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Weighted integrability and L^1-convergence of multiple trigonometric series

by

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Abstract. We prove that if $c_{jk}\to 0$ as $\max(|j|,|k|)\to\infty$, and

$$\sum_{|j|=0\pm}^{\infty}\sum_{|k|=0\pm}^{\infty}\theta(|j|^{\top})\vartheta(|k|^{\top})|\Delta_{12}c_{jk}|<\infty,$$

then $f(x,y)\phi(x)\psi(y)\in L^1(T^2)$ and $\iint_{T^2}|s_{mn}(x,y)-f(x,y)|\cdot|\phi(x)\psi(y)|\,dx\,dy\to 0$ as $\min(m,n)\to\infty$, where $f(x,y)$ is the limiting function of the rectangular partial sums $s_{mn}(x,y)$, (ϕ,θ) and (ψ,ϑ) are pairs of type I. A generalization of this result concerning L^1-convergence is also established. Extensions of these results to double series of orthogonal functions are also considered. These results can be extended to n-dimensional case. The aforementioned results generalize work of Balashov [1], Boas [2], Chen [3, 4, 5], Marzuq [9], Móricz [11], Móricz–Schipp–Wade [14], and Young [16].

1. Introduction. Let $T^2=\{(x,y)\in\mathbb{R}^2:-\pi\le x,y<\pi\}$. Consider the double trigonometric series

$$\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}c_{jk}e^{i(jx+ky)}. \tag{1.1}$$

We assume that there are two positive, nondecreasing functions $\theta(t)$ and $\vartheta(t)$ defined on $[1,\infty)$ such that

$$c_{jk}\to 0\quad\text{as }\max(|j|,|k|)\to\infty, \tag{1.2}$$

$$\sum_{|j|=0\pm}^{\infty}\sum_{|k|=0\pm}^{\infty}\theta(|j|^{\top})\vartheta(|k|^{\top})|\Delta_{12}c_{jk}|<\infty, \tag{1.3}$$

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where $\xi^{\top}=\max(1,\xi)$, the sum "$\sum_{|j|=0\pm}^{\infty}$" means "$\sum_{0+\le j\le\infty}+\sum_{-\infty\le j\le 0-}$", and the differences $\Delta_1 c_{jk}$, $\Delta_2 c_{jk}$, $\Delta_{12} c_{jk}$ are defined by

$$\Delta_1 c_{jk} = c_{jk} - c_{\tau(j),k}, \qquad \Delta_2 c_{jk} = c_{jk} - c_{j,\tau(k)},$$
$$\Delta_{12} c_{jk} = \Delta_1\Delta_2 c_{jk} = \Delta_2\Delta_1 c_{jk}.$$

Here $c_{0+,k} = c_{0-,k} = c_{0k}$, $c_{j,0+} = c_{j,0-} = c_{j0}$, and the function $\tau(j)$ is defined by $\tau(0+) = 1$, $\tau(0-) = -1$, $\tau(j) = j+1$ for $j \ge 1$, and $\tau(j) = j-1$ for $j \le -1$.

Without loss of generality, we assume that $\theta(t) \ge 1$ and $\vartheta(t) \ge 1$ for all t. It is obvious that (1.3) implies

$$\sum_{|j|=0\pm}^{\infty}\sum_{|k|=0\pm}^{\infty} |\Delta_{12} c_{jk}| < \infty, \tag{1.4}$$

which is equivalent to

$$\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} |\Delta_{11} c_{jk}| < \infty,$$

where $\Delta_{11} c_{jk} = c_{jk} - c_{j,k+1} - c_{j+1,k} + c_{j+1,k+1}$. In [10], Móricz proved that the rectangular partial sums

$$s_{mn}(x,y) = \sum_{|j|\le m}\sum_{|k|\le n} c_{jk} e^{i(jx+ky)}$$

converge pointwise to a measurable function $f(x,y)$. Moreover, $f \in L^p(T^2)$ for all $0<p<1$, and $s_{mn}(x,y)$ converges in $L^p(T^2)$-metric to f as $\min(m,n) \to \infty$.

In this paper, we are concerned with the validity of the following two statements:

$$f(x,y)\phi(x)\psi(y) \in L^1(T^2), \tag{1.5}$$

$$\iint_{T^2} |s_{mn}(x,y) - f(x,y)|\cdot|\phi(x)\psi(y)|\,dx\,dy \to 0 \quad \text{as } \min(m,n)\to\infty, \tag{1.6}$$

where ϕ and ψ are two measurable functions on T.

2. Weighted integrability theorem. We say that (ϕ,θ) is a pair of *type* I if there is a constant M such that

$$\varrho\Big(\int_{|t|\le\pi/\varrho} |\phi(t)|\,dt\Big) + \int_{\pi/\varrho\le|t|\le\pi} |\phi(t)/t|\,dt \le M\theta(\varrho) \quad \text{for all } \varrho \ge 1.$$

The main result in this section reads as follows.

THEOREM 1. *Let $\theta(t)$ and $\vartheta(t)$ be positive, nondecreasing functions defined on $[1,\infty)$ such that* (1.2) *and* (1.3) *are satisfied. Assume that (ϕ,θ) and (ψ,ϑ) are of type* I. *Then the assertions* (1.5) *and* (1.6) *hold.*

This theorem will apply to many particular cases, which generalize [2, 4, 9, 11, 16]. The pairs (ϕ,θ) and (ψ,ϑ) in Theorem 1 can be chosen from any of (i)–(vii), stated below:

(i) $(\phi(t), 1)$ $(\phi(t)/t \in L^\infty(T))$;
(ii) $((\log 1/|t|)^{-\varepsilon}, 1)$ $(\varepsilon > 1)$;
(iii) $((\log 1/|t|)^{-1}, (\log\log t)^{\top})$;
(iv) $((\log 1/|t|)^{-1}(\log\log 1/|t|)^{-1}, (\log\log\log t)^{\top})$;
(v) $((\log 1/|t|)^{-\varepsilon}, \{(\log t)^{\top}\}^{1-\varepsilon})$ $(0<\varepsilon<1)$;
(vi) $(1, (\log t)^{\top})$;
(vii) $(|t|^{-\alpha}, t^{\alpha})$ $(0<\alpha<1)$.

Any of (i)–(vii) is of type I. The logarithm functions given in (ii)–(vi) are defined in the extended sense. This means that they have the original value whenever they are well defined; otherwise, they are defined as 0. It is known that the functions

$$(\log 1/|t|)^{-1} \quad\text{and}\quad (\log 1/|t|)^{-1}(\log\log 1/|t|)^{-1}$$

are closely related to the Dini–Lipschitz test (cf. [17, Vol. I, p. 303]). We focus our attention on the following three cases. The first one is $\theta(t) = \vartheta(t) = 1$. Then (1.3) is exactly (1.4). From (i) and (ii), we get

COROLLARY 1. *Assume that* (1.4) *holds. Then* (1.5) *and* (1.6) *remain true for all ϕ and ψ chosen from* (i) *and* (ii).

The second case we consider is $\theta(t) = \{(\log t)^{\top}\}^{1-\varepsilon}$ and $\vartheta(t) = \{(\log t)^{\top}\}^{1-\delta}$, where $0 \le \varepsilon, \delta < 1$. In this case, (1.3) reduces to

$$\sum_{|j|=0\pm}^{\infty}\sum_{|k|=0\pm}^{\infty} \{(\log|j|)^{\top}\}^{1-\varepsilon}\{(\log|k|)^{\top}\}^{1-\delta}|\Delta_{12}c_{jk}| < \infty, \tag{2.1}$$

which is equivalent to

$$\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \{(\log|j|)^{\top}\}^{1-\varepsilon}\{(\log|k|)^{\top}\}^{1-\delta}|\Delta_{11}c_{jk}| < \infty.$$

Therefore, (v) and (vi) imply

COROLLARY 2. *If both of* (1.2) *and* (2.1) *are satisfied for some $0 \le \varepsilon, \delta < 1$, then $|f(x,y)|/\{|\log(1/|x|)|^{\varepsilon}|\log(1/|y|)|^{\delta}\} \in L^1(T^2)$ and*

$$\iint_{T^2} \frac{|s_{mn}(x,y) - f(x,y)|}{|\log(1/|x|)|^{\varepsilon}|\log(1/|y|)|^{\delta}}\,dx\,dy = o(1) \quad \text{as } \min(m,n)\to\infty.$$

The particular case of (2.1) with $\varepsilon = \delta = 0$ is

$$\sum_{|j|=0\pm}^{\infty} \sum_{|k|=0\pm}^{\infty} (\log |j|)^{\top} (\log |k|)^{\top} |\Delta_{12} c_{jk}| < \infty. \tag{2.2}$$

In this case, Corollary 2 has the following form:

COROLLARY 3. *If both of* (1.2) *and* (2.2) *are satisfied, then the sum f of series* (1.1) *is integrable*, (1.1) *is the Fourier series of f, and $\|s_{mn} - f\|_1 \to 0$ as $\min(m,n) \to \infty$.*

This corollary was proved in [4], which generalizes [11, Theorems 2, 4, 5] and [16]. Set $\Delta_{12}^* c_{jk} = (\log |j|)^{\top} (\log |k|)^{\top} \Delta_{12} c_{jk}$. Then by the Hölder inequality, we find that (2.2) can be replaced by the condition

$$A_p^* = \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} \Big\{ \sum_{[2^{u-1}] \le |j| < 2^u} \sum_{[2^{v-1}] \le |k| < 2^v} (|j|^{\top} |k|^{\top})^{p-1} |\Delta_{12}^* c_{jk}|^p \Big\}^{1/p} < \infty, \tag{2.3}$$

where $1 \le p \le \infty$. For $p = \infty$, A_p^* is defined as

$$A_{\infty}^* = \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} \max_{[2^{u-1}] \le |j| < 2^u} \max_{[2^{v-1}] \le |k| < 2^v} |j|^{\top} |k|^{\top} |\Delta_{12}^* c_{jk}|.$$

Here $[\xi]$ denotes the integral part of ξ, in particular, $[\xi] = 0\pm$ for $0 < \xi < 1$. Condition (2.3) is closely related to condition (1.11) given in [11].

COROLLARY 4. *If both of* (1.2) *and* (2.3) *are satisfied for some $1 \le p \le \infty$, then the sum f of series* (1.1) *is integrable*, (1.1) *is the Fourier series of f, and $\|s_{mn} - f\|_1 \to 0$ as $\min(m,n) \to \infty$.*

Given $\varepsilon > 0$ and $1 < p < \infty$, we have $\{(\log |j|)^{\top}\}^p \le M(|j|^{\top})^{\varepsilon}$ for all j, where M is a suitable constant depending only on ε and p. Therefore, the following condition implies (2.3):

$$\sum_{u=0}^{\infty} \sum_{v=0}^{\infty} \Big\{ \sum_{[2^{u-1}] \le |j| < 2^u} \sum_{[2^{v-1}] \le |k| < 2^v} (|j|^{\top} |k|^{\top})^{p-1+\varepsilon} |\Delta_{12} c_{jk}|^p \Big\}^{1/p} < \infty. \tag{2.4}$$

COROLLARY 5. *If both of* (1.2) *and* (2.4) *are satisfied for some $\varepsilon > 0$ and for some $1 < p < \infty$, then the sum f of series* (1.1) *is integrable*, (1.1) *is the Fourier series of f, and $\|s_{mn} - f\|_1 \to 0$ as $\min(m,n) \to \infty$.*

The conclusion of Corollary 5 involves three types of results: the sum f of series (1.1) is integrable, (1.1) is the Fourier series of f, and $\|s_{mn} - f\|_1 \to 0$ as $\min(m,n) \to \infty$. For $|j| \le m$ and $|k| \le n$, we have $|c_{jk} - \widehat{f}(j,k)| \le \|s_{mn} - f\|_1$. Thus, the last one implies the others. It should be noticed that the conclusion of Corollary 5 may not hold for the case $\varepsilon = 0$, in general. This

phenomenon may happen even for the first two results of the conclusion. Let c_{jk} be the coefficients of the series

$$\sum_{j=2}^{\infty} \sum_{k=2}^{\infty} \frac{\sin jx \sin ky}{(\log j)(\log k)}. \tag{2.5}$$

Then for all j and for all k,

$$|\Delta_{12} c_{jk}| \le M(|j|^{\top} |k|^{\top})^{-1} \{(\log |j|)^{\top} (\log |k|)^{\top}\}^{-2},$$

where M is an absolute constant. From this, we see that condition (2.4) with $\varepsilon = 0$ is satisfied for all $p > 1$, but series (2.5) is not a double Fourier series (cf. [17, Vol. I, p. 253] or [4]). This gives us a counterexample to Corollary 5 for the case $\varepsilon = 0$. It is easy to check that condition (2.4) with $\varepsilon = 0$ is equivalent to condition (1.11) given in [11]. Hence, [11, Corollary 1] tells us that the first two results in Corollary 5 can be extended to the case $\varepsilon = 0$ for double cosine series. However, the example

$$f(x,y) = \sum_{j=2}^{\infty} \frac{\cos jx}{\log j} \tag{2.6}$$

indicates that the third result in Corollary 5 still fails for the case $\varepsilon = 0$ (cf. [3, Corollary 3.3]). To guarantee such a result, some additional conditions are required (cf. [11, Corollary 2] or [5]).

The third case we investigate is $\theta(t) = t^{\alpha}$ and $\vartheta(t) = t^{\beta}$, where $0 < \alpha$, $\beta < 1$. In this case, (1.3) is of the form

$$\sum_{|j|=0\pm}^{\infty} \sum_{|k|=0\pm}^{\infty} (|j|^{\top})^{\alpha} (|k|^{\top})^{\beta} |\Delta_{12} c_{jk}| < \infty. \tag{2.7}$$

Moreover, $(|t|^{-\alpha}, t^{\alpha})$ and $(|t|^{-\beta}, t^{\beta})$ are of type I. Hence,

COROLLARY 6. *If both of* (1.2) *and* (2.7) *are satisfied for some $0 < \alpha$, $\beta < 1$, then $|x|^{-\alpha} |y|^{-\beta} |f(x,y)| \in L^1(T^2)$ and*

$$\int\int_{T^2} |s_{mn}(x,y) - f(x,y)| (|x|^{-\alpha} |y|^{-\beta})\, dx\, dy \to 0 \quad \text{as } \min(m,n) \to \infty.$$

This corollary is false for the case $\alpha = \beta = 0$. Series (2.5) provides a counterexample (cf. [4]). Assume that $\Delta_{12} c_{jk} \ge 0$ for all j and for all k. Then we have

$$\Delta_1 c_{jk} \ge 0, \quad \Delta_2 c_{jk} \ge 0, \quad \text{and} \quad c_{jk} \ge 0.$$

Applying a double summation by parts, we get

$$\sum_{|j|=0\pm}^{\infty} \sum_{|k|=0\pm}^{\infty} (|j|^{\top})^{\alpha-1} (|k|^{\top})^{\beta-1} |c_{jk}| \ge \sum_{|j|=0\pm}^{\infty} \sum_{|k|=0\pm}^{\infty} (|j|^{\top})^{\alpha} (|k|^{\top})^{\beta} |\Delta_{12} c_{jk}|.$$

COROLLARY 7. *Assume that $\Delta_{12}c_{jk} \geq 0$ for all j and for all k. If* (1.2) *and*

$$\sum_{|j|=0\pm}^{\infty} \sum_{|k|=0\pm}^{\infty} (|j|^{\top})^{\alpha-1}(|k|^{\top})^{\beta-1}|c_{jk}| < \infty \tag{2.8}$$

are satisfied for some $0 < \alpha, \beta < 1$, then the conclusion of Corollary 6 *holds.*

This corollary for double cosine series was proved in [9, Theorem 4], which extends [2, Theorem 4.2] from the one-dimensional to two-dimensional case. Corollary 7 also generalizes [2, Theorem 4.1].

3. L^1-convergence. As indicated by series (2.6) (or [11, Example 4]), the assertion that $\|s_{mn}(f) - f\|_1 = o(1)$ as $\min(m,n) \to \infty$ may not follow from the assumption that $f \in L^1(T^2)$. To ensure such a conclusion, a certain kind of conditions are needed. The purpose of this section is to provide such conditions. The conditions involved here are weaker than (2.2)–(2.4). The main result in this section is the following.

THEOREM 2. *Let $f \in L^1(T^2)$ and c_{jk} be its Fourier coefficients. If*

$$\sum_{|j|=0\pm}^{\infty} (\log|j|)^{\top}(\log|k|)^{\top}|\Delta_1 c_{jk}| \to 0 \quad \text{as } |k| \to \infty, \tag{3.1}$$

$$\sum_{|k|=0\pm}^{\infty} (\log|j|)^{\top}(\log|k|)^{\top}|\Delta_2 c_{jk}| \to 0 \quad \text{as } |j| \to \infty, \tag{3.2}$$

$$\lim_{\lambda\downarrow 1} \overline{\lim_{m\to\infty}} \sum_{|j|=m+1}^{[\lambda m]} \sum_{|k|=0\pm}^{\infty} (\log|j|)^{\top}(\log|k|)^{\top}|\Delta_{12} c_{jk}| = 0, \tag{3.3}$$

and

$$\lim_{\lambda\downarrow 1} \overline{\lim_{n\to\infty}} \sum_{|j|=0\pm}^{\infty} \sum_{|k|=n+1}^{[\lambda n]} (\log|j|)^{\top}(\log|k|)^{\top}|\Delta_{12} c_{jk}| = 0, \tag{3.4}$$

then $\|s_{mn}(f) - f\|_1 = o(1)$ as $\min(m,n) \to \infty$.

Theorem 2 is a two-dimensional analogue of [3, Corollary 3.3]. It extends [5, Corollary 2] from double Fourier cosine series to general double Fourier series. Obviously, (2.2) implies (3.3) and (3.4). For any j and any k, we have

$$|\Delta_1 c_{jk}| \leq \sum_{|v|=|k|}^{\infty} |\Delta_{12} c_{jv}| \tag{3.5}$$

and

$$|\Delta_2 c_{jk}| \leq \sum_{|u|=|j|}^{\infty} |\Delta_{12} c_{uk}|. \tag{3.6}$$

Hence (2.2) implies (3.1) and (3.2). The above argument indicates that Theorem 2 generalizes Corollary 3. It is not hard to check that

$$f(x,y) = \sum_{j=27}^{\infty} \frac{\cos jx}{(\log j)(\log\log j)} \tag{3.7}$$

defines an integrable function on T^2. The coefficients of series (3.7) satisfy (3.1)–(3.4), but condition (2.2) fails. This example distinguishes Theorem 2 from Corollary 3.

Employing the same argument as given in Section 2, we get the following two consequences of Theorem 2, which generalize Corollaries 4 and 5, respectively.

COROLLARY 8. *Let $f \in L^1(T^2)$ and c_{jk} be its Fourier coefficients. Assume that*

$$(\log|k|)^{\top} \sum_{u=0}^{\infty} \Big[\sum_{[2^{u-1}]\leq|j|<2^u} (|j|^{\top})^{p-1}|\Delta_1^* c_{jk}|^p \Big]^{1/p} \to 0 \quad \text{as } |k| \to \infty, \tag{3.8}$$

$$(\log|j|)^{\top} \sum_{v=0}^{\infty} \Big[\sum_{[2^{v-1}]\leq|k|<2^v} (|k|^{\top})^{q-1}|\Delta_2^* c_{jk}|^q \Big]^{1/q} \to 0 \quad \text{as } |j| \to \infty, \tag{3.9}$$

$$\lim_{\lambda\downarrow 1} \overline{\lim_{m\to\infty}} \sum_{v=0}^{\infty} \Big[\sum_{|j|=m+1}^{[\lambda m]} \sum_{[2^{v-1}]\leq|k|<2^v} (|j|^{\top}|k|^{\top})^{r-1}|\Delta_{12}^* c_{jk}|^r \Big]^{1/r} = 0, \tag{3.10}$$

and

$$\lim_{\lambda\downarrow 1} \overline{\lim_{n\to\infty}} \sum_{u=0}^{\infty} \Big[\sum_{[2^{u-1}]\leq|j|<2^u} \sum_{|k|=n+1}^{[\lambda n]} (|j|^{\top}|k|^{\top})^{s-1}|\Delta_{12}^* c_{jk}|^s \Big]^{1/s} = 0 \tag{3.11}$$

for some $1 < p, q, r, s \leq \infty$. Then $\|s_{mn}(f) - f\|_1 = o(1)$ as $\min(m,n) \to \infty$.

Here $\Delta_1^* c_{jk} = (\log|j|)^{\top}\Delta_1 c_{jk}$, $\Delta_2^* c_{jk} = (\log|k|)^{\top}\Delta_2 c_{jk}$, and $\Delta_{12}^* c_{jk}$ are defined in Section 2. For $p = q = r = s = \infty$, the conditions (3.8)–(3.11) are defined in the same way as we did for (2.3).

COROLLARY 9. *Let $f \in L^1(T^2)$ and c_{jk} be its Fourier coefficients. Assume that*

$$(\log|k|)^{\top} \sum_{u=0}^{\infty} \Big[\sum_{[2^{u-1}]\leq|j|<2^u} (|j|^{\top})^{p-1+\varepsilon}|\Delta_1 c_{jk}|^p \Big]^{1/p} \to 0 \quad \text{as } |k| \to \infty, \tag{3.12}$$

$$(\log|j|)^{\top} \sum_{v=0}^{\infty} \Big[\sum_{[2^{v-1}]\leq|k|<2^v} (|k|^{\top})^{q-1+\varepsilon}|\Delta_2 c_{jk}|^q \Big]^{1/q} \to 0 \quad \text{as } |j| \to \infty, \tag{3.13}$$

$$\text{(3.14)}\quad \lim_{\lambda\downarrow 1}\overline{\lim_{m\to\infty}}\sum_{v=0}^{\infty}\Big[\sum_{|j|=m+1}^{[\lambda m]}\sum_{[2^{v-1}]\le|k|<2^v}(|j|^{\top}|k|^{\top})^{r-1+\varepsilon}|\Delta_{12}c_{jk}|^r\Big]^{1/r}=0,$$

and

$$\text{(3.15)}\quad \lim_{\lambda\downarrow 1}\overline{\lim_{n\to\infty}}\sum_{u=0}^{\infty}\Big[\sum_{[2^{u-1}]\le|j|<2^u}\sum_{|k|=n+1}^{[\lambda n]}(|j|^{\top}|k|^{\top})^{s-1+\varepsilon}|\Delta_{12}c_{jk}|^s\Big]^{1/s}=0$$

for some $\varepsilon>0$ *and for some* $1<p,q,r,s<\infty$. *Then* $\|s_{mn}(f)-f\|_1=o(1)$ *as* $\min(m,n)\to\infty$.

The conditions (3.12)–(3.15) with $\varepsilon=0$ reduce to the conditions of [5, Corollary 1]. As shown in [5, Corollary 1], the conclusion of Corollary 9 for this case should be replaced by the following statement:

$$\text{(3.16)}\quad \|s_{mn}(f)-f\|_1\to 0 \text{ as } \min(m,n)\to\infty$$
$$\text{if and only if } c_{jk}(\log|j|)(\log|k|)=o(1) \text{ as } \min(|j|,|k|)\to\infty.$$

It is still an open problem whether Corollary 9 holds for this case. If so, it will extend [11, Corollary 2] and [5, Corollary 1] from double cosine series to any double series of type (1.1).

4. Auxiliary results. Consider the functions $\Psi_j(t)$ defined by $\Psi_{0+}(t)=\Psi_{0-}(t)=1/2$, $\Psi_j(t)=1/2+(e^{it}+e^{2it}+\ldots+e^{ijt})$ for $j\ge 1$, and $\Psi_{-j}(t)=\Psi_j(-t)$ for $j\ge1$. We have $|\Psi_j(t)|\le\min(2|j|^{\top},\pi/|t|)$ for all j and for all t.

LEMMA 1. *Let* (ϕ,θ) *be of type* I *and* M *the constant given in the definition of the pair* (ϕ,θ). *Then for all* j,

$$\int_{-\pi}^{\pi}|\Psi_j(t)\phi(t)|\,dt\le 4M\theta(|j|^{\top}).$$

Proof. From the definition, we find that $(\widetilde{\phi},\theta)$ is of type I, where $\widetilde{\phi}(t)=\phi(-t)$. Since $\Psi_{-j}(t)=\Psi_j(-t)$, it suffices to prove the case $j\ge 0+$. Let $\varrho=|j|^{\top}$. Since $|\Psi_j(t)|\le\min(2|j|^{\top},\pi/|t|)$ for all j and for all t, we get

$$\int_{-\pi}^{\pi}|\Psi_j(t)\phi(t)|\,dt\le 2|j|^{\top}\int_{|t|\le\pi/\varrho}|\phi(t)|\,dt+\pi\int_{\pi/\varrho\le|t|\le\pi}|\phi(t)/t|\,dt\le 4M\theta(|j|^{\top}).$$

Lemma 1 generalizes [4, Lemma 1]. Due to the structure of $\Psi_j(t)$, the following summation by parts formula holds. Its proof is left to the reader.

LEMMA 2.

$$\sum_{|j|\le m}a_je^{ijt}=\sum_{|j|=0\pm}^{m}\Delta a_j\Psi_j(t)+\sum_{|j|=m}a_{\tau(j)}\Psi_j(t).$$

With the help of Lemma 2, the following representation for $s_{mn}(x,y)$ can be easily derived.

LEMMA 3.

$$\begin{aligned}s_{mn}(x,y)&=\sum_{|j|=0\pm}^{m}\sum_{|k|=0\pm}^{n}\Delta_{12}c_{jk}\Psi_j(x)\Psi_k(y)\\&+\sum_{|j|=0\pm}^{m}\sum_{|k|=n}\Delta_1c_{j,\tau(k)}\Psi_j(x)\Psi_k(y)\\&+\sum_{|j|=m}\sum_{|k|=0\pm}^{n}\Delta_2c_{\tau(j),k}\Psi_j(x)\Psi_k(y)\\&+\sum_{|j|=m}\sum_{|k|=n}c_{\tau(j),\tau(k)}\Psi_j(x)\Psi_k(y).\end{aligned}$$

Denote by $\sigma_{mn}(x,y)$ the first Cesàro mean of the sequence $\{s_{mn}(x,y)\}$, that is,

$$\sigma_{mn}(x,y)=\frac{1}{m+1}\cdot\frac{1}{n+1}\sum_{j=0}^{m}\sum_{k=0}^{n}s_{jk}(x,y).$$

It was pointed out in [6, 7] that the following identity holds.

LEMMA 4 (Chen and Hsieh [6]). *Let* $\xi^{\perp}=\min(1,\xi)$. *Assume that* $\lambda m\ge m+1$ *and* $\lambda n\ge n+1$. *Then*

$$\begin{aligned}s_{mn}-\sigma_{mn}&=\frac{[\lambda m]+1}{[\lambda m]-m}(\sigma_{[\lambda m],n}-\sigma_{mn})+\frac{[\lambda n]+1}{[\lambda n]-n}(\sigma_{m,[\lambda n]}-\sigma_{mn})\\&+\frac{[\lambda m]+1}{[\lambda m]-m}\cdot\frac{[\lambda n]+1}{[\lambda n]-n}(\sigma_{[\lambda m],[\lambda n]}-\sigma_{[\lambda m],n}-\sigma_{m,[\lambda n]}+\sigma_{mn})\\&+R^{\lambda}(m,n;x,y),\end{aligned}$$

where

$$\begin{aligned}&R^{\lambda}(m,n;x,y)\\&=-\sum_{|j|\le\lambda m}\sum_{|k|\le\lambda n}\Big(\frac{[\lambda m]+1-|j|}{[\lambda m]-m}\Big)^{\perp}\Big(\frac{[\lambda n]+1-|k|}{[\lambda n]-n}\Big)^{\perp}c_{jk}e^{i(jx+ky)}\\&\quad+\sum_{|j|\le m}\sum_{|k|\le n}\Big(\frac{[\lambda m]+1-|j|}{[\lambda m]-m}\Big)^{\perp}\Big(\frac{[\lambda n]+1-|k|}{[\lambda n]-n}\Big)^{\perp}c_{jk}e^{i(jx+ky)}.\end{aligned}$$

Performing a double summation by parts (i.e., applying Lemma 2 twice), and simplifying the final result, we get the second representation for $R^{\lambda}(m,n;x,y)$ below.

LEMMA 5.

$$R^\lambda(m,n;x,y) = R_0(m,n;x,y) - R_1^\lambda(m,n;x,y) - R_2^\lambda(m,n;x,y) - R_3^\lambda(m,n;x,y)\,,$$

where

$$R_0(m,n;x,y) = \sum_{|j|=m}\sum_{|k|=n} c_{\tau(j),\tau(k)}\Psi_j(x)\Psi_k(y)\,,$$

$$R_1^\lambda(m,n;x,y) = \frac{1}{[\lambda m]-m}\cdot\frac{1}{[\lambda n]-n}\sum_{|j|=m+1}^{[\lambda m]}\sum_{|k|=n+1}^{[\lambda n]} c_{\tau(j),\tau(k)}\Psi_j(x)\Psi_k(y)\,,$$

$$\begin{aligned}R_2^\lambda(m,n;x,y) &= -\sum_{|j|=0\pm}^{m}\sum_{|k|=n}\Delta_1 c_{j,\tau(k)}\Psi_j(x)\Psi_k(y) - \sum_{|j|=m}\sum_{|k|=0\pm}^{n}\Delta_2 c_{\tau(j),k}\Psi_j(x)\Psi_k(y)\\ &+ \frac{1}{[\lambda n]-n}\sum_{|j|=0\pm}^{[\lambda m]}\sum_{|k|=n+1}^{[\lambda n]}\left(\frac{[\lambda m]+1-|j|}{[\lambda m]-m}\right)^{\perp}\Delta_1 c_{j,\tau(k)}\Psi_j(x)\Psi_k(y)\\ &+ \frac{1}{[\lambda m]-m}\sum_{|j|=m+1}^{[\lambda m]}\sum_{|k|=0\pm}^{[\lambda n]}\left(\frac{[\lambda n]+1-|k|}{[\lambda n]-n}\right)^{\perp}\Delta_2 c_{\tau(j),k}\Psi_j(x)\Psi_k(y)\,,\end{aligned}$$

$$\begin{aligned}R_3^\lambda(m,n;x,y) &= \sum_{|j|=m+1}^{[\lambda m]}\sum_{|k|=0\pm}^{n}\frac{[\lambda m]+1-|j|}{[\lambda m]-m}\Delta_{12}c_{jk}\Psi_j(x)\Psi_k(y)\\ &+ \sum_{|j|=0\pm}^{m}\sum_{|k|=n+1}^{[\lambda n]}\frac{[\lambda n]+1-|k|}{[\lambda n]-n}\Delta_{12}c_{jk}\Psi_j(x)\Psi_k(y)\\ &+ \sum_{|j|=m+1}^{[\lambda m]}\sum_{|k|=n+1}^{[\lambda n]}\frac{[\lambda m]+1-|j|}{[\lambda m]-m}\cdot\frac{[\lambda n]+1-|k|}{[\lambda n]-n}\Delta_{12}c_{jk}\Psi_j(x)\Psi_k(y)\,.\end{aligned}$$

5. Proof of Theorem 1. It is clear that

$$|c_{jk}| \le \sum_{|u|=|j|}^{\infty}\sum_{|v|=|k|}^{\infty}|\Delta_{12}c_{uv}|\,. \tag{5.1}$$

We have $|\Psi_j(t)| \le \pi/|t|$ for all j and for all $0<|t|\le\pi$. By (1.4), (3.5), (3.6), (5.1), and Lemma 3, we infer that for $0<|x|,|y|\le\pi$,

$$\sum_{|j|=0\pm}^{m}\sum_{|k|=0\pm}^{n}\Delta_{12}c_{jk}\Psi_j(x)\Psi_k(y)\to f(x,y)\quad\text{as }\min(m,n)\to\infty\,. \tag{5.2}$$

It follows from (1.3), Lemma 1, and Fatou's lemma that

$$\int_{T^2}\!\!\int |f(x,y)\phi(x)\psi(y)|\,dx\,dy \le M^*\Big\{\lim_{m\to\infty}\sum_{|j|=0\pm}^{m}\sum_{|k|=0\pm}^{m}\theta(|j|^\top)\vartheta(|k|^\top)|\Delta_{12}c_{jk}|\Big\}<\infty\,,$$

where M^* is a suitable constant coming from the definitions of (ϕ,θ) and (ψ,ϑ). Thus, (1.5) holds. For (1.6), putting (5.2), Lemma 1, and Lemma 3 together yields

$$\begin{aligned}\int_{T^2}\!\!\int |s_{mn}(x,y)-f(x,y)|\cdot|\phi(x)\psi(y)|\,dx\,dy &\le M^*\Big\{\sum_{(j,k)\in Q(m,n)}\theta(|j|^\top)\vartheta(|k|^\top)|\Delta_{12}c_{jk}|\\ &+\sum_{|j|=0\pm}^{\infty}\sum_{|k|=n}\theta(|j|^\top)\vartheta(|k|^\top)|\Delta_1 c_{j,\tau(k)}|\\ &+\sum_{|j|=m}\sum_{|k|=0\pm}^{\infty}\theta(|j|^\top)\vartheta(|k|^\top)|\Delta_2 c_{\tau(j),k}|\\ &+\sum_{|j|=m}\sum_{|k|=n}\theta(|j|^\top)\vartheta(|k|^\top)|c_{\tau(j),\tau(k)}|\Big\}\,,\end{aligned}$$

where $Q(m,n)$ consists of all (j,k) with $|j|>m$ or $|k|>n$. From this, we see that the desired result follows from (1.3), (3.5), (3.6), and (5.1).

6. Proof of Theorem 2. We adopt the notations of Lemmas 4 and 5. The particular case of Lemma 1 for the pair $(1,(\log t)^\top)$ says that

$$\int_{-\pi}^{\pi}|\Psi_j(t)|\,dt \le 4M(\log|j|)^\top\,. \tag{6.1}$$

We have

$$|c_{jk}| \le \sum_{|u|=|j|}^{\infty}|\Delta_1 c_{uk}|\,. \tag{6.2}$$

By (3.1) and (6.2), we find that

$$c_{jk}(\log|j|)(\log|k|) = o(1) \quad \text{as } \min(|j|,|k|) \to \infty.$$

Combining this with (6.1), we conclude that $\|R_0(m,n;x,y)\|_1 \to 0$ as $\min(m,n) \to \infty$. Moreover, we infer that

$$\|R_1^\lambda(m,n;x,y)\|_1 \le \max_{\substack{m<u\le\lambda m\\ n<v\le\lambda n}} \|R_0(u,v;x,y)\|_1 \to 0 \quad \text{uniformly in } \lambda, \text{ as } \min(m,n)\to\infty.$$

Write

$$R_2^\lambda(m,n;x,y) = -R_{21}(m,n;x,y) - R_{22}(m,n;x,y) + R_{23}^\lambda(m,n;x,y) + R_{24}^\lambda(m,n;x,y), \quad \text{say}.$$

By (3.1) and (6.1), we obtain $\|R_{21}(m,n;x,y)\|_1 \to 0$ as $\min(m,n) \to \infty$, and

$$\|R_{23}^\lambda(m,n;x,y)\|_1 \le \max_{\substack{m<u\le\lambda m\\ n<v\le\lambda n}} \|R_{21}(u,v;x,y)\|_1 \to 0 \quad \text{uniformly in } \lambda, \text{ as } \min(m,n)\to\infty.$$

Similarly, (3.2) implies that

$$\|R_{22}(m,n;x,y)\|_1 \to 0 \quad \text{as } \min(m,n)\to\infty,$$

and

$$\|R_{24}^\lambda(m,n;x,y)\|_1 \to 0 \quad \text{uniformly in } \lambda, \text{ as } \min(m,n)\to\infty.$$

Thus, $\|R_2^\lambda(m,n;x,y)\|_1 \to 0$ uniformly in λ, as $\min(m,n) \to \infty$. For $\|R_3^\lambda(m,n;x,y)\|_1$, it follows from (6.1) that

$$\|R_3^\lambda(m,n;x,y)\|_1 \le (16M^2)\Big\{\sum_{|j|=m+1}^{[\lambda m]}\sum_{|k|=0\pm}^{\infty}(\log|j|)^\top(\log|k|)^\top|\Delta_{12}c_{jk}| + \sum_{|j|=0\pm}^{\infty}\sum_{|k|=n+1}^{[\lambda n]}(\log|j|)^\top(\log|k|)^\top|\Delta_{12}c_{jk}|\Big\}.$$

Thus, by (3.3) and (3.4), we get

$$\lim_{\lambda\downarrow 1}\overline{\lim_{m,n\to\infty}}\|R_3^\lambda(m,n;x,y)\|_1 = 0.$$

Here the symbol "$\overline{\lim}_{m,n\to\infty}$" means the limit superior, defined as

$$\overline{\lim_{m,n\to\infty}} c_{jk} = \inf_{m,n\ge 1}\sup_{j\ge m,k\ge n} c_{jk} = \lim_{n\to\infty}\sup_{j\ge n,k\ge n} c_{jk}.$$

Since $f\in L^1(T^2)$, we have $\|\sigma_{mn}(f)-f\|_1 \to 0$ as $\min(m,n)\to\infty$. Combining all what we have done so far, we conclude that the desired result follows from Lemmas 4 and 5.

7. Generalizations. It was pointed out in [7] that Lemma 4 still holds for the n-dimensional case. Inspecting the proofs given in Sections 4–6, we find that the results established in the preceding sections can be extended to higher dimensions without difficulty. The only thing we have to do is to modify the corresponding conditions from the two-dimensional to the n-dimensional case. We leave these to the reader. Following the proofs given above, we find that the theory developed here can be extended to any double series of the type

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} c_{jk}w_j(x)w_k(y), \tag{7.1}$$

where $w_j(t)$ are measurable functions defined on $[0,\alpha]$. The prerequisite is the existence of a sequence $\{\Psi_j(t)\}$ of measurable functions such that

$$|\Psi_j(t)| \le A(|j|)^\top \quad (\text{all } j), \tag{7.2}$$

$$|\Psi_j(t)| \le B/|t| \quad (\text{all } j, \text{ all } 0<t\le\alpha), \tag{7.3}$$

$$\Psi_j(t) - \Psi_{j-1}(t) = w_j(t) \quad (\text{all } j), \tag{7.4}$$

where A and B are two absolute constants. For instance, Theorem 1 can be extended in the following way. Let $s_{mn}(x,y)$ be the corresponding rectangular partial sums of (7.1). Assume that the following analogues of conditions (1.2) and (1.3) hold:

$$c_{jk} \to 0 \quad \text{as } \max(j,k)\to\infty, \tag{7.5}$$

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\theta(j^\top)\vartheta(k^\top)|\Delta_{12}c_{jk}| < \infty. \tag{7.6}$$

Then $s_{mn}(x,y)$ converges pointwise to a measurable function $f(x,y)$. We say that (ϕ,θ) is a pair of *type* I* if there is a constant M such that

$$\varrho\int_0^{\alpha/\varrho}|\phi(t)|\,dt + \int_{\alpha/\varrho}^{\alpha}|\phi(t)/t|\,dt \le M\theta(\varrho) \quad \text{for all } \varrho\ge 1.$$

Then Theorem 1 has the following form.

THEOREM 1*. *Let $\theta(t)$ and $\vartheta(t)$ be positive, nondecreasing functions defined on $[1,\infty)$ such that (7.5) and (7.6) are satisfied. Assume that (ϕ,θ) and (ψ,ϑ) are of type* I*. *Then $f(x,y)\phi(x)\psi(y)\in L^1([0,\alpha]\times[0,\alpha])$ and*

$$\int_0^\alpha\int_0^\alpha |s_{mn}(x,y)-f(x,y)|\cdot|\phi(x)\psi(y)|\,dx\,dy \to 0 \quad \textit{as } \min(m,n)\to\infty.$$

We know that if $\{w_j(t): j\ge 0\}$ is the Walsh orthonormal system defined on the interval $[0,1)$ in the Paley enumeration, then the corresponding

Walsh–Dirichlet kernels $D_j(t)$ (chosen for $\Psi_j(t)$) have the properties (7.2)–(7.4) (cf. [8] or [15]). Hence, the conclusion of Theorem 1* holds for double Walsh series, or more generally, for n-dimensional Walsh series. Its corollaries generalize the corresponding results in [1] and [14]. Other generalizations are left to the reader.

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Nonaccessible filters in measure algebras and functionals on $L^\infty(\lambda)^*$

by

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Abstract. In a nonatomic measure algebra, we construct a nonprincipal filter with the inaccessibility property considered by Kunen [7]. Using that filter we define two "pathological" functionals on $L^\infty(\lambda)^*$. It follows that the Banach space $L^\infty(\lambda)$ is not realcompact whenever the measure λ is not separable.

The main aim of this paper is to prove that a Banach space $L^\infty(\lambda)$ is not realcompact in its weak topology whenever the measure λ is not separable. According to a characterization of realcompact Banach spaces due to Corson (see the next section), it suffices to find a functional from $L^\infty(\lambda)^{**} \setminus L^\infty(\lambda)$ which, roughly speaking, behaves like an element of $L^\infty(\lambda)$, when considered on countable subsets of $L^\infty(\lambda)^*$.

Actually, we shall be dealing with the usual measure λ on the Cantor cube 2^{ω_1}. We find it convenient to treat $C(S)$, the space of continuous functions on the Stone space of λ, rather than the space $L^\infty(\lambda)$ itself. We shall show that one may define a functional with the required properties putting $\mu \to \mu(F)$ for $\mu \in C(S)^*$, where F is a certain closed subset of S. In fact, F will be defined as the set of all ultrafilters from S that extend a suitably chosen filter in $\mathbf{B}$. To make the idea work, we consider a property of filters in measure algebras that has been invented by Kunen [7] for another purpose. It is rather technical; we call it Kunen's property (see Section 2).

It is shown in [7] that in measure algebras of cardinality $\mathfrak{c}$ there are ultrafilters with Kunen's property provided Martin's Axiom holds. However, as explained in Section 2, given such an ultrafilter one may construct a p-point in $\beta\omega \setminus \omega$, which indicates that the existence of ultrafilters with Kunen's property is independent of the usual axioms. Therefore we present a

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