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The Taylor transformation of analytic functionals with non-bounded carrier

by

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Abstract. Let L be a closed convex subset of some proper cone in $\mathbb{C}$. The image of the space of analytic functionals $Q'(L)$ with non-bounded carrier in L under the Taylor transformation as well as the representation of analytic functionals from $Q'(L)$ as the boundary values of holomorphic functions outside L are given. Multipliers and operators in $Q'(L)$ are described.

Let Ω be an $\mathbb{R}_-$-connected open subset of $\mathbb{C}$, i.e. a subset such that together with any point $z \in \Omega$ it contains the half line $z+\mathbb{R}_-$. In the paper [7] (see also [4]) B. Ziemian introduced the class $M(\Omega)$ of distributions $u \in \mathcal{D}'(\mathbb{R}_+)$ extendible onto $\mathbb{R}$ such that the local Mellin transform

$$\mathcal{M}u(z) = u_x[\chi(x)x^{-z-1}],$$

where χ is some cut-off function and $\operatorname{Re} z$ is sufficiently small, extends holomorphically to Ω, and used it to describe regularity of solutions to Fuchsian type partial differential equations. For $u \in M(\Omega)$ the set $\mathbb{C} \setminus \Omega$ may be regarded as the set of those exponents which enter into the asymptotic expansion of u at zero in powers x^α.

It follows from our paper that in the case $\Omega = \mathbb{C} \setminus L$, where $L \in \mathcal{L}$ (see notation), a distribution $u \in M(\Omega)$ whose local Mellin transform satisfies some growth conditions in fact belongs to the space $\widetilde{\mathcal{O}}^L$ of holomorphic functions on some subset of the universal covering space of $\mathbb{C} \setminus \{0\}$. To prove that, we extend the definition (given in [3]) of the space of analytic functionals $Q'(L)$ with carrier in L to the case when $L \in \mathcal{L}$ has a non-bounded imaginary part. Then the Taylor transformation gives an isomorphism of $Q'(L)$ onto $\widetilde{\mathcal{O}}^L$. It is natural to call an element of $\widetilde{\mathcal{O}}^L$ a generalized analytic function ([8]). On the other hand, the (modified) Cauchy transformation is an isomorphism of $\widetilde{\mathcal{O}}^L$ onto the cohomology class $\widetilde{H}^1_L$ and allows us to treat $\widetilde{\mathcal{O}}^L$ as a subspace of $M(\mathbb{C} \setminus L)$.

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In the final section we describe multipliers and operators acting on $Q'(L)$ and $\widetilde{\mathcal{O}}^L$. We also give an example showing that functions from $\widetilde{\mathcal{O}}^L$ appear as solutions to singular differential equations (Example 5).

Since the facts given in Sections 1–3 are a straightforward generalization of those given in [3] we state them without proofs.

0. Notation. Let $t > 0$. We denote by $\widetilde{B}(t)$ the universal covering space of the punctured disc $B(t) \setminus \{0\} \subset \mathbb{C}$ and by $\widetilde{\mathbb{C}}$ the universal covering space of $\mathbb{C} \setminus \{0\}$. Recall that any point $x \in \widetilde{B}(t)$ can be written in the form $x = |x| \exp(i \arg x)$, where $|x| < t$.

For $A \subset \mathbb{C}$ we set

$$A_\varepsilon = \{z \in \mathbb{C} : \operatorname{dist}(z, A) < \varepsilon\}, \qquad \varepsilon > 0,$$
$$A + s = \{z \in \mathbb{C} : z = a + s \text{ for some } a \in A\}, \qquad s \in \mathbb{R}.$$

The supporting function of A is given by

$$H_A(\xi, \eta) = \sup_{z \in A}(\xi \operatorname{Re} z + \eta \operatorname{Im} z), \qquad \xi, \eta \in \mathbb{R}.$$

Let $-\infty < k_1 \le k_2 < \infty$. Denote by $\mathcal{L}^{k_1,k_2}$ the collection of all closed convex subsets L of $\mathbb{C}$ satisfying: for every $\delta > 0$ there exists C_δ such that

$$(k_1 - \delta) \operatorname{Re} \zeta - C_\delta \le \operatorname{Im} \zeta \le (k_2 + \delta) \operatorname{Re} \zeta + C_\delta \qquad \text{for } \zeta \in L.$$

We put

$$\mathcal{L} = \bigcup_{k_1 \le k_2} \mathcal{L}^{k_1,k_2}.$$

For unbounded $L \in \mathcal{L}$ we define

$$k_1^L = \sup\{k_1 : L \in \mathcal{L}^{k_1,k_2}\}, \qquad k_2^L = \inf\{k_2 : L \in \mathcal{L}^{k_1,k_2}\}.$$

By rotation of L we can assume that $k_1^L \le 0 \le k_2^L$. In this case we put

$$b_1^L = \begin{cases} (k_1^L)^{-1} & \text{if } k_1^L < 0, \\ -\infty & \text{if } k_1^L = 0; \end{cases} \qquad b_2^L = \begin{cases} (k_2^L)^{-1} & \text{if } k_2^L > 0, \\ \infty & \text{if } k_2^L = 0. \end{cases}$$

We set

$$L^{\mathrm{ap}} = \{z \in \mathbb{C} : k_1^L \operatorname{Re} z \le \operatorname{Im} z \le k_2^L \operatorname{Re} z\},$$
$$L^0 = \{z \in \mathbb{C} : \operatorname{Re} z \operatorname{Re} \zeta + \operatorname{Im} z \operatorname{Im} \zeta \ge 0 \text{ for } \zeta \in L, \operatorname{Re} \zeta \ge 0\}$$

and call them the *approximative* and the *dual cone* of L, respectively. Obviously $(L^0)^0 = L^{\mathrm{ap}}$.

We also set

$$\Gamma_t^{k_1,k_2} = \{x \in \widetilde{\mathbb{C}} : |x| < t \exp(\min(k_1 \arg x, k_2 \arg x))\}, \qquad \Gamma_t^L = \Gamma_t^{k_1^L, k_2^L}.$$

Observe that Γ_t^L is the sum of all $\Gamma_t^{k_1,k_2}$ such that $L \in \mathcal{L}^{k_1,k_2}$.

We abbreviate

$$D = \frac{d}{dx}, \quad \widetilde{D} = x \frac{d}{dx} \quad \text{and} \quad t_{-\kappa} = t \exp(-\kappa) \quad \text{for } t > 0,\ \kappa > 0.$$

$\mathcal{O}(W)$ denotes the set of holomorphic functions on an open subset W of some Riemann manifold. The value of a functional S on a test function φ is denoted by $S[\varphi]$.

1. The space $Q(L,t)$. In this section we define the fundamental space $Q(L,t)$ and state some basic properties of the dual space $Q'(L,t)$.

DEFINITION. Let $L \in \mathcal{L}$, $\delta > 0$, $\kappa > 0$ and $t > 0$. We define

$$Q_{\mathrm{b}}(L_\delta, t_{-\kappa}) = \{\varphi \in C^0(\overline{L}_\delta) \cap \mathcal{O}(L_\delta) : \varrho_{\delta,\kappa}(\varphi) = \sup_{\zeta \in L_\delta} |\varphi(\zeta) t_{-\kappa}^{-\zeta}| < \infty\}.$$

$Q(L,t)$ is the inductive limit of the Banach spaces $Q_{\mathrm{b}}(L_\delta, t_{-\kappa})$ over $\delta > 0$ and $\kappa > 0$.

EXAMPLE 1. If $L \in \mathcal{L}$ and $x \in \Gamma_t^L$ then the function $\zeta \to x^\zeta$ belongs to $Q(L,t)$.

Proof. Indeed, if $\arg x \ge 0$ and $|x| = t_{-\kappa_1} \exp(k_1^L \arg x)$ for some $\kappa_1 > 0$, then for any $\delta > 0$ and $\kappa < \kappa_1$,

$$\varrho_{\delta,\kappa}(x^\zeta) = \sup_{\zeta \in L_\delta} \exp\{(\kappa - \kappa_1)\xi + \arg x(k_1^L \xi - \eta)\}$$
$$\le \sup_{\zeta \in L_\delta} \exp\{(\kappa - \kappa_1)\xi + \arg x(\varrho \xi + C)\} < \infty$$

for ϱ sufficiently small. The same is true if $\arg x < 0$ and $|x| < t \exp(k_2^L \arg x)$.

Observe that if L is a compact subset of $\mathbb{C}$, then the function $\zeta \to x^\zeta$ belongs to $Q(L,t)$ with arbitrary $t > 0$ for every $x \in \widetilde{\mathbb{C}}$.

We call an element of the dual space $Q'(L,t)$ an *analytic functional with carrier in L of exponential type t*. A linear functional S on $Q(L,t)$ is in $Q'(L,t)$ if and only if for every $\delta > 0$ and $\kappa > 0$ there exists a constant C such that

$$|S[\varphi]| \le C \varrho_{\delta,\kappa}(\varphi) \qquad \text{for } \varphi \in Q_{\mathrm{b}}(L_\varepsilon, t_{-\kappa}).$$

If L_0 is a compact convex subset of $L \in \mathcal{L}$ then by the Runge theorem $Q(L,t)$ is dense in $\mathcal{O}(L_0)$. Hence by the Hahn–Banach theorem we may consider the space of analytic functionals with carrier in L_0 as a subspace of $Q'(L,t)$.

PROPOSITION 1. *Let $t_1 < t_2$. Then the natural mapping $Q'(L, t_2) \to Q'(L, t_1)$ is injective.*

Proof. Since multiplication by t^ζ is an isomorphism of $Q(L,1)$ onto $Q(L,t)$ we can assume that $t_1 = 1$ and put $t_2 = t > 1$. By the Hahn–Banach theorem it is sufficient to show that $Q(L,1)$ is a dense subset of $Q(L,t)$. To this end take $\varphi \in Q(L,t)$, put $\varphi_n(\zeta) = t^{-\zeta/n}\varphi(\zeta)$ for $\zeta \in \overline{L}_\varepsilon$, $n \in \mathbb{N}$ and observe that $Q(L,1) \ni \varphi_n \to \varphi$ in $Q(L,t)$.

2. The spaces $R^L(\mathbb{C}\setminus L,t)$, $R^L(\mathbb{C},t)$, $H^1_L(\mathbb{C},t)$ and $\widetilde{H}^1_L(\mathbb{C},t)$. In this section we use the following version of the Phragmén–Lindelöf theorem (see [1], [5]).

THEOREM 1 (Maximum principle). *Let $L \in \mathcal{L}$ and $R > 0$. Suppose that F is a holomorphic function on L_R and continuous on $\overline{L}_R$ such that for some $M > 0$ the function*

$$\overline{L}_R \ni z \to \exp(-M|z|)F(z) \text{ is bounded.}$$

If $|F(z)| \le K$ on the boundary of $\overline{L}_R$, then $|F(z)| \le K$ on $\overline{L}_R$.

DEFINITION. Let $L \in \mathcal{L}$, $0 < \varepsilon < R$, $\kappa > 0$ and $t > 0$. We define

$$R_b(\overline{L_R\setminus L_\varepsilon}, t_{-\kappa}) = \{F \in C^0(\overline{L_R\setminus L_\varepsilon}) \cap \mathcal{O}(\mathrm{int}(L_R\setminus L_\varepsilon)) : \sup_{z\in\overline{L_R\setminus L_\varepsilon}} |F(z)t^z_{-\kappa}| < \infty\},$$

$$R_b(\overline{L}_R, t_{-\kappa}) = \{F \in C^0(\overline{L}_R) \cap \mathcal{O}(L_R) : \sup_{z\in\overline{L}_R} |F(z)t^z_{-\kappa}| < \infty\}.$$

$R_b(\overline{L_R\setminus L_\varepsilon}, t_{-\kappa})$ and $R_b(\overline{L}_R, t_{-\kappa})$ are Banach spaces, and by the restriction mapping the second space is contained in the first one. The following lemma follows from Theorem 1.

LEMMA 1. *$R_b(\overline{L}_R, t_{-\kappa})$ is a closed subspace of $R_b(\overline{L_R\setminus L_\varepsilon}, t_{-\kappa})$.*

Let $0 < \varepsilon_1 < \varepsilon < R < R_1$ and $0 < \kappa_1 < \kappa$. Then the diagram of restriction mappings

$$\begin{array}{ccc} R_b(\overline{L_{R_1}\setminus L_{\varepsilon_1}}, t_{-\kappa_1}) & \to & R_b(\overline{L_R\setminus L_\varepsilon}, t_{-\kappa}) \\ \uparrow & & \uparrow \\ R_b(\overline{L}_{R_1}, t_{-\kappa_1}) & \to & R_b(\overline{L}_R, t_{-\kappa}) \end{array}$$

is commutative and the horizontal mappings are compact.

Thus, we can define the spaces

$$R^L_b(\mathbb{C}\setminus L, t_{-\kappa}) = \operatorname*{limproj}_{R\to\infty,\varepsilon\to 0} R_b(\overline{L_R\setminus L_\varepsilon}, t_{-\kappa}),$$

$$R^L_b(\mathbb{C}, t_{-\kappa}) = \operatorname*{limproj}_{R\to\infty} R_b(\overline{L}_R, t_{-\kappa}).$$

It follows from Lemma 1 that $R^L_b(\mathbb{C}, t_{-\kappa})$ is a closed subspace of $R^L_b(\mathbb{C}\setminus L, t_{-\kappa})$ and we can define

$$H^1_{L,b}(\mathbb{C}, t_{-\kappa}) = \frac{R^L_b(\mathbb{C}\setminus L, t_{-\kappa})}{R^L_b(\mathbb{C}, t_{-\kappa})}.$$

LEMMA 2. *Let $\kappa_1 < \kappa$. Then the natural mapping $i : H^1_{L,b}(\mathbb{C}, t_{-\kappa_1}) \to H^1_{L,b}(\mathbb{C}, t_{-\kappa})$ is injective.*

The proof follows by Theorem 1.

Further, we define

$$R^L(\mathbb{C}\setminus L, t) = \operatorname*{limproj}_{\kappa\to 0} R^L_b(\mathbb{C}\setminus L, t_{-\kappa}),$$

$$R^L(\mathbb{C}, t) = \operatorname*{limproj}_{\kappa\to 0} R^L_b(\mathbb{C}, t_{-\kappa}).$$

It is easy to show that $R^L(\mathbb{C},t)$ is a closed subspace of $R^L(\mathbb{C}\setminus L,t)$.

DEFINITION. (i) We define the space

$$H^1_L(\mathbb{C},t) = \frac{R^L(\mathbb{C}\setminus L,t)}{R^L(\mathbb{C},t)}.$$

(ii) Further, we define

$$\widetilde{H}^1_L(\mathbb{C},t) = \operatorname*{limproj}_{\kappa\to 0} H^1_{L,b}(\mathbb{C}, t_{-\kappa}).$$

By the definition an element f of $H^1_L(\mathbb{C},t)$ is defined by a function $F \in \mathcal{O}(\mathbb{C}\setminus L)$ such that for every $R > \varepsilon > 0$ and $\kappa > 0$,

$$\sup_{z\in\overline{L_R\setminus L_\varepsilon}} |F(z)t^z_{-\kappa}| < \infty.$$

On the other hand, an element g of $\widetilde{H}^1_L(\mathbb{C},t)$ is given by a set of functions $G_\kappa \in \mathcal{O}(\mathbb{C}\setminus L)$, $\kappa > 0$, such that for every $R > \varepsilon > 0$,

$$\sup_{z\in\overline{L_R\setminus L_\varepsilon}} |G_\kappa(z)t^z_{-\kappa}| < \infty,$$

and for every $R > 0$ and $\kappa_1 < \kappa$,

$$\sup_{z\in L_R} |G_\kappa(z) - G_{\kappa_1}(z)| t^{\operatorname{Re} z}_{-\kappa} < \infty.$$

In that case g will be denoted by $[G(\cdot,+0)]$ or $[G_{+0}]$.

Theorem 1 yields the following

LEMMA 3. *The natural mapping $i : H^1_L(\mathbb{C},t) \to \widetilde{H}^1_L(\mathbb{C},t)$ is injective.*

3. The boundary value transformation. Let $\varphi \in Q(L,t)$. Take $\varepsilon_0 > 0$ and $\kappa_0 > 0$ such that $\varphi \in Q_b(L_{\varepsilon_0}, t_{-\kappa_0})$. Then for $F \in R(\mathbb{C}\setminus L,t)$ the integral

$$\int_{\partial L_\varepsilon} \varphi(\zeta)F(\zeta)\,d\zeta$$

converges absolutely and uniformly in ε with $\varepsilon_1 < \varepsilon < \varepsilon_0$. So it does not depend on $\varepsilon < \varepsilon_0$ and can be written as

$$\int_{\partial L_+} \varphi(\zeta)F(\zeta)\,d\zeta\,.$$

For $F \in R(\mathbb{C} \setminus L, t)$ we define a linear functional $b(F)$ on $Q(L,t)$ by

$$b(F)[\varphi] = \frac{1}{2\pi i} \int_{\partial L_+} \varphi(\zeta)F(\zeta)\,d\zeta\,.$$

Since for $\varphi \in Q_{\mathrm{b}}(L_{\varepsilon_0}, t_{-\kappa_0})$, $\varepsilon_1 < \varepsilon < \varepsilon_0$ and $\kappa < \kappa_0$,

$$\begin{aligned} |b(F)[\varphi]| &\le \frac{1}{2\pi} \int_{\partial L_\varepsilon} |\varphi(\zeta)F(\zeta)|\,d\zeta \\ &\le \frac{1}{2\pi} \int_{\partial L_\varepsilon} |F(\zeta)|t_{-\kappa_0}^{\operatorname{Re}\zeta}\,d\zeta \cdot \sup_{\zeta \in L_{\varepsilon_0}} |\varphi(\zeta)|t_{-\kappa_0}^{-\operatorname{Re}\zeta} \le C\varrho_{\varepsilon_0,\kappa_0}(\varphi) \end{aligned}$$

with

$$C = \frac{1}{2\pi} \sup_{\zeta \in \overline{L_{\varepsilon_0} \setminus L_{\varepsilon_1}}} |F(\zeta)|t_\kappa^{\operatorname{Re}\zeta} \Big/ \int_{\partial L_\varepsilon} \exp(\kappa - \kappa_0)\operatorname{Re}\zeta\,d\zeta\,,$$

we get the following:

PROPOSITION 2. *For $F \in R(\mathbb{C} \setminus L, t)$ the functional $b(F)$ is continuous on $Q(L,t)$. If $F \in R^L(\mathbb{C}, t)$ then $b(F) = 0$.*

Thus, we can define a continuous linear mapping

$$b : H_L^1(\mathbb{C}, t) \to Q'(L,t)\,.$$

We call it the *boundary value transformation*.

Suppose that $[G_{+0}] \in \widetilde{H}_L^1(\mathbb{C}, t)$ is given and let $\varphi \in Q(L,t)$. Then $\varphi \in Q(L_{\varepsilon_0}, t_{-\kappa_0})$ for some $\varepsilon_0 > 0$ and $\kappa_0 > 0$. By the Cauchy integral theorem the integral

$$\int_{\partial L_\varepsilon} \varphi(\zeta)G_\kappa(\zeta)\,d\zeta$$

is independent of $0 < \varepsilon < \varepsilon_0$, $0 < \kappa < \kappa_0$, and the choice of G_κ from the cohomology class $[G_{+0}]$. We write it

$$\int_{\partial L_+} \varphi(\zeta)[G(\zeta, +0)]\,d\zeta\,.$$

We define the functional

$$\widetilde{b}[G_{+0}][\varphi] = \frac{1}{2\pi i} \int_{\partial L_+} \varphi(\zeta)[G(\zeta, +0)]\,d\zeta\,.$$

As in the case of Proposition 2 we get

PROPOSITION 3. *If $g \in \widetilde{H}_L^1(\mathbb{C}, t)$, then $\widetilde{b}(g)$ is a continuous linear functional on $Q(L,t)$.*

We call the mapping

$$\widetilde{b} : \widetilde{H}_L^1(\mathbb{C}, t) \to Q'(L,t)$$

the *modified boundary value transformation.*

Immediately from the definitions of the mappings $i, b, \widetilde{b}$ we get

PROPOSITION 4. *The following diagram of continuous linear mappings is commutative:*

$$\begin{array}{ccc} & Q'(L,t) & \\ {\scriptstyle b} \nearrow & & \nwarrow {\scriptstyle \widetilde{b}} \\ H_L^1(\mathbb{C}, t) & \xrightarrow{\ i\ } & \widetilde{H}_L^1(\mathbb{C}, t) \end{array}$$

Remark. It can be proved that all the mappings in the diagram are topological isomorphisms (see [3]).

4. The modified Cauchy transformation. Denote by $\chi_{\widetilde{r},r}$, $0 < \widetilde{r} < r$, any cut-off function (not necessarily smooth) supported by $[0,r]$ and equal to one on $[0,\widetilde{r}]$. Put

$$G_{\widetilde{r},r}(\zeta) = \int_0^r \chi_{\widetilde{r},r}(x)x^{-\zeta-1}\,dx \qquad \text{for } \operatorname{Re}\zeta < 0\,. \tag{1}$$

It is easy to note that $G_{\widetilde{r},r}$ extends to a holomorphic function on $\mathbb{C} \setminus \{0\}$ with simple pole at zero with residue -1. Furthermore, it satisfies

$$|G_{\widetilde{r},r}(\zeta)| \le \begin{cases} C\widetilde{r}^{-\operatorname{Re}\zeta} & \text{for } \operatorname{Re}\zeta \ge 0,\ |\zeta| \ge 1, \\ Cr^{-\operatorname{Re}\zeta} & \text{for } \operatorname{Re}\zeta \le 0,\ |\zeta| \ge 1. \end{cases} \tag{2}$$

The function $G_{\widetilde{r},r}$ will be called a *modified Cauchy kernel*. In the above definition we can also take $\widetilde{r} = r$ and put

$$G_{r,r}(\zeta) = \frac{r^{-\zeta}}{-\zeta}, \qquad \zeta \ne 0\,.$$

LEMMA 4. *Let $G_{\widetilde{r},r}$ be the modified Cauchy kernel with $r < t$. Then*

$$\mathbb{C} \setminus L \ni z \to G_{\widetilde{r},r}(z - \zeta)$$

is a $Q(L,t)$-valued holomorphic function. Furthermore, it belongs to $R_{\mathrm{b}}^L(\mathbb{C} \setminus L, \widetilde{r})$.

Proof. In fact, for $z \in \mathbb{C} \setminus L$, $\operatorname{Re} z \ge -R$, $\varepsilon < \operatorname{dist}(z, L)$ and κ such that $r < t_{-\kappa}$ we get

$$\sup_{\zeta \in L_\varepsilon} |G_{\widetilde{r},r}(z - \zeta)t_{-\kappa}^{-\zeta}| \le C_R \frac{\widetilde{r}^{-\operatorname{Re} z}}{\varepsilon} \cdot \sup_{\zeta \in L_\varepsilon} |\widetilde{r}^{\zeta} t_{-\kappa}^{-\zeta}| = C_{\varepsilon,\kappa,R}\widetilde{r}^{-\operatorname{Re} z}\,.$$

THEOREM 2 (The Cauchy integral formula). *Let $L \in \mathcal{L}$ and $\varphi \in Q(L,t)$. Choose ε_0 and κ_0 such that $\varphi \in Q_{\mathrm{b}}(L_{\varepsilon_0}, t_{-\kappa_0})$. Let $G_{\tilde{r},r}$ be a modified Cauchy kernel with $t_{-\kappa_0} < \tilde{r} < t$. Then*

$$(3) \qquad \varphi(\zeta) = \frac{-1}{2\pi i} \int_{\partial L_\varepsilon} \varphi(z) G_{\tilde{r},r}(z-\zeta)\, dz \quad \text{for } \zeta \in L_\varepsilon,\ \varepsilon < \varepsilon_0\,,$$

and the integral converges in the topology of $Q(L,t)$.

Proof. Fix $\zeta \in L_\varepsilon$ and denote by γ_d the boundary of the set $\{z \in L_\varepsilon : \mathrm{Re}\, z \le d\}$, where d is big enough. Then by the usual Cauchy integral formula we have

$$\varphi(\zeta) = \frac{-1}{2\pi i} \int_{\gamma_d} \varphi(z) G_{\tilde{r},r}(z-\zeta)\, dz \quad \text{for } \zeta \in L_\varepsilon \text{ and } \mathrm{Re}\,\zeta < d\,.$$

The integral over the segment $\{z \in \gamma_d : \mathrm{Re}\, z = d\}$ is bounded by $Cdt^d_{-\kappa_0} \times r^{\mathrm{Re}\,\zeta - d}$, and hence, converges to zero as $d \to \infty$. Now for κ_1 such that $r < t_{-\kappa_1}$ we have, by Lemma 4,

$$\sup_{\zeta \in L_{\varepsilon/2}} \Big| \int_{\partial L_{\varepsilon,d}} \varphi(z) G_{\tilde{r},r}(z-\zeta)\, dz \Big| t^{\mathrm{Re}\,\zeta}_{-\kappa_1} \le C_{\varepsilon,\kappa_1} \Big| \int_{\partial L_{\varepsilon,d}} \varphi(z) r^{-\mathrm{Re}\, z}\, dz \Big| \to 0 \quad \text{as } d \to \infty\,,$$

where $\partial L_{\varepsilon,d} = \{z \in \partial L_\varepsilon : \mathrm{Re}\, z \ge d\}$.

DEFINITION. Let $G_{\tilde{r},r}$ be a modified Cauchy kernel. We define the $(\tilde{r},r)$-*Cauchy transformation* of $S \in Q'(L,t)$ by

$$\mathcal{C}_{\tilde{r},r} S(z) = \frac{1}{2\pi i} S[G_{\tilde{r},r}(z - \cdot)] \quad \text{for } z \notin L\,.$$

It follows by Lemma 4 that $\mathcal{C}_{\tilde{r},r} S \in R^L_{\mathrm{b}}(\mathbb{C} \setminus L, \tilde{r})$.

LEMMA 5. *Let $\tilde{r} < \tilde{r}_1 < t$ and $S \in Q'(L,t)$. Then the function*

$$F(z) = \mathcal{C}_{\tilde{r},r} S(z) - \mathcal{C}_{\tilde{r}_1,r_1} S(z), \quad z \notin L\,,$$

extends to an entire function which belongs to $R^L_{\mathrm{b}}(\mathbb{C}, \tilde{r})$.

Proof. Indeed, the holomorphic extension is given by

$$F(z) = \frac{1}{2\pi i} S[(G_{\tilde{r},r} - G_{\tilde{r}_1,r_1})(z - \cdot)], \quad z \in \mathbb{C}\,,$$

where $G_{\tilde{r},r} - G_{\tilde{r}_1,r_1}$ is an entire function bounded by $C_R \tilde{r}^{-\mathrm{Re}\,\zeta}$ for $\mathrm{Re}\,\zeta \ge -R$.

By Lemmas 4 and 5 the set of $(\tilde{r},r)$-Cauchy transforms of a functional $S \in Q'(L,t)$ defines a cohomology class $\mathcal{C}S = [\{\mathcal{C}_{\tilde{r},r} S\}_{\tilde{r}<t}]$ in $\widetilde{H}^1_L(\mathbb{C},t)$, which we call the *Cauchy transform* of S.

THEOREM 3. *Let $\mathcal{C}S \in \widetilde{H}^1_L(\mathbb{C},t)$ be the Cauchy transform of $S \in Q'(L,t)$. Then the following inversion formula holds:*

$$S[\varphi] = -\int_{\partial L_+} \varphi(z)[\mathcal{C}S_{+0}(z)]\, dz \quad \text{for } \varphi \in Q(L,t)\,.$$

In other words, $\tilde{b} \circ \mathcal{C} = \mathrm{id}$. Furthermore, $\mathcal{C} \circ \tilde{b} = \mathrm{id}$.

Proof. By the Cauchy integral formula (3) for $S \in Q'(L,t)$, $\varphi \in Q(L,t)$ and $\tilde{r}$ close to t, we derive

$$\begin{aligned} -\int_{\partial L_+} \varphi(z)[\mathcal{C}S_{+0}(z)]\, dz &= -\int_{\partial L_\varepsilon} \varphi(z) \mathcal{C}_{\tilde{r},r} S(z)\, dz \\ &= \frac{-1}{2\pi i} \int_{\partial L_\varepsilon} \varphi(z) S[G_{\tilde{r},r}(z - \cdot)]\, dz \\ &= \frac{-1}{2\pi i} S\Big[\int_{\partial L_\varepsilon} \varphi(z) G_{\tilde{r},r}(z - \cdot)\, dz\Big] = S[\varphi]\,. \end{aligned}$$

Take now $g \in \widetilde{H}^1_L(\mathbb{C},t)$. It is given by a set of functions $G_\kappa \in R^L_{\mathrm{b}}(\mathbb{C} \setminus L, t_{-\kappa})$, $\kappa > 0$, such that $G_\kappa - G_{\kappa_1} \in R^L_{\mathrm{b}}(\mathbb{C}, t_{-\kappa})$ for $\kappa_1 < \kappa$. Let $\varphi \in Q(L,t)$. Then $\varphi \in Q_{\mathrm{b}}(L_{\varepsilon_0}, t_{\kappa_0})$ for some $\varepsilon_0 > 0$ and $\kappa_0 > 0$. By the definition, $\tilde{b}(g) = S$ is given by

$$S[\varphi] = -\int_{\partial L_\varepsilon} \varphi(z) G_\kappa(z)\, dz \quad \text{for } \varepsilon < \varepsilon_0,\ \kappa < \kappa_0\,.$$

By the first part of the theorem, for $\tilde{r} > t_{-\kappa}$ we have

$$S[\varphi] = -\int_{\partial L_\varepsilon} \varphi(z) S[G_{\tilde{r},r}(z - \cdot)]\, dz\,.$$

So

$$(4) \qquad \int_{\partial L_\varepsilon} \varphi(z) \psi_\kappa(z)\, dz = 0\,,$$

where

$$\psi_\kappa(z) = G_\kappa(z) - S[G_{\tilde{r},r}(z - \cdot)] \quad \text{for } z \in \mathbb{C} \setminus L\,.$$

We now show that $\psi_\kappa \in R^L_{\mathrm{b}}(\mathbb{C} \setminus L, t_{-\kappa})$ and that ψ_κ extends holomorphically to a function $\tilde{\psi}_\kappa \in R^L_{\mathrm{b}}(\mathbb{C}, t_{-\kappa})$. To this end put

$$\widetilde{G}_\kappa(z) = \frac{1}{2\pi i} \int_{\partial L_{\varepsilon_0}} \psi(\zeta) G_{\tilde{r},r}(\zeta - z)\, d\zeta, \quad z \in L_{\varepsilon_0}\,.$$

Then $|\widetilde{G}_\kappa(z)| \le C\tilde{r}^{-\mathrm{Re}\, z}$ for $z \in L_{\varepsilon_0}$. For a fixed $z \in L_{\varepsilon_0} \setminus \overline{L}_\varepsilon$, by (4) and

Lemma 4 we have $\widetilde{G}_\kappa(z) = \psi_\kappa(z)$. Thus, if we put

$$\widetilde{\psi}_\kappa = \begin{cases} \psi_\kappa(z) & \text{for } z \in \mathbb{C} \setminus L, \\ \widetilde{G}_\kappa(z) & \text{for } z \in L_{\varepsilon_0}, \end{cases}$$

then $\widetilde{\varphi}_\kappa$ is an entire function, which by Theorem 1 belongs to $R^L_{\rm b}(\mathbb{C}, t_{-\kappa})$. Hence $[\widetilde{\psi}_\kappa] = 0$ in $\widetilde{H}^1_L(\mathbb{C}, t)$ and $g = \mathcal{C} \circ \widetilde{b}(g)$.

COROLLARY 1. *Let $L_1, L_2 \in \mathcal{L}$, $L_1 \subset L_2$, and $t_1 < t_2$. Then the natural mapping $Q'(L_1, t_1) \to Q'(L_2, t_2)$ is injective.*

Proof. By Proposition 1 we may suppose $t_1 = t_2 = t$. Now it follows by Theorem 3 that it is sufficient to show the injectivity of the natural mapping $\widetilde{H}^1_{L_1}(\mathbb{C}, t) \to \widetilde{H}^1_{L_2}(\mathbb{C}, t)$. But this is obvious since for every $\kappa > 0$,

$$R^{L_1}_{\rm b}(\mathbb{C} \setminus L_1, t_{-\kappa}) \cap R^{L_2}_{\rm b}(\mathbb{C}, t_{-\kappa}) \subset R^{L_1}_{\rm b}(\mathbb{C}, t_{-\kappa}).$$

5. The Taylor transformation

DEFINITION. Let $L \in \mathcal{L}$. We define the *Taylor transform* TS of $S \in Q'(L, t)$ by

$$TS(x) = S_\alpha[x^\alpha] \quad \text{for } x \in \Gamma^L_t.$$

By Example 1 and the formula

$$\widetilde{D}(TS) = T(\alpha S) \tag{5}$$

it is a well defined holomorphic function on Γ^L_t.

In order to describe the image of $Q'(L, t)$ under the Taylor transformation we define the space

$$\widetilde{\mathcal{O}}^L_t = \{u \in \mathcal{O}(\Gamma^L_t) : \text{for every } \varepsilon > 0, \kappa > 0 \text{ there exists a constant } C = C(\varepsilon, \kappa) \text{ such that } |u(x)| \le C \exp H_{-L_\varepsilon}(-\ln|x|, \arg x) \text{ for } x \in \Gamma^L_{t_{-\kappa}}\}.$$

THEOREM 4. *Let $S \in Q'(L, t)$ and $u(x) = TS(x)$ for $x \in \Gamma^L_t$. Then $u \in \widetilde{\mathcal{O}}^L_t$.*

Proof. Indeed, for any $\varepsilon > 0$, $\kappa > 0$ and $x \in \Gamma^L_{t_{-\kappa}}$ we have

$$|u(x)| \le C \sup_{\zeta \in L_\varepsilon} |x^\zeta t_{-\kappa}^{-\operatorname{Re}\zeta}| \le C_1 \exp H_{-L_\varepsilon}(-\ln|x|, \arg x).$$

6. The Mellin transformation.

By Theorem 4, $TQ'(L, t) \subset \widetilde{\mathcal{O}}^L_t$. In fact, the Taylor transformation is an isomorphism between $TQ'(L, t)$ and $\widetilde{\mathcal{O}}^L_t$. The inverse mapping is given by the composition of the Mellin and boundary value transformations. To show this we need to recall the definition of the first one (cf. [6]). In this section we assume that $k^L_1 \le 0 \le k^L_2$ which can be obtained by rotation of the original L.

LEMMA 6. *Let $u \in \widetilde{\mathcal{O}}^L_t$, $0 < r < t$ and $b^L_1 < b < b^L_2$. Put*

$$\gamma_{r,b} = \{x \in \Gamma^L_t : x = r\exp(-(1 + ib)\varphi),\ 0 < \varphi < \infty\},$$

$$\mathcal{M}_{r,b}u(z) = \int_{\gamma_{r,b}} u(x)x^{-z-1}\,dx \quad \text{for } z \in \Omega_b,$$

where $\Omega_b := \{z \in \mathbb{C} : b\operatorname{Im} z > \operatorname{Re} z + H_L(-1, b)\}$. Then $\mathcal{M}_{r,b}u \in \mathcal{O}(\Omega_b)$ and for every $a > H_L(-1, b)$,

$$|\mathcal{M}_{r,b}u(z)| \le C_{a,b} r^{-\operatorname{Re} z} \quad \text{for } b\operatorname{Im} z \ge \operatorname{Re} z + a.$$

Proof. Let $z \in \Omega_b$. Take $\varepsilon > 0$ and $\delta > 0$ such that $\varepsilon|b| + \delta + H_L(-1, b) + \operatorname{Re} z - b\operatorname{Im} z < 0$. Then

$$\begin{aligned} |\mathcal{M}_{r,b}u(z)| &= \Big| r^{-z} \int_0^\infty (-1 - ib)u(r\exp(-1 - ib)\varphi)\exp(1 + ib)\varphi z\,d\varphi \Big| \\ &\le Cr^{-\operatorname{Re} z} \int_0^\infty \exp[H_{-L_\varepsilon}(-\ln r + \varphi, -b\varphi) + \varphi(\operatorname{Re} z - b\operatorname{Im} z)]\,d\varphi \\ &\le C_1 r^{-\operatorname{Re} z} \end{aligned}$$

since by the properties of the supporting function, for φ large we have

$$H_{-L_\varepsilon}(-\ln r + \varphi, -b\varphi) \le (H_L(-1, b) + \varepsilon(1 + |b|))\varphi + \delta.$$

Observe also that the constant C_1 is uniformly bounded on $\{b\operatorname{Im} z \ge \operatorname{Re} z + a\}$, $a > H_L(-1, b)$.

To study the relations among different $\mathcal{M}_{r,b}u$ we need the following corollary from Theorem 1.

COROLLARY 2. *For $b_1 < b_2$ put*

$$\Gamma_{b_1,b_2,r} = \bigcup_{b_1 < b < b_2} \gamma_{r,b}.$$

Let $v \in \mathcal{O}(\Gamma_{b_1,b_2,r}) \cap C^0(\overline{\Gamma}_{b_1,b_2,r})$ be such that

$$|v(x)| \le C|x|^{-A} \quad \text{for } x \in \Gamma_{b_1,b_2,r} \text{ with some } C > 0, A > 0$$

and that

$$|v(x)| \le C|x|^{\delta-1} \quad \text{for } x \in \gamma_{r,b_1} \cup \gamma_{r,b_2} \text{ with } \delta > 0.$$

Then

$$|v(x)| \le C|x|^{\delta-1} \quad \text{for } x \in \overline{\Gamma}_{b_1,b_2,r}.$$

Proof. Indeed, the function

$$F(\varphi+ib)=(r\exp(-1-ib)\varphi)^{1-\delta}v(r\exp(-1-ib)\varphi)$$

is holomorphic on the strip

$$L_{b_1,b_2}:=\{\varphi+ib:0<\varphi<\infty,\ b_1<b<b_2\}$$

and satisfies the assumptions of Theorem 1. By that theorem we get the conclusion.

LEMMA 7. *Let* $u\in\widetilde{\mathcal{O}}_t^L$, $0<r<t$ *and* $b_1^L<b_1<b_2<b_2^L$. *Then*

$$\mathcal{M}_{r,b_1}u(z)=\mathcal{M}_{r,b_2}u(z)\quad \text{for } z\in\Omega_{b_1}\cap\Omega_{b_2}.$$

Proof. It is sufficient to show that for some $a>\max(H_L(-1,b_1), H_L(-1,b_2))$ and z satisfying $b_j\operatorname{Im}z>\operatorname{Re}z+a$, $j=1,2$, we have

$$\mathcal{M}_{r,b_1}u(z)=\mathcal{M}_{r,b_2}u(z).$$

To this end, put $v(x)=u(x)x^{-z-1}$ for $x\in\Gamma_t^L$. Then $v\in\mathcal{O}(\Gamma_{b_1,b_2,r})\cap C^0(\overline{\Gamma}_{b_1,b_2,r})$ and for $x\in\Gamma_{b_1,b_2,r}$,

$$\begin{aligned}|v(x)|&\le C\exp H_{-L_\varepsilon}(-\ln|x|,\arg x)r^{-\operatorname{Re}z-1}\exp(\operatorname{Re}z+1-b\operatorname{Im}z)\varphi\\&\le Cr^{-\operatorname{Re}z-1}\exp[(\operatorname{Re}z-b\operatorname{Im}z+1+H_L(-1,b)+\varepsilon(1+|b|))\varphi+\delta]\\&\le C_1|x|^{-A}\quad\text{with } A=\sup_{b_1\le b\le b_2}(\operatorname{Re}z-b\operatorname{Im}z+a+1).\end{aligned}$$

For $x\in\gamma_{r,b_j}$, $j=1,2$, we obtain

$$|v(x)|\le C_1|x|^{\delta-1}\quad\text{with some }\delta>0. \tag{6}$$

Thus, by Corollary 2, (6) holds for $x\in\Gamma_{b_1,b_2,r}$. For $d<r$ put

$$\gamma_d=\{x\in\Gamma_{b_1,b_2,r}:x=d\exp ib,\ b_1<b<b_2\}.$$

Then $|\int_{\gamma_d}v(x)\,dx|\le C_1d^\delta\to0$ as $d\to0$ and the hypothesis follows by the Cauchy integral theorem.

It follows by Lemmas 6 and 7 that for $u\in\widetilde{\mathcal{O}}_t^L,\kappa>0$ and $b_1^L<b<b_2^L$, the function $\mathcal{M}_{t_{-\kappa},b}u$ extends to a function $\mathcal{M}_{t_{-\kappa}}u$ holomorphic on $\mathbb{C}\setminus L$ such that for every $\delta<b^L=\min(b_2^L,-b_1^L)$,

$$|\mathcal{M}_{t_{-\kappa}}u(z)|\le C_{\delta,\varepsilon}t_{-\kappa}^{-\operatorname{Re}z}\quad\text{for } z\in\mathbb{C}\setminus L_\varepsilon^\delta,\quad\text{where } L^\delta=\mathbb{C}\setminus(\Omega_{b_1^L+\delta}\cup\Omega_{b_2^L-\delta}).$$

LEMMA 8. *Let* $\kappa_1<\kappa$. *Then*

$$G(z)=\mathcal{M}_{t_{-\kappa}}u(z)-\mathcal{M}_{t_{-\kappa_1}}u(z)$$

is an entire function satisfying

$$|G(z)|\le\begin{cases}Ct_{-\kappa_1}^{-\operatorname{Re}z} & \text{for } \operatorname{Re}z<0,\\ Ct_{-\kappa}^{-\operatorname{Re}z} & \text{for } \operatorname{Re}z\ge0.\end{cases}$$

Proof. In fact, we have

$$G(z)=\int_{t_\kappa}^{t_{\kappa_1}}u(x)x^{-z-1}\,dx\quad\text{for } z\in\mathbb{C}.$$

Thus, for every $\delta>0$, the set of functions $\{\mathcal{M}_{t_{-\kappa}}u\}_{\kappa>0}$ defines an element g_δ of the cohomology class $\widetilde{H}^1_{L^\delta}(\mathbb{C},t)$. Since

$$\bigcap_{\delta>0}\widetilde{H}^1_{L^\delta}(\mathbb{C},t)\simeq\bigcap_{\delta>0}Q'(L^\delta,t)=Q'(L,t)\simeq\widetilde{H}^1_L(\mathbb{C},t)$$

we obtain the mapping

$$\mathcal{M}:\widetilde{\mathcal{O}}_t^L\to\widetilde{H}^1_L(\mathbb{C},t).$$

EXAMPLE 2. The power function $x\to x^\alpha$ belongs to $\widetilde{\mathcal{O}}_t^L$ if and only if $\alpha\in L$. In that case

$$\mathcal{M}_{t_{-\kappa}}(x^\alpha)(z)=\frac{t_{-\kappa}^{\alpha-z}}{\alpha-z}\quad\text{for } z\ne\alpha.$$

Now we are in a position to prove the main result of the paper.

THEOREM 5. *Let* $u\in\widetilde{\mathcal{O}}_t^L$, $f=[\mathcal{M}u]$ *and* $S=\widetilde{b}(f)$. *Then*

$$u(x)=TS(x)\quad\text{for } x\in\Gamma_t^L.$$

Proof. Take $x\in\Gamma_t^L$. Then $x\in\Gamma_t^{L^\delta}$ and $|x|=t_{-\kappa_0}$ for some $\delta>0$ and $\kappa_0>0$. For simplicity suppose that $b_1^L>-\infty$ and $b_2^L<\infty$. Let

$$\kappa<\min\left(\frac{-\delta}{2b_1^L+\delta},\frac{\delta}{2b_2^L-\delta}\right)\kappa_0,$$

and $\varepsilon>0$. By the definition of $\widetilde{b}$ we have

$$TS(x)=-\int_{\partial L_\varepsilon}\mathcal{M}_{t_{-\kappa}}u(z)x^z\,dz=-\int_{\partial L_\varepsilon^{\delta/2}}\mathcal{M}_{t_{-\kappa}}u(z)x^z\,dz$$

since the function $z\to\mathcal{M}_{t_{-\kappa}}u(z)x^z$ decreases exponentially on $L_\varepsilon^{\delta/2}\setminus L_\varepsilon$ as $\operatorname{Re}z\to\infty$. Indeed, we have either $0\le\arg x<\kappa_0(b_1^L+\delta)$ or $0\ge\arg x>\kappa_0(b_2^L-\delta)$. In the first case we derive, for $z\notin L_\varepsilon$,

$$\begin{aligned}|\mathcal{M}_{t_{-\kappa}}u(z)x^z|&\le C\exp((\kappa-\kappa_0)\operatorname{Re}z-\operatorname{Im}z\arg x)\\&\le\begin{cases}C\exp(\kappa-\kappa_0)\operatorname{Re}z & \text{if } \operatorname{Im}z\ge0,\\ C\exp((\kappa-\kappa_0)\operatorname{Re}z+\kappa_0(b_1^L+\delta)\operatorname{Im}z) & \text{if } \operatorname{Im}z<0,\end{cases}\end{aligned}$$

and for $z\in L_\varepsilon^{\delta/2}$,

$$(\kappa-\kappa_0)\operatorname{Re}z+\kappa_0(b_1^L+\delta)\operatorname{Im}z\le\left(\kappa+\kappa_0\frac{\delta}{2b_1^L+\delta}\right)\operatorname{Re}z+C_1.$$

In the second case we get the analogous estimate. Decompose $\partial L_\varepsilon^{\delta/2}$ into the union of two half lines $L_1 \cup L_2$, where L_1 lies above and L_2 under L. Denote by w the common starting point of L_1 and L_2. Then for $z \in L_j$, $j = 1, 2$, we have

$$\mathcal{M}_{t_{-\kappa}} u(z) = \int_{\gamma_j} u(\zeta)\zeta^{-z-1}\, d\zeta ,$$

where $\gamma_1 = \gamma_{t_{-\kappa}, b_2^L - \delta/2}$ and $\gamma_2 = \gamma_{t_{-\kappa}, b_1^L + \delta/2}$. Thus

$$\begin{aligned} TS(x) &= -\int_{L_1}\int_{\gamma_1} u(\zeta)\zeta^{-z-1}\, d\zeta\, x^z\, dz + \int_{L_2}\int_{\gamma_2} u(\zeta)\zeta^{-z-1}\, d\zeta\, x^z\, dz \\ &= \int_{\gamma_1} \frac{u(\zeta)}{\zeta}\Big(-\int_{L_1}\Big(\frac{x}{\zeta}\Big)^z dz\Big) d\zeta + \int_{\gamma_2} \frac{u(\zeta)}{\zeta}\Big(\int_{L_2}\Big(\frac{x}{\zeta}\Big)^z dz\Big) d\zeta \\ &= \int_{\gamma_2-\gamma_1} \frac{u(\zeta)}{\zeta}\Big(\frac{x}{\zeta}\Big)^w \frac{1}{\ln x - \ln\zeta}\, d\zeta = u(x) \end{aligned}$$

by the Cauchy integral formula in logarithmic variables.

COROLLARY 3. *Let $L \in \mathcal{L}$ and $t > 0$. Then we have the following diagram of linear topological isomorphisms*:

$$\begin{array}{ccc} Q'(L,t) & \xrightarrow{T} & \widetilde{\mathcal{O}}_t^L \\ \tilde{b} \nwarrow \searrow \mathcal{C} & & \swarrow \mathcal{M} \\ & \widetilde{H}_L^1(\mathbb{C}, t). & \end{array}$$

Proof. By Theorems 3 and 5 we only have to prove that $\mathcal{C} = \mathcal{M} \circ T$. To this end take $S \in Q'(L,t)$. Then for $\operatorname{Re} z$ small enough and any $\kappa > 0$ we derive

$$\begin{aligned} \mathcal{M}_{t_{-\kappa}} \circ TS(z) &= \int_0^{t_{-\kappa}} S[x^{\cdot}]x^{-z-1}\, dx \\ &= S\Big[\int_0^{t_{-\kappa}} x^{\cdot - z - 1}\, dx\Big] = \mathcal{C}_{t_{-\kappa}, t_{-\kappa}} S(z) . \end{aligned}$$

Next by uniqueness of holomorphic extension both sides are equal on $\mathbb{C} \setminus L$.

7. Multipliers and operators. In order to describe multipliers in Q' we introduce the following:

DEFINITION. A function P holomorphic in some neighbourhood L_δ of L is said to be of *infraexponential growth* in L if for every $\varepsilon > 0$ there exists $C_\varepsilon < \infty$ such that

$$|P(\zeta)| \le C_\varepsilon \exp \varepsilon|\zeta| \quad \text{for } \zeta \in L_\delta .$$

We denote the space of such functions by $\operatorname{Infr}(L)$.

If $L_1 \subset L_2$ then $\operatorname{Infr}(L_2) \subset \operatorname{Infr}(L_1)$. For a compact set L, $\operatorname{Infr}(L)$ consists of functions holomorphic in some neighbourhood of L.

PROPOSITION 5. *The multiplication operator*

$$Q(L,t) \ni \varphi \to P\varphi \in Q(L,t)$$

is continuous if and only if $P \in \operatorname{Infr}(L)$. In this case by duality the multiplication operator

$$Q'(L,t) \ni S \to PS \in Q'(L,t)$$

is also continuous.

Proof. The sufficiency is obvious. To prove the necessity observe that if $P \notin \operatorname{Infr}(L)$ then the function $\zeta \to P(\zeta)x^\zeta$ is not in $Q(L,t)$ for x sufficiently close to t.

DEFINITION. We define the action of the differential operator of infinite order with symbol $P \in \operatorname{Infr}(L)$ on a function $u \in \widetilde{\mathcal{O}}_t^L$ as follows:

$$P(\widetilde{D})u = T(P(\alpha)\tilde{b}(\mathcal{M}u)) .$$

By Proposition 5, $P(\widetilde{D}) : \widetilde{\mathcal{O}}_t^L \to \widetilde{\mathcal{O}}_t^L$ is a well defined continuous operation. If P is a polynomial, then $P(\widetilde{D})$ is a usual differential operator. However, in the general case it may not be local as shown by

EXAMPLE 3. Let $a \in -L^0$ and $u \in \widetilde{\mathcal{O}}_t^L$. Then

$$e^{a\widetilde{D}}u(x) = u(e^a x) \quad \text{for } x \in \Gamma_t^L .$$

Proof. Indeed, the function $\alpha \to e^{a\alpha}$ belongs to $\operatorname{Infr}(L)$ if and only if $a \in -L^0$. In this case

$$T(e^{a\alpha}\tilde{b}(\mathcal{M}u))(x) = e^{a\alpha}\tilde{b}(\mathcal{M}u)[x^{\cdot}] = u(e^a x), \quad x \in \Gamma_t^L .$$

EXAMPLE 4. Since

$$\frac{1}{-\ln x} = \int_0^\infty x^\alpha\, d\alpha \quad \text{for } |x| < 1 ,$$

for $P \in \operatorname{Infr}([0,\infty))$ we have

$$P(\widetilde{D})\Big(\frac{1}{-\ln x}\Big) = \int_0^\infty P(\alpha)x^\alpha\, d\alpha \quad \text{for } |x| < 1 .$$

By the definition of the Γ-function we also have (see [2], [4])

$$\frac{1}{\Gamma(-\theta-1)}\widetilde{D}^{-\theta-1}\Big(\frac{1}{-\ln x}\Big) = (-\ln x)^\theta \quad \text{for } |x| < 1,\ \operatorname{Re}\theta < 0 .$$

EXAMPLE 5. Let $\delta > 0$ and $\lambda > 0$. The function

$$u(x) = \exp\left[-\frac{\lambda}{1+\delta}(-\ln x)^{1+\delta}\right]$$

belongs to $\widetilde{\mathcal{O}}_1^L$ with

$$L = \left\{z \in \mathbb{C} : |\operatorname{Im} z| \le \cot\left(\frac{\pi}{2(1+\delta)}\right)\operatorname{Re} z\right\}$$

and solves the equation

$$\left((-\ln x)^{-\delta} x\frac{d}{dx} - \lambda\right)u = 0\,.$$

Proof. Indeed,

$$|u(x)| = \exp\left[-\frac{\lambda}{1+\delta}\operatorname{Re}(\exp(1+\delta)\ln(-\ln x))\right]$$
$$= \exp\left[-\frac{\lambda}{1+\delta}\exp((1+\delta)\ln\sqrt{\ln^2|x| + \arg^2 x}) \times \cos\left((1+\delta)\arctan\left(\frac{\arg x}{\ln|x|}\right)\right)\right]$$

and this is bounded by

$$CH_{-L_\varepsilon}(-\ln|x|, \arg x) \quad \text{for } x \in \Gamma^L_{1-\kappa} \text{ with some } C < \infty$$

if and only if

$$\Gamma_1^L \supset \left\{x \in \widetilde{B}(1) : \frac{|\arg x|}{-\ln|x|} < \tan\frac{\pi}{2(1+\delta)}\right\},$$
$$L \supset \left\{z \in \mathbb{C} : |\operatorname{Im} z| < \cot\left(\frac{\pi}{2(1+\delta)}\right)\operatorname{Re} z\right\}.$$

In order to describe multipliers in $\widetilde{\mathcal{O}}_t^L$ we introduce the space

$$\operatorname{Infr}(\Gamma_t^L) = \{f \in \mathcal{O}(\Gamma_t^L) : \text{for every } \varepsilon > 0 \text{ and } \kappa > 0 \text{ there exists } C = C(\varepsilon, \kappa) \text{ such that } |f(x)| \le C|x|^{-\varepsilon}\exp(\varepsilon|\arg x|) \text{ for } x \in \Gamma^L_{t-\kappa}\}.$$

PROPOSITION 6. *The multiplication operator*

$$\widetilde{\mathcal{O}}_t^L \ni u \to fu \in \widetilde{\mathcal{O}}_t^L$$

is continuous if and only if $f \in \operatorname{Infr}(\Gamma_t^L)$.

Proof. The sufficiency is obvious. To show the necessity take $f \notin \operatorname{Infr}(\Gamma_t^L)$. By rotation we can assume that either

$$L^{\text{ap}} = \left\{|\operatorname{Im} z| \le \cot\left(\frac{\pi}{2(1+\delta)}\right)\operatorname{Re} z\right\} \quad \text{with some } \delta > 0 \quad \text{or} \quad L^{\text{ap}} = \overline{\mathbb{R}}_+.$$

Put

$$a = \inf\{b \in \mathbb{R} : b + L^{\text{ap}} \subset L\},$$
$$u(x) = \begin{cases} x^a\exp(-(-\ln x)^{1+\delta}) & \text{in the first case,} \\ x^a & \text{in the second case.}\end{cases}$$

By the proof of Example 5 observe that $fu \notin \widetilde{\mathcal{O}}_t^L$.

LEMMA 9. *If* $f \in \operatorname{Infr}(\Gamma_t^L)$ *and* $R(y) = f(\exp(-y))$ *for* $y \in L^0 - \ln t$, *then* $R \in \operatorname{Infr}(L^0 - \ln t + \kappa)$ *for every* $\kappa > 0$. *Conversely, if*

$$R \in \bigcap_{\kappa>0}\operatorname{Infr}(L^0 - \ln t + \kappa), \quad f(x) = R(-\ln x) \quad \textit{for } x \in \Gamma_t^L,$$

then $f \in \operatorname{Infr}(\Gamma_t^L)$.

Proof. Indeed, if $f \in \operatorname{Infr}(\Gamma_t^L)$ and $y \in L^0 - \ln t + \kappa$, then $\exp(-y) \in \Gamma^L_{t-\kappa}$ and

$$|R(y)| \le C|\exp\varepsilon y|\exp|\arg e^{-y}|$$
$$\le C\exp[\varepsilon(\operatorname{Re} y + |\operatorname{Im} y|)] \le C\exp(\varepsilon|y|)\,.$$

Hence, $R \in \operatorname{Infr}(L^0 - \ln t + \kappa_1)$ with $\kappa_1 > \kappa$. On the other hand, if $R \in \operatorname{Infr}(L^0 - \ln t + \kappa)$, then

$$|f(x)| = |R(-\ln x)| \le C\exp(\varepsilon|-\ln x|) \le C|x|^{-\varepsilon}\exp(\varepsilon|\arg x|)$$

for $x \in \Gamma^L_{t-\kappa}$.

DEFINITION. Let $R \in \bigcap_{\kappa>0}\operatorname{Infr}(L^0 - \ln t + \kappa)$, $f(x) = R(-\ln x)$ for $x \in \Gamma_t^L$ and $S \in Q'(L,t)$. We define the action of the operator $R(D)$ on S by

$$R(D)S = \widetilde{b}(\mathcal{M}(fTS))\,.$$

By Lemma 9 and Proposition 6, $R(D)S$ is a well defined element of $Q'(L,t)$.

EXAMPLE 6. Let $R(y) = \exp ay$ for $y \in L^0 - \ln t$ with $a \in -(L^0)^0 = -L^{\text{ap}}$. Then by Example 3,

$$R \in \bigcap_{\kappa>0}\operatorname{Infr}(L^0 - \ln t + \kappa)$$

and for $S \in Q'(L,t)$ we have

$$\exp(aD)S = \widetilde{b}(\mathcal{M}(x^{-a}TS)) = S(\cdot - a) \in Q'(L,t)\,.$$

EXAMPLE 7. Let $f(x) = (-\ln x)^k$ for $x \in \Gamma_1^L$, $L \supset [0,\infty)$ and $k \in \mathbb{N}$. Then $f \in \operatorname{Infr}(\Gamma_1^L)$ and

$$D^kS = \widetilde{b}(\mathcal{M}((-\ln x)^kTS))\,.$$

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Weighted integrability and L^1-convergence of multiple trigonometric series

by

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Abstract. We prove that if $c_{jk} \to 0$ as $\max(|j|,|k|) \to \infty$, and

$$\sum_{|j|=0\pm}^{\infty} \sum_{|k|=0\pm}^{\infty} \theta(|j|^{\top})\vartheta(|k|^{\top})|\Delta_{12}c_{jk}| < \infty,$$

then $f(x,y)\phi(x)\psi(y) \in L^1(T^2)$ and $\iint_{T^2} |s_{mn}(x,y) - f(x,y)| \cdot |\phi(x)\psi(y)|\,dx\,dy \to 0$ as $\min(m,n) \to \infty$, where $f(x,y)$ is the limiting function of the rectangular partial sums $s_{mn}(x,y)$, (ϕ,θ) and (ψ,ϑ) are pairs of type I. A generalization of this result concerning L^1-convergence is also established. Extensions of these results to double series of orthogonal functions are also considered. These results can be extended to n-dimensional case. The aforementioned results generalize work of Balashov [1], Boas [2], Chen [3, 4, 5], Marzuq [9], Móricz [11], Móricz–Schipp–Wade [14], and Young [16].

1. Introduction. Let $T^2 = \{(x,y) \in \mathbb{R}^2 : -\pi \le x, y < \pi\}$. Consider the double trigonometric series

$$\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{jk} e^{i(jx+ky)}. \tag{1.1}$$

We assume that there are two positive, nondecreasing functions $\theta(t)$ and $\vartheta(t)$ defined on $[1,\infty)$ such that

$$c_{jk} \to 0 \quad \text{as } \max(|j|,|k|) \to \infty, \tag{1.2}$$

$$\sum_{|j|=0\pm}^{\infty} \sum_{|k|=0\pm}^{\infty} \theta(|j|^{\top})\vartheta(|k|^{\top})|\Delta_{12}c_{jk}| < \infty, \tag{1.3}$$

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