STUDIA MATHEMATICA 108 (2) (1994)

On weighted Bergman kernels of bounded domains

by

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Abstract. We build on work by Z. Pasternak-Winiarski [PW2], and study a-Bergman kernels of bounded domains $\Omega \subset \mathbb{C}^N$ for admissible weights $a \in L^1(\Omega)$.

1. Admissible weights and a-Bergman kernels. Let $\Omega \subseteq \mathbb{C}^N$ be an open subset, $\Omega \neq \emptyset$. Let $W(\Omega)$ be the set of all *weights* on Ω, i.e. an element $a \in W(\Omega)$ is a Lebesgue measurable function $a : \Omega \to \mathbb{R}$ so that $a \geq 0$ a.e. in Ω. Given $a \in W(\Omega)$ let $L^2(\Omega, a)$ denote the Hilbert space of complex functions on Ω for which $\int_\Omega |f|^2 a\, d\mu < \infty$, $d\mu$ denoting the Lebesgue measure on $\mathbb{R}^{2N}$. The inner product on $L^2(\Omega, a)$ is

$$\langle f, g\rangle_a = \int_\Omega \overline{f(z)} g(z) a(z)\, d\mu(z)$$

and as usual the norm is defined by $\|f\|_a = \langle f, f\rangle_a^{1/2}$.

Let $L^2H(\Omega, a)$ denote the set of functions in $L^2(\Omega, a)$ which are holomorphic in Ω. For $z \in \Omega$ fixed define $E_z(f) = f(z)$, for any $f \in L^2H(\Omega, a)$. Then $a \in W(\Omega)$ is an *admissible weight* if $L^2H(\Omega, a)$ is a closed subspace of $L^2(\Omega, a)$ and E_z is continuous on $L^2H(\Omega, a)$ for any $z \in \Omega$. Let $AW(\Omega)$ be the set of all admissible weights on Ω. If $a \in AW(\Omega)$ then, by the Riesz representation theorem, there is a unique $e_{z,a} \in L^2H(\Omega, a)$ so that $E_z(f) = \langle e_{z,a}, f\rangle_a$ for any $f \in L^2H(\Omega, a)$. The function $K_a : \Omega \times \Omega \to \mathbb{C}$ given by $K_a(z, w) = \overline{e_{z,a}(w)}$ is the *a-Bergman kernel* of Ω. For $a \equiv 1$ this is the Bergman kernel $K(z, w)$ of Ω (cf. [Be]). If Ω is bounded then K is known to give rise to a Kählerian metric g on Ω so that each holomorphic diffeomorphism is an isometry. Moreover, if Ω is homogeneous (i.e. the group of holomorphic diffeomorphisms of Ω acts transitively on Ω) then $K(z, z)$

1991 *Mathematics Subject Classification*: Primary 32H10; Secondary 32C17.
Key words and phrases: admissible weight, a-Bergman kernel, a-Bergman metric.
Research partially supported by G.N.S.A.G.A., Italy.

is proportional to the density $G(x,y)^{1/2}$ where $G = \det(g_{ij})$ (cf. Prop. 3.6 of [He], p. 371).

It is our purpose in the present paper to establish weighted analogues of the above results. If Ω is bounded and $a \in L^1(\Omega) \cap AW(\Omega)$ then

$$g_a = \operatorname{Re}\{H_a|_{\mathcal{X}(\Omega)\times\mathcal{X}(\Omega)}\} \tag{1}$$

is a Kählerian metric on Ω (cf. our Theorem 1) where H_a is given by

$$H_a = \sum_{1\le i,j\le N} \left(\frac{\partial}{\partial z_i}\frac{\partial}{\partial \overline{z}_j}\log K_a(z,z)\right) dz_i \otimes d\overline{z}_j \tag{2}$$

and $\mathcal{X}(\Omega)$ is the $C^\infty(\Omega)$-module of all real tangent vector fields on Ω. The proof of Theorem 1 relies on the representation of $K_a(z,w)$ in terms of a complete orthonormal system in $L^2H(\Omega,a)$ (cf. Th. 2.1 of [PW2], p. 3). Let $\Omega = \mathbb{D}^N$ be the unit polidisc and $a(z) = \exp(|z|^{-1/2})$, $z \neq 0$, $a(0) = 0$. Then $K_a(z,z)$ is shown to be proportional to the density $G_a(x,y)^{1/2}$ on any N-dimensional torus in Ω (cf. our Theorem 2). The lack of generality of Theorem 2 (as opposed to Prop. 3.6 of [He], p. 371) may be justified as follows. Let $\mathrm{Hol}(\Omega)$ be the group of all holomorphic diffeomorphisms of Ω. There is a natural action of $\mathrm{Hol}(\Omega)$ on $AW(\Omega)$ (cf. Section 3). Let I_a be the isotropy group of $a \in AW(\Omega)$. Then I_a acts on $L^2H(\Omega,a)$. However, I_a may be calculated only for specific choices of a. In particular, if a is the admissible weight in Theorem 2 then I_a acts transitively on any N-dimensional torus in the unit polidisc in $\mathbb{C}^N$. Finally, in Section 4 we mention an open problem in connection with work in [Ke].

2. The behaviour of a-Bergman kernels under holomorphic diffeomorphisms and the a-Bergman metrics. Let $a \in AW(\Omega)$ and let $K_a(z,w)$ be the a-Bergman kernel of Ω. For any complete orthonormal system $\{\phi_k\}$ in $L^2H(\Omega,a)$,

$$K_a(z,z) = \sum_k \phi_k(z)\overline{\phi_k(z)} \tag{3}$$

for any $z \in \Omega$ (cf. Th. 2.1(i) of [PW2], p. 3). The series $\sum_k \phi_k(z)\overline{\phi_k(w)}$ converges uniformly on any compact subset of $\Omega\times\Omega$ (cf. Prop. 2.1 of [PW2], p. 4) so that (3) may be differentiated term by term. We obtain

$$\frac{\partial}{\partial z_i}\frac{\partial}{\partial \overline{z}_j}\log K_a(z,z) \tag{4}$$
$$= K_a(z,z)^{-2}\sum_{k>l}\left\{\phi_k\frac{\partial\phi_l}{\partial z_i} - \phi_l\frac{\partial\phi_k}{\partial z_i}\right\}\overline{\left\{\phi_k\frac{\partial\phi_l}{\partial z_j} - \phi_l\frac{\partial\phi_k}{\partial z_j}\right\}}.$$

Let H_a be given by (2) and $Z, W \in \Gamma^\infty(T(\Omega)\otimes\mathbb{C})$ two complex vector fields on Ω. Then (4) yields $H_a(Z,\overline{Z}) \ge 0$ and $H_a(Z,W) = \overline{H_a(\overline{W},\overline{Z})}$, i.e.

g_a is positive and symmetric. To show that g_a is a Riemannian metric on Ω, it remains to be checked that g_a is definite. Let $X \in \mathcal{X}(\Omega)$ and assume that $g_a(X,X) = 0$ at $z \in \Omega$. If $X = \sum_{i=1}^N(\xi_i\partial/\partial z_i + \overline{\xi}_i\partial/\partial\overline{z}_i)$ then (4) gives

$$\sum_{i=1}^N\left(\phi_k\frac{\partial\phi_l}{\partial z_i} - \phi_l\frac{\partial\phi_k}{\partial z_i}\right)\xi_i = 0 \tag{5}$$

at $z \in \Omega$. If Ω is bounded there is $M > 0$ so that $|z_j| \le M$ (where $z_j : \Omega \to \mathbb{C}$, $1 \le j \le N$, are the coordinate functions). Thus $\int_\Omega |z_j|^2 a(z)\,d\mu(z) \le M^2\int_\Omega a(z)\,d\mu(z) < \infty$, provided that $a \in L^1(\Omega)$, so that $1, z_1, \ldots, z_N \in L^2H(\Omega,a)$. The rest of our argument reproduces that in [He], p. 368. Indeed, let

$$\widetilde{\phi}_{j+1} = z_{j+1} - \sum_{k=0}^j \langle z_{j+1},\phi_k\rangle_a\phi_k\,, \qquad \widetilde{\phi}_0 \equiv 1\,,$$

where $\phi_j \in L^2H(\Omega,a)$ is given by $\phi_j = \|\widetilde{\phi}_j\|_a^{-1}\widetilde{\phi}_j$, $0 \le j \le N$. Set $b_{ij} = \partial\phi_j/\partial z_i$, $1 \le i,j \le N$. Then $\det(b_{ij}) = b_{11}\ldots b_{NN} \neq 0$ together with (5) for $k = 0$ lead to $\xi_i = 0$ at $z \in \Omega$.

THEOREM 1. *Let $a \in AW(\Omega)$ and $\varphi \in \mathrm{Hol}(\Omega)$. If Ω is bounded and $a \in L^1(\Omega)$ then g_a is a Kählerian metric on Ω. Moreover, φ is an isometry of (Ω, g_a) into $(\Omega, g_{a\circ\varphi^{-1}})$, provided that $a' = a\circ\varphi^{-1} \in L^1(\Omega)$.*

Proof. Note that g_a has complex components $(g_a)_{ij} = (g_a)_{\overline{i}\overline{j}} = 0$, $(g_a)_{i\overline{j}} = \frac{1}{2}\partial^2\log K_a(z,z)/\partial z_i\partial\overline{z}_j$, as a consequence of (2). Thus (by Lemma 2.2 of [He], p. 360), g_a is Kählerian. We shall need the following:

LEMMA 1. *Let Ω, Ω' be domains in $\mathbb{C}^N$ and $\varphi : \Omega \to \Omega'$ a holomorphic diffeomorphism. Let $a \in AW(\Omega)$ and set $a' = a\circ\varphi^{-1}$. Then:*

(i) *$a' \in AW(\Omega')$.*

(ii) *The following identity holds:*

$$K_a(z,w) = K_{a'}(\varphi(z),\varphi(w))J_\varphi(z)\overline{J_\varphi(w)} \tag{6}$$

for any $z, w \in \Omega$.

Here, if $\varphi(z_1,\ldots,z_N) = (\zeta_1(z_1,\ldots,z_N),\ldots,\zeta_N(z_1,\ldots,z_N))$ then J_φ denotes the Jacobian determinant $J_\varphi = \partial(\zeta_1,\ldots,\zeta_N)/\partial(z_1,\ldots,z_N)$. To prove (i) let $Y \subset \Omega'$ be a compact subset and $w_0 \in Y$. Set $X = \varphi^{-1}(Y) \subset \Omega$ and $z_0 = \varphi^{-1}(w_0)$. Next, let $f \in L^2H(\Omega',a')$ and set $g = (f\circ\varphi)J_\varphi$. Then

$$\|g\|_a^2 = \int_\Omega |f(\varphi(z))|^2 a(z)|J_\varphi(z)|^2\,d\mu(z)$$
$$= \int_{\Omega'} |f(w)|^2 a'(w)\,d\mu(w) = \|f\|_{a'}^2 < \infty\,.$$

Also J_φ is holomorphic in Ω, so that $g \in L^2H(\Omega, a)$. By Theorem 2.2 of [PW2], p. 4, as a is admissible and X compact, there is $C_X > 0$ so that $|E_{z_0}g| \le C_X\|g\|_a$. Thus $|f(w_0)||J_\varphi(z_0)| \le C_X\|f\|_{a'}$, which yields

$$(7)\qquad |E_{w_0}f| \le C_Y\|f\|_{a'}$$

where $C_Y = C_X \sup_{z\in X}|J_\varphi(z)|^{-1}$. The estimate (7) holds for arbitrary $f \in L^2H(\Omega', a')$ so that, again by Theorem 2.2 of [PW2], it follows that a' is admissible.

Next, the identity $E_z((f\circ\varphi)J_\varphi) = E_{\varphi(z)}(f)J_\varphi(z)$ may be written as

$$(8)\qquad \int_{\Omega'} \{K_a(z, \varphi^{-1}(\zeta))(\overline{J_\varphi(\varphi^{-1}(\zeta))})^{-1} - K_{a'}(\varphi(z), \zeta)J_\varphi(z)\}f(\zeta)a'(\zeta)\,d\mu(\zeta) = 0\,.$$

Note that $\zeta \mapsto e_{z,a}(\varphi^{-1}(\zeta))(J_\varphi(\varphi^{-1}(\zeta)))^{-1}$ is holomorphic in Ω' and

$$\|(e_{z,a}\circ\varphi^{-1})(J_\varphi\circ\varphi^{-1})^{-1}\|^2_{a'} = \int_{\Omega'} |e_{z,a}(\varphi^{-1}(\zeta))|^2|J_\varphi(\varphi^{-1}(\zeta))|^{-2}a'(\zeta)\,d\mu(\zeta)$$
$$= \int_{\Omega'} |e_{z,a}(\varphi^{-1}(\zeta))|^2|J_{\varphi^{-1}}(\zeta)|^2a'(\zeta)\,d\mu(\zeta) = \|e_{z,a}\|^2_a < \infty$$

so that $(e_{z,a}\circ\varphi^{-1})(J_\varphi\circ\varphi^{-1})^{-1} \in L^2H(\Omega', a')$ and (8) becomes

$$\langle (e_{z,a}\circ\varphi^{-1})(J_\varphi\circ\varphi^{-1})^{-1} - \overline{J_\varphi(z)}e_{\varphi(z),a'}, f\rangle_{a'} = 0$$

for any $f \in L^2H(\Omega', a')$. This yields (6). ∎

In particular, $K_a(z,z) = K_{a\circ\varphi^{-1}}(\varphi(z), \varphi(z))|J_\varphi(z)|^2$ for any $z \in \Omega$, so that the proof of the second statement in Theorem 1 is similar to that of Proposition 3.5 of [He], p. 370 (and is therefore left as an exercise to the reader). Cf. also [M2]. The Kählerian metric g_a is the *a-Bergman metric* of Ω. It is an open problem to study curvature properties of (Ω, g_a) (cf. e.g. [Kl], when $a \equiv 1$).

As a byproduct of our Lemma 1, if $\Omega = \mathbb{D}^1$ is the unit disc in $\mathbb{C}$, one may estimate $K_a(z,w)$ in terms of the unweighted Bergman kernel of Ω, i.e. $K_1(z,w) = \pi^{-1}(1-z\overline{w})^{-2}$ (cf. e.g. [Ho], p. 147). Let $z \in \Omega$ be fixed and φ_z the automorphism which takes z to 0, i.e. $\varphi_z(w) = (z-w)(1-w\overline{z})^{-1}$. Since $\varphi_z'(w) = (|z|^2-1)(1-w\overline{z})^{-2}$ the identity (6) becomes

$$K_a(z,w) = K_{a\circ\varphi_z^{-1}}(0, \varphi_z(w))\frac{1}{(1-\overline{w}z)^2}\,.$$

Let $X \subset \Omega$ be a compact subset and $Y_z = \varphi_z(X)$. Then

$$|K_a(z,w)| \le C_z|K_1(z,w)|$$

for any $|z| < 1$, $w \in X$, where $C_z = \pi\sup_{w\in Y_z}|K_a(0,w)| < \infty$.

3. Isotropy groups of the natural action of $\mathrm{Hol}(\Omega)$ on $AW(\Omega)$. Let $H(\Omega)$ be the space of all holomorphic functions on Ω. Consider the actions $\alpha : H(\Omega)\times\mathrm{Hol}(\Omega) \to H(\Omega)$, $\alpha(f,\varphi) = (f\circ\varphi)J_\varphi$, and $\beta : \mathrm{Hol}(\Omega)\times AW(\Omega) \to AW(\Omega)$, $\beta(\varphi, a) = a\circ\varphi^{-1}$. Then β is well defined, by Lemma 1. Let I_a be the isotropy group of $a \in AW(\Omega)$ with respect to β. Then α induces an action $H(\Omega)\times I_a \to H(\Omega)$ for each $a \in AW(\Omega)$. This descends to an action $L^2H(\Omega,a)\times I_a \to L^2H(\Omega,a)$ since $\|\alpha(f,\varphi)\|_a = \|f\|_{\beta(\varphi,a)} = \|f\|_a < \infty$, for any $f \in L^2H(\Omega,a)$, $\varphi \in I_a$. Set $\mathbb{D}^N = \{z \in \mathbb{C}^N : |z_j| < 1, 1 \le j \le N\}$ and define $a \in W(\mathbb{D}^N)$ by

$$a(z) = \begin{cases} \exp(|z|^{-1/2}), & z \ne 0,\\ 0 & z = 0,\end{cases}$$

where $|z|^2 = |z_1|^2 + \ldots + |z_N|^2$, $z \in \mathbb{D}^N$.

THEOREM 2. (i) $a \in AW(\mathbb{D}^N)$.

(ii) *Let $0 < r < 1$ and consider the torus $T^N(r) = S^1(r)\times\ldots\times S^1(r)$ (N factors). Then there is $C_a > 0$ so that $K_a(z,z) = C_aG_a(x,y)^{1/2}$ for any $z \in T^N(r)$.*

Proof. (i) follows from Corollary 3.1 of [PW2], p. 6. Let $\Omega = \mathbb{D}^N$. To prove (ii) recall (cf. e.g. [N], p. 68) that $\mathrm{Hol}(\Omega) = \{\varphi_{\theta,\alpha,p} : \theta \in \mathbb{R}^N, \alpha \in \mathbb{C}^N, |\alpha_j| < 1, 1 \le j \le N, p \in \sigma_N\}$ where

$$\varphi_{\theta,\alpha,p}(z) = \left(e^{i\theta_1}\frac{z_{p(1)}-\alpha_1}{1-\overline{\alpha}_1z_{p(1)}}, \ldots, e^{i\theta_N}\frac{z_{p(N)}-\alpha_N}{1-\overline{\alpha}_Nz_{p(N)}}\right), \qquad z \in \Omega\,,$$

and σ_N denotes the permutation group of order $N!$.

Step 1. $I_a = \{\varphi_{\theta,0,p} : \theta \in \mathbb{R}^N, p \in \sigma_N\}$.

Proof. We must solve

$$(9)\qquad a\circ\varphi_{\theta,\alpha,p} = a$$

for θ, α and p. Apply (9) to $z = 0$. This gives $\alpha = 0$. On the other hand, let $z \in \Omega$, $z \ne 0$. Then $|\varphi_{\theta,0,p}(z)|^2 = \sum_{j=1}^N|e^{i\theta_j}z_{p(j)}|^2 = |z|^2$, i.e. $\varphi_{\theta,0,p}$ satisfies the functional equation (9).

Step 2. *I_a acts transitively on $T^N(r)$.*

Proof. Let $I_a\times T^N(r) \to T^N(r)$, $(\varphi, z) \mapsto \varphi(z)$. Then $|\varphi(z)_j| = |e^{i\theta_j}z_{p(j)}| = r$ so that the action is well defined. Next, for any $z, w \in T^N(r)$ the equation $\varphi_{\theta,0,p}(z) = w$ may be solved for $\theta \in \mathbb{R}^N$, $p \in \sigma_N$ (e.g. take $p = \mathrm{id}$ and $\theta_j \in \{2n\pi + \arg(w_j/z_j) : n \in \mathbb{Z}\}$).

Set $z_j = x_j + iy_j$, $1 \le j \le N$, and let $(g_a)_{AB}$ be the real components of g_a, $1 \le A, B \le 2N$ (with respect to $\partial/\partial x_j$, $\partial/\partial y_j$). Let $G_a(x,y) = \det((g_a)_{AB}(x,y))$. We finish the proof of (ii) in Theorem 2 by showing:

Step 3. $K_a(z,z)/G_a(x,y)^{1/2} = \mathrm{const.}$ on $T^N(r)$.

Proof. Let $z, w \in T^N(r)$. By Step 2 there is $\varphi \in I_a$ so that $\varphi(z) = w$. Then (6) yields $K_a(z,z) = K_a(w,w)|J_\varphi(z)|^2$. Set $w_j = u_j + iv_j$. Finally, $G_a(x,y) = G_a(u,v)|J_\varphi(z)|^4$ so that

$$K_a(z,z)G_a(x,y)^{-1/2} = K_a(w,w)G_a(u,v)^{-1/2}. \blacksquare$$

We end Section 3 by looking at yet another example. Let $\Omega = \mathbb{D}^1$ be the unit disc in $\mathbb{C}$ and $a \in W(\mathbb{D}^1)$ given by $a(z) = |\mathrm{Im}(z)|^{1/(1-|z|)}$, $|z| < 1$.

PROPOSITION 1. (i) $a \in AW(\mathbb{D}^1)$.

(ii) I_a *is a subgroup of* $G_a = \{\psi_{k,\alpha} : k \in \mathbb{Z},\ \alpha \in \mathbb{R},\ |\alpha| < 1\}$, *where*

$$\psi_{k,\alpha}(z) = (-1)^k \frac{z-\alpha}{1-\alpha z}, \qquad |z| < 1.$$

Proof. Note that

$$\psi_{k,\alpha} \circ \psi_{l,\beta} = \psi_{k+l,\gamma}, \quad \text{where} \quad \gamma = ((-1)^l\alpha + \beta)(1 + (-1)^l\alpha\beta)^{-1},$$

and $(\psi_{k,\alpha})^{-1} = \psi_{k,\beta}$, where $\beta = (-1)^{k+1}\alpha$, so that G_a is a group. To prove $I_a \subset G_a$ recall that $\mathrm{Hol}(\mathbb{D}^1) = \{\varphi_{\theta,\alpha} : \theta \in \mathbb{R},\ \alpha \in \mathbb{C},\ |\alpha| < 1\}$, where

$$\varphi_{\theta,\alpha}(z) = e^{i\theta}\frac{z-\alpha}{1-\overline{\alpha}z}.$$

If $\varphi_{\theta,\alpha} \in I_a$ then

$$a(\varphi_{\theta,\alpha}(z)) = a(z) \tag{10}$$

for any $|z| < 1$. Let $z = 0$ in (10). This gives $\mathrm{Re}(\alpha)\sin\theta + \mathrm{Im}(\alpha)\cos\theta = 0$ and we distinguish the following cases: 1) $\cos\theta \neq 0$, $\mathrm{Re}(\alpha) \neq 0$, and then $\tan\theta = -\mathrm{Im}(\alpha)/\mathrm{Re}(\alpha)$, or 2) $\cos\theta \neq 0$, $\mathrm{Re}(\alpha) = 0$, and then $\alpha = 0$, or 3) $\cos\theta = 0$, $\mathrm{Re}(\alpha) = 0$. Clearly the case $\cos\theta = 0$, $\mathrm{Re}(\alpha) \neq 0$ cannot occur. Let $z = \alpha$ in (10). This yields $\alpha \in \mathbb{R}$. Now, according to cases 1) to 3) above, we obtain the following sets of holomorphic diffeomorphisms: 1) $\psi_{k,\alpha}(z) = \varphi_{k\pi,\alpha}(z) = (-1)^k\frac{z-\alpha}{1-\alpha z}$, $\alpha \in \mathbb{R}$, $|\alpha| < 1, k \in \mathbb{Z}$, 2) $\varphi_{\theta,0}(z) = \exp(i\theta)z$, and 3) $\psi_k(z) = \varphi_{k\pi+\pi/2,0}(z) = i(-1)^k z$, $k \in \mathbb{Z}$. Note that $a(\varphi_{\theta,0}(1/2)) = 0$ yields $\theta \in \{k\pi : k \in \mathbb{Z}\}$ so that 2) is contained in 1) for $\alpha = 0$. Finally, $a(\psi_k(1/2)) \neq 0$ so that $\psi_k \notin I_a$.

4. Derivatives of a-Bergman kernels and an open problem. Let Δ be the Laplace operator on $\mathbb{R}^{2N} \approx \mathbb{C}^N$, $N \geq 2$, and

$$\Gamma_w(z) = \frac{|z-w|^{2(1-N)}}{2(1-N)\omega_{2N}}, \quad \text{where} \quad \omega_{2N} = 2\pi^N\Gamma(N)^{-1}.$$

Let $\Omega \subset \mathbb{C}^N$ be a bounded domain, $a = e^{-\phi}$, and assume $\phi \in C^0(\overline{\Omega})$ throughout Section 4. Then $a \in AW(\Omega)$ (cf. [Ho], p. 145). Let $P_\phi : L^2(\Omega, a) \to L^2H(\Omega, a)$ be the orthogonal projection (with respect to $\langle\ ,\ \rangle_a$). For $w \in \Omega$ fixed there exist (cf. [Ke], p. 156) open sets U_j, $1 \leq j \leq 4$, so that $w \in U_4$ and $\overline{U}_{j+1} \subset U_j$, $0 \leq j \leq 3$ (here $U_0 = \Omega$), and real-valued functions φ_w, ψ_w with the properties $\varphi_w \in C_0^\infty(U_2)$, $\varphi_w|_{U_3} \equiv 1$, $\psi_w \in C_0^\infty(U_1)$, $\psi_w|_{U_4} \equiv 0$ and $\psi_w|_{U_2-U_3} \equiv 1$. Let $K_a(z,w)$ be the a-Bergman kernel of Ω. We seek for a weighted analogue of Lemma 1 of [Ke], p. 152. We set ourselves under the hypothesis of Theorem 3.5.1 of [Ho], p. 145, i.e. we assume that the weak maximal operator $\overline{\partial} : L^2(\Omega, a) \to L^2_{(0,1)}(\Omega, a)$ has a closed range, and that $\partial\Omega$ is of class C^2 and strictly pseudoconvex (i.e. the Levi form $\sum \varrho_{jk}(z)t_j\overline{t}_k$ is positive definite in the plane $\sum \varrho_j(z)t_j = 0$, for any $z \in \partial\Omega$, cf. conventions and notations of [Ho], p. 127). We establish the following:

THEOREM 3. (i) $D_{\overline{w}}^\beta K_a(\cdot, w) \in L^2H(\Omega, a)$ *provided that* $|\varrho|^{-2(N+1)}a \in L^1(\Omega)$.

(ii) $D_{\overline{w}}^\beta K_a(\cdot, w) = (-1)^{|\beta|}P_\phi e^\phi D_{\overline{z}}^\beta(\varphi_w \Delta(\psi_w \Gamma_w))$, *i.e.* $D_{\overline{w}}^\beta K_a(z,w)$ *can be represented as an orthogonal projection.*

Here $D_{\overline{z}}^\beta = \partial^{|\beta|}/\partial\overline{z}_1^{\beta_1}\ldots\partial\overline{z}_N^{\beta_N}$. To prove (i) fix $w \in \Omega$ and a polyradius $r = (r_1, \ldots, r_N)$ so that $\overline{D}_r(w) \subset \Omega$, where $\overline{D}_r(w) = \{\zeta \in \mathbb{C}^N : |\zeta_j - w_j| \leq r_j,\ 1 \leq j \leq N\}$. As $K_a(\cdot, z) \in H(\Omega)$ for any $z \in \Omega$, we may use the Cauchy integral formula and Theorem 2.1(ii) of [PW2], p. 3, to obtain

$$D_{\overline{w}}^\beta K_a(z,w) = (-1)^N\beta!\left(\frac{1}{2\pi i}\right)^N \int\limits_{\partial_0 D} \frac{K_a(z,\zeta)}{(\overline{\zeta}-\overline{w})^{\beta+1}}\,d\mu(\zeta) \tag{11}$$

where $D = D_r(w)$ and $\partial_0 D$ denotes its distinguished boundary. Clearly $D_{\overline{w}}^\beta K_a(\cdot, w) \in H(\Omega)$. On the other hand, set $\zeta - w = (r_1e^{i\theta_1}, \ldots, r_Ne^{i\theta_N})$ and use $|K_a(\zeta, z)| \leq C_X\|e_{z,a}\|_a$ (for some $C_X > 0$ depending only on the compact set $X = \partial_0 D \subset \Omega$, and any $\zeta \in X$) to obtain

$$\Bigg|\int\limits_{\partial_0 D} \frac{K_a(\zeta,z)}{(\zeta-w)^{\beta+1}}\,d\mu(\zeta)\Bigg| \leq C_X\|e_{z,a}\|_a\frac{(2\pi)^N}{r^\beta}. \tag{12}$$

Next (11)–(12) yield the estimate

$$\|D_{\overline{w}}^\beta K_a(\cdot, w)\|_a^2 \leq C_X^2 r^{-2\beta}\int\limits_\Omega \|e_{z,a}\|_a^2 a(z)\,d\mu(z). \tag{13}$$

We need to estimate the integral $\iint_{\Omega\times\Omega}|K_a(z,w)|^2 a(z)a(w)\,dz\,dw$. Define $F_a : \overline{\Omega} \to \mathbb{R}$ by

$$F_a(z) = \begin{cases} |\varrho(z)|^{N+1}K_a(z,z), & z \in \Omega, \\ \lambda(z)e^{\phi(z)}N!/(4\pi^N), & z \in \partial\Omega, \end{cases}$$

where $\lambda(z)$ is the product of the $N-1$ eigenvalues of the Levi form of $\partial\Omega$ in the plane $\sum \varrho_j t_j = 0$. By Theorem 3.5.1 of [Ho], p. 145, F_a is continuous. Finally, by Theorem 2.1(i) of [PW2], p. 3, $|K_a(z,w)| \leq K_a(z,z) + K_a(w,w)$

so that we may perform the following estimates:

$$\iint_{\Omega\times\Omega} |K_a(z,w)|^2 a(z)a(w)\,dz\,dw$$

$$\le 2A\int_\Omega K_a(z,z)^2 a(z)\,d\mu(z) + 2\Big(\int_\Omega K_a(z,z)a(z)\,d\mu(z)\Big)^2$$

$$= 2A\int_\Omega F_a(z)^2|\varrho(z)|^{-2(N+1)}a(z)\,d\mu(z)$$

$$+\,2\Big(\int_\Omega F_a(z)|\varrho(z)|^{-(N+1)}a(z)\,d\mu(z)\Big)^2$$

$$\le 2(\sup_{\overline{\Omega}} F_a)^2\Big\{A\int_\Omega |\varrho|^{-2(N+1)}a\,d\mu + \Big(\int_\Omega |\varrho|^{-(N+1)}a\,d\mu\Big)^2\Big\} < \infty$$

where $A = \int_\Omega a\,d\mu < \infty$, so that $D_{\overline{w}}^\beta K_a(\cdot,w) \in L^2(\Omega,a)$, and (i) of Theorem 3 is completely proved. It is easy to see that $|\varrho|^{-2(N+1)}a \in L^1(\Omega)$ yields $|\varrho|^{-(N+1)}a \in L^1(\Omega)$ as well (e.g. let $0<\varepsilon<1$ and $\overline{\Omega}_\varepsilon = \{z\in\Omega : \varrho(z)\ge\varepsilon\}$ and note that $\int_\Omega |\varrho|^{-(N+1)}a\,d\mu \le \int_{\overline{\Omega}_\varepsilon}|\varrho|^{-(N+1)}a\,d\mu + \int_{\Omega-\overline{\Omega}_\varepsilon}|\varrho|^{-2(N+1)}a\,d\mu < \infty$).

The proof of the second statement in Theorem 3 is similar to that of Lemma 1 of [Ke], p. 152, so that we allow ourselves to be somewhat sketchy. Let $g\in H(\Omega)$. Using (1.18) of [J], p. 97, and $\psi_w\Gamma_w \in C_0^\infty(\Omega)$ (to integrate by parts) we have

$$g(w) = \int_\Omega \psi_w(z)\Gamma_w(z)\Delta(\varphi_w g)(z)\,d\mu(z)$$

$$= \int_\Omega g(z)\varphi_w(z)\Delta(\psi_w\Gamma_w)(z)\,d\mu(z).$$

Let $g = D^\beta f$, $f\in L^2H(\Omega,e^{-\phi})$, and integrate again by parts to get

$$D^\beta f(w) = (-1)^{|\beta|}\langle P_\phi e^\phi D_{\overline{z}}^\beta(\varphi_w\Delta(\psi_w\Gamma_w)), f\rangle_a. \tag{14}$$

Next apply D_w^β to $f(w) = \int_\Omega K_a(w,z)f(z)a(z)\,d\mu(z)$ to obtain

$$D^\beta f(w) = \langle \overline{D_{\overline{w}}^\beta K_a(w,\cdot)}, f\rangle_a. \tag{15}$$

Finally, (14)–(15) together with Theorem 3(i) yield (ii). ∎

It is an open problem to prove differentiability up to the boundary of the a-Bergman kernel of Ω; cf. Theorem 1 of [Ke], p. 151, where $\phi\equiv 0$. There, essential use is made of a formula for $P_0 : L^2(\Omega,1)\to L^2H(\Omega,1)$ in terms of the Neumann operator (cf. the solution of the $\overline{\partial}$-Neumann problem, [Ko], p. 140) and the Sobolev lemma (a weighted version of which is already known, cf. [Ku]).

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Received October 8, 1992
Revised version July 4, 1993 (3006)