{B\cap D}) \setminus V(\widehat{D\setminus B})$ so $g = 1$ on F. Since g vanishes outside $V(d)$, $\mu_n^2(g) = 0$ for every n. Moreover,

$$\begin{aligned}\mu_n^1(g) &= \int \chi(B)\chi(D)\varphi_n \, d\lambda - \int \chi(B^c)\chi(D)\varphi_n \, d\lambda \\ &= (\lambda(B) - \lambda(B')) \int \chi(D)\varphi_n \, d\lambda = 0 ,\end{aligned}$$

since B is independent of D and φ_n. Therefore $\mu_n(g) = 0$ and the proof is complete.

LEMMA 3.3. *The Banach space $L^\infty(\lambda)$, where λ is the usual product measure on 2^{ω_1}, is not realcompact.*

Proof. The functional z described at the beginning of this section witnesses non-realcompactness. Indeed, given a sequence $(\mu_n) \subseteq M^+(S)$, we can write $\mu_n = \mu_n^1 + \mu_n^2$, where $\mu_n^1(F) = 0$. By Lemma 3.2 there exists $g \in C(S)$ such that $\mu_n^1(g) = 0$ and $g = 1$ on F. Then $z(\mu_n) = \mu_n^2(F) = \mu_n(g)$ for every n. Therefore $L^\infty(\lambda)$ is not realcompact by Theorem 1(c).

THEOREM 3. *If μ is any finite measure then the Banach space $L^\infty(\mu)$ is realcompact if and only if λ is separable.*

Proof. As we mentioned in Section 1, if μ is a separable measure then $L^\infty(\mu)$ is weak* separable and hence realcompact.

By the Maharam structure theorem (see e.g. [6], Theorem 3.9), the Banach space $L^\infty(\mu)$, where μ is a nonseparable measure, contains a subspace isometric to $L^\infty(\lambda)$ (as above, λ stands for the usual measure on 2^{ω_1}). Now the assertion follows from Lemma 3.3 and the fact that realcompactness is preserved when we pass to closed subspaces (see e.g. Edgar [3]).

4. Another functional. The space $C(K)^{**}$ for K compact has a rather complicated description (see [11], 27.2.2)—every functional from $C(K)^{**}$ can be represented by a family of Borel functions on K. Of all elements of $C(K)^{**}$, those which are relatively easy to handle are given by the formula $z(\mu) = \mu(\varphi)$, where φ is a bounded real-valued function which is universally measurable (that is, measurable with respect to every Radon measure on K).

It seems that all known examples of functionals from $C(K)^{**} \setminus C(K)$ that are weak* continuous on weak* separable subspaces of $C(K)^*$ are given by universally measurable functions (see 2-4-4 of [14] and [13]). Using a filter with Kunen's property, in this section we describe a functional from the realcompactification of $C(S)$ of a different nature. We follow the notation of the previous sections. In particular, $\mathbf{F}$ is a filter in $\mathbf{B}$ as in Proposition 2.3 and $F \subseteq S$ is given by $F = \{p \in S : \mathbf{F} \subseteq p\}$.

Let $\mathcal{C}(\omega_1)$ denote the family of cocountable subsets of ω_1. For any set $E \subseteq \omega_1$ we let $\theta(C)$ be the closed set in S given by

$$\theta(C) = \bigcap\{V(\widehat{B}) : \widehat{B} \in \mathbf{F},\ B \sim C\}.$$

We say that a measure $\mu \in M^+(S)$ is *close to F* if there is $C \in \mathcal{C}(\omega_1)$ such that $\mu(\theta(C)) = \mu(S)$. Further, μ is *far from F* if $\mu(\theta(C)) = 0$ for every $C \in \mathcal{C}(\omega_1)$.

LEMMA 4.1. *Every measure $\mu \in M^+(S)$ can be written as $\mu = \mu_c + \mu_f$, where μ_c is close to F and μ_f is far from F.*

Proof. It is easy to see that there is $C_0 \in \mathcal{C}(\omega_1)$ with

$$\mu(\theta(C_0)) = \sup\{\mu(\theta(C)) : C \in \mathcal{C}(\omega_1)\}.$$

We define μ_c to be the restriction of μ to the set $\theta(C_0)$ and $\mu_f = \mu - \mu_c$. This gives the required decomposition. Indeed, if $\mu(\theta(C)) > 0$ for some $C \in \mathcal{C}(\omega_1)$ then $\mu(\theta(C_0)) = \mu(\theta(C \cap C_0)) \geq \mu(\theta(C))$, so $\mu(\theta(C) \setminus \theta(C_0)) = 0$, and thus $\mu_f(\theta(C)) = 0$.

LEMMA 4.2. *Let $(\mu_n) \subseteq M_1^+(S)$ be a sequence of measures that are far from F and let $(\nu_n) \subseteq M_1^+(S)$ be a sequence of measures that are close to F. Then there is a function $g \in C(S)$ such that $\mu_n(g) = 0$ and $\nu_n(g) = 1$ for every n.*

Proof (sketch). For every n choose $C_n \in \mathcal{C}(\omega_1)$ such that $\nu_n(\theta(C_n)) = \nu_n(S)$ and put $C = \bigcap_n C_n$.

Now we proceed as in the proofs of Lemmas 3.1 and 3.2 but instead of $\mathbf{B}$ consider the subalgebra $\mathbf{B}(C) = \{\widehat{B} : B \in \mathcal{B}(2^{\omega_1}),\ B \sim C\}$. Thus we can find a function $g \in C(S)$ such that $g = 1$ on F and $\mu_n(g) = 0$ for every n. Again g is the element of $C(S)$ corresponding to $\chi(B \cap D) - \chi(D \setminus B)$. The point is that now B and D are Borel sets depending on C. It follows that $g = 1$ on $\theta(C)$, so $\nu_n(g) = 1$ for every n.

We define a functional on $C(S)^*$ by the formulas $w(\mu) = \mu_c(S)$, where $\mu \in M^+(S)$, and $w(\mu) = w(\mu^+) - w(\mu^-)$, where $\mu \in M(S)$ and $\mu^+, \mu^- \in M^+(S)$.

PROPOSITION 4.3. *The functional w is weak* continuous on weak* separable subspaces of $C(S)^*$. Moreover, w is not given by a single universally measurable function.*

Proof. The first part follows from Lemma 4.2, Lemma 4.1 and Theorem 1.

Suppose that $w(\mu) = \mu(\varphi)$ for every $\mu \in M^+(S)$, where φ is a universally measurable function on S. Let Q be the set of all $p \in S$ such that the Dirac measure δ_p is close to F.

Let $K \subseteq S$ be a compact set with $\lambda(K) > 0$. Then there is $a \in \mathbf{B}$ such that $V(a) \subseteq K$. Choose a set $A \in \mathcal{B}(2^{\omega_1})$ with $a = \widehat{A}$, A depending on a countable set $I \subseteq \omega_1$. Since the family

$$\{a\} \cup \{\widehat{B} \in F : B \sim \omega_1 \setminus I\}$$

has the finite intersection property, it can be extended to an ultrafilter $p \in S$. It follows that $p \in V(a) \cap Q$. Thus we have proved that Q is the set of full outer measure λ.

Now $\varphi(p) = w(\delta_p) = 1$ for every $p \in Q$ so $\varphi = 1$ λ-almost everywhere. On the other hand, $w(\lambda) = 0$, since λ is far from F, a contradiction.

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