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Convolution algebras with weighted rearrangement-invariant norm

by

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Abstract. Let X be a rearrangement-invariant space of Lebesgue-measurable functions on $\mathbb{R}^n$, such as the classical Lebesgue, Lorentz or Orlicz spaces. Given a nonnegative, measurable (weight) function on $\mathbb{R}^n$, define $X(w) = \{F : \mathbb{R}^n \to \mathbb{C} : \infty > \|F\|_{X(w)} := \|Fw\|_X\}$. We investigate conditions on such a weight w that guarantee $X(w)$ is an algebra under the convolution product $F*G$ defined at $x \in \mathbb{R}^n$ by $(F*G)(x) = \int_{\mathbb{R}^n} F(x-y)G(y)\,dy$; more precisely, when $\|F * G\|_{X(w)} \le \|F\|_{X(w)}\|G\|_{X(w)}$ for all $F, G \in X(w)$.

1. Introduction. A *weight function* on $\mathbb{R}^n$ is a Lebesgue-measurable function w for which $0 < w < \infty$ a.e. with respect to Lebesgue measure. Given $1 \le p \le \infty$, define

$$L^p(w) = \Big\{F : \mathbb{R}^n \to \mathbb{C} : \infty > \|F\|_{L^p(w)} = \Big[\int_{\mathbb{R}^n} |F(x)w(x)|^p\,dx\Big]^{1/p}\Big\}.$$

When $w \equiv 1$ we use the abbreviated notations L^p and $\| \ \|_p$. As usual, $p' = p/(p-1)$.

This paper was motivated by the problem of determining when $L^p(w)$ is an algebra under the convolution product $F * G$ defined at $x \in \mathbb{R}^n$ by

$$(F * G)(x) = \int_{\mathbb{R}^n} F(x-y)G(y)\,dy\,;$$

more precisely, when

$$\|F * G\|_{L^p(w)} \le \|F\|_{L^p(w)}\|G\|_{L^p(w)} \qquad \text{for } F, G \in L^p(w)\,. \tag{1}$$

The problem was solved in the case $p = 1$ by Beurling [2] who showed (1) holds if and only if

$$w(x+y) \le w(x)w(y) \qquad \text{for } x, y \in \mathbb{R}^n\,, \tag{2}$$

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or, equivalently, setting $w(x) = e^{\Phi(x)}$,

$$\Phi(x+y) \le \Phi(x) + \Phi(y) \quad \text{for } x, y \in \mathbb{R}^n . \tag{3}$$

We observe that a natural class of weights for which (2) holds is the class $\mathcal{C}$ consisting of those $w = e^{\Phi}$, where $\Phi(x) = \Phi(|x|)$ is radial and, considered as a function on $\mathbb{R}_+ = (0,\infty)$, Φ is increasing and concave with $\Phi(0+) = 0$. Examples of such weights are $(1+|x|)^\alpha$, $\alpha \ge 0$, and $e^{|x|^\beta}$, $0 \le \beta \le 1$. Here $|x|$ can be any norm on $\mathbb{R}^n$. However, for $n \ge 2$, the methods used below require the norm $|x| = |x_1| + \ldots + |x_n|$ for $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$, which we adopt from now on. Given $x_0 \in \mathbb{R}^n$, $r > 0$, we denote by $B_r(x_0)$ the set $\{x \in \mathbb{R}^n : |x - x_0| < r\}$.

Another case readily dealt with is $p = \infty$. The weights w satisfying (1) are those for which $w(w^{-1} * w^{-1}) \le 1$; that is,

$$\int_{\mathbb{R}^n} \frac{w(x)}{w(x-y)w(y)}\, dy \le 1 \quad \text{for } x \in \mathbb{R}^n .$$

This, together with (2) written in the form

$$\frac{w(x)}{w(x-y)w(y)} \le 1 \quad \text{for } x \in \mathbb{R}^n ,$$

suggests, for $1 < p < \infty$, the condition

$$\left[\int_{\mathbb{R}^n} \left(\frac{w(x)}{w(x-y)w(y)}\right)^{p'} dy\right]^{1/p'} \le 1 \quad \text{for } x \in \mathbb{R}^n . \tag{4}$$

Nikol'skiĭ [12] showed (4) is sufficient for (1) in the context of sequence spaces. See also Grabiner [7]. The short proof, which it will be convenient for us to reproduce here, is a clever application of Hölder's inequality. Observe first that, writing $F = f/w$, $G = g/w$, (1) becomes

$$\left\| w\left(\frac{f}{w} * \frac{g}{w}\right)\right\|_p \le \|f\|_p \|g\|_p .$$

Now,

$$\begin{aligned} &\left[\int_{\mathbb{R}^n} \left| w\left(\frac{f}{w} * \frac{g}{w}\right)\right|^p dx\right]^{1/p} \quad (5)\\ &\quad \le \left[\int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \frac{w(x)}{w(x-y)w(y)} |f(x-y)||g(y)|\, dy\right]^p dx\right]^{1/p} \end{aligned}$$

$$\leq \left[\int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n}\left(\frac{w(x)}{w(x-y)w(y)}\right)^{p'}dy\right]^{p-1}\left[\int_{\mathbb{R}^n}|f(x-y)g(y)|^p\,dy\right]dx\right]^{1/p}$$

$$\leq \operatorname*{ess\,sup}_{x\in\mathbb{R}^n}\left[\int_{\mathbb{R}^n}\left(\frac{w(x)}{w(x-y)w(y)}\right)^{p'}dy\right]^{1/p'}\|\,\|f(x-y)g(y)\|_{L^p(dy)}\|_{L^p(dx)}\,.$$

The first factor in the last line of (5) is, by (4), at most 1, while Fubini's theorem can be applied to the second factor to yield

$$\|\,\|f(x-y)g(y)\|_{L^p(dy)}\|_{L^p(dx)} = \|f\|_p\|g\|_p\,. \tag{6}$$

This proves (1).

Condition (4) is not, in general, necessary for (1). But, as we will show in Section 4, it is if w is in a certain class containing $\mathcal{C}$ (cf. [10] for a result similar to this in the case $n = 1$).

The main purpose of this paper is to investigate when

$$X(w) = \{F : \mathbb{R}^n \to \mathbb{C} : \infty > \|F\|_{X(w)} = \|Fw\|_X\}$$

is closed under convolution, where X is a rearrangement-invariant (r.i.) space of functions on $\mathbb{R}^n$ with Köthe dual X'. See Section 2 for definitions and some properties of such spaces. For more background we recommend [1].

Inequalities (5) with X and X' in place of L^p and $L^{p'}$, respectively, suggest the condition

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^n}\left\|\frac{w(x)}{w(x-y)w(y)}\right\|_{X'(dy)} \leq C \tag{7}$$

is sufficient for

$$\|F*G\|_{X(w)} \leq C\|F\|_{X(w)}\|G\|_{X(w)}\,, \tag{8}$$

which would certainly, by Nikol'skiĭ's argument (5), be true if the following (weaker) analogue of (6) held:

$$\|\,\|f(x-y)g(y)\|_{X(dy)}\|_{X(dx)} \leq C\|f\|_X\|g\|_X\,. \tag{9}$$

(We would like to point out that (8) is equivalent to $X(w)$ being closed under convolution, see [8], p. 471, and that one can take $C = 1$ if w is replaced by w/C.) Now, on the one hand, (7) is no longer sufficient for (8); in particular, as shown in Section 5, (7) guarantees (8) for the Lorentz space $X = L^{pq}(\mathbb{R})$, $q \geq p$, if and only if $p = q$. On the other hand, as shown in Section 2, (9) does hold for nonnegative f and g in the class $\mathcal{R.D.}$ of radially decreasing functions; that is, $f(x) = f(|x|)$ and $g(x) = g(|x|)$ are decreasing functions of $|x|$. This, then, raises the question of characterizing those weights for which it is enough to test (8) for nonnegative functions in $\mathcal{R.D.}$

To this end, we introduce the class $\mathcal{M}$ of weights $w(x) = w(|x|)$ for which $w(y) \le Cw(z)$, $0 < y < z$, and

$$B(r,s) = \frac{w(r+s)}{w(r)w(s)} \qquad (r,s > 0)$$

is essentially decreasing in each variable separately, i.e. $B(r_1,s) \le CB(r_2,s)$, whenever $s > 0$ and $r_1 \ge r_2 > 0$. (This class contains $\mathcal{C}$, since for $w = e^{\Phi}$, Φ concave on $\mathbb{R}_+$, $\partial B/\partial r = (\Phi'(r+s) - \Phi'(r))B(r,s) \le 0$.) We prove that given $w \in \mathcal{M}$ there holds the following weighted analogue of an inequality of F. Riesz [13] and S. L. Sobolev [14]:

$$(10) \qquad \int_{\mathbb{R}^n} \left(\frac{f}{w} * \frac{g}{w}\right) hw \le C \int_{\mathbb{R}^n} \left(\frac{f^+}{w} * \frac{g^+}{w}\right) h^{++} w \qquad \text{for } f,g,h \ge 0\,.$$

Here, for example, h^+ is the (a.e.) unique nonnegative function in $\mathcal{R}.\mathcal{D}.$ on $\mathbb{R}^n$ satisfying

$$|\{x \in \mathbb{R}^n : h^+(|x|) > \lambda\}| = |\{x \in \mathbb{R}^n : |h(x)| > \lambda\}|$$

for all $\lambda > 0$, and h^{++} is the (larger) nonnegative $\mathcal{R}.\mathcal{D}.$ function on $\mathbb{R}^n$ given by

$$h^{++}(x) = h^{++}(|x|) = (C_n|x|)^{-n} \int_{|y| \le |x|} h^+(|y|)\, dy\,,$$

where $C_n^n = |B_1(0)|$.

In sum, we are able to prove the following

THEOREM 1. *Let $w \in \mathcal{M}$ and suppose X is an r.i. space of functions on $\mathbb{R}^n$ for which the mapping $f \to f^{++}$ is bounded on X'. Then a necessary and sufficient condition for $X(w)$ to be closed under convolution is*

$$(11) \qquad \left\| \frac{w(x)}{w(x-y)w(y)} \right\|_{X'(dy)} \le C \qquad \textit{for } x \in \mathbb{R}^n\,.$$

The requirement that $f \to f^{++}$ be bounded on X' eliminates those r.i. spaces X near L^∞ (see Lemma 6 below). To include such spaces in our theory requires a stronger weighted analogue of the Riesz inequality, namely (10) with h^+ in place of h^{++}; that is,

$$(12) \qquad \int_{\mathbb{R}^n} \left(\frac{f}{w} * \frac{g}{w}\right) hw \le C \int_{\mathbb{R}^n} \left(\frac{f^+}{w} * \frac{g^+}{w}\right) h^{+} w \qquad \text{for } f,g,h \ge 0\,.$$

We show that for $w \in \mathcal{M}_\infty$, where

$$\mathcal{M}_\infty = \left\{ w \in \mathcal{M} : \frac{w(x+y)}{w(x)w(y)} \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n) \right\},$$

(12) holds if and only if w satisfies the additional condition

$$\int_0^{r/2} \frac{w(r)}{w(r-s)w(s)} s^{n-1}\, ds \le C \int_0^r \frac{s^{n-1}}{w(s)^2}\, ds \qquad \text{for } r > 0\,. \tag{13}$$

(We note in passing that $w(x) = e^{|x|^\alpha}$, $x \in \mathbb{R}^n$, belongs to $\mathcal{C} \subset \mathcal{M}_\infty$ for $0 \le \alpha \le 1$, but satisfies (13) if and only if $\alpha = 1$. Indeed, if $\alpha = 1$ the left side is $\approx r^n$, while the right side is $O(1)$.) We can now obtain the following result having no restriction on X.

THEOREM 2. *Let $w(x) = w(|x|)$ belong to $\mathcal{M}_\infty$ and satisfy* (13). *Suppose X is an r.i. space of functions on $\mathbb{R}^n$. Then* (11) *is a necessary and sufficient condition for $X(w)$ to be closed under convolution.*

The sufficiency of (11) is related to (10) and (12) in Section 2 and proofs of the latter are given in the following section. The necessity of (11) is the subject of Section 4 and, as mentioned above, the question of the general sufficiency of (11) is considered for the L^{pq} spaces, $q \ge p$ (when $n = 1$), in Section 5.

The referee has pointed out that it should be possible to extend some of our results to the setting of locally compact Abelian groups.

2. Rearrangement-invariant function spaces. Let (Ω, Σ, μ) be a complete σ-finite measure space. A *Banach lattice* $X = X(\Omega)$ is a Banach space of (equivalence classes of μ-a.e. equal) complex-valued measurable functions on Ω such that if $|g| \le |f|$ μ-a.e., where $f \in X$ and g is measurable, then $g \in X$ and $\|g\|_X \le \|f\|_X$. If, in addition, X has the *Fatou property*:

$$0 \le f_n \uparrow f \ \mu\text{-a.e.}, \ \sup_n \|f_n\|_X < \infty \ \Rightarrow \ f \in X \text{ and } \|f\|_X = \lim_{n\to\infty} \|f_n\|_X$$

together with the property that whenever $E \in \Sigma$ with $\mu(E) < \infty$ we have $\chi_E \in X$ and $\int_\Omega |f|\chi_E\, d\mu < \infty$ for all $f \in X$, then X is said to be a *Banach function space*. Such a space is a *saturated* Banach lattice in the sense that every $E \in \Sigma$ with $\mu(E) > 0$ has a measurable subset F of finite positive measure for which $\chi_F \in X$.

The Banach function space $X = X(\Omega)$ is called a *rearrangement-invariant function space* (r.i. space) if $f \in X$ implies $g \in X$ and $\|g\|_X = \|f\|_X$, whenever g is equimeasurable with f, that is,

$$\begin{aligned} \mu_f(t) &:= \mu(\{x \in \Omega : |f(x)| > t\}) \\ &= \mu(\{x \in \Omega : |g(x)| > t\}) =: \mu_g(t) \qquad \text{for } t > 0\,. \end{aligned}$$

Important examples of r.i. spaces are the Lorentz spaces $L^{pq}(\Omega)$, $1 < p < \infty$,

$1 \le q \le \infty$, with norms given by

$$\|f\|_{pq} = \begin{cases} \{\int_0^\infty (s\mu_f(s)^{1/p})^q s^{-1}\,ds\}^{1/q} & \text{for } q < \infty, \\ \sup_{s>0} s\mu_f(s)^{1/p} & \text{for } q = \infty. \end{cases}$$

In case $p = q$, $L^{pq}(\Omega) = L^p(\Omega)$, the usual Lebesgue space, and we shorten $\|f\|_{pp}$ to $\|f\|_p$. The smallest of all r.i. spaces is the intersection, $L^1 \cap L^\infty$, of $L^1(\Omega)$ and $L^\infty(\Omega)$, with $\|f\|_{L^1\cap L^\infty} = \max\{\|f\|_1, \|f\|_\infty\}$.

The *Köthe dual* or *associate space* $X' = X'(\Omega)$ of a Banach lattice $X = X(\Omega)$ consists of those complex-valued measurable functions f on Ω such that $fg \in L^1(\Omega)$ for all $g \in X$. We define

$$\|f\|_{X'} = \sup\Big\{\Big|\int\limits_\Omega fg\,d\mu\Big| : \|g\|_X \le 1\Big\}.$$

This is a norm provided X has the Fatou property. In this case, X' is a Banach lattice which is both saturated and has the Fatou property; moreover, $X'' = X$ isometrically, so that

$$\|f\|_X = \sup\Big\{\Big|\int\limits_\Omega fg\,d\mu\Big| : \|g\|_{X'} \le 1\Big\}. \tag{14}$$

The *generalized Hölder inequality* asserts the consequence of (14) that when $f \in X$, $g \in X'$, the function $fg \in L^1(\Omega)$ and $\|fg\|_1 \le \|f\|_X\|g\|_{X'}$. Theorem 5.2 of [1] shows that for X an r.i. space on $\mathbb{R}^n$,

$$\|\chi_{B_r(0)}\|_X\|\chi_{B_r(0)}\|_{X'} = C_n r^n \quad \text{for } r > 0. \tag{15}$$

Given Banach lattices $X = X(\Omega)$, $Y = Y(\Omega)$ and $0 < \theta < 1$, the *Calderón product* $Z = X^{1-\theta}Y^\theta$ consists of all measurable h on Ω such that $|h| \le \lambda f^{1-\theta}g^\theta$ μ-a.e. for some $\lambda > 0$, $0 \le f \in X$, $0 \le g \in Y$, $\|f\|_X$, $\|g\|_Y \le 1$. In this case, $\|h\|_Z = \inf\lambda$. It is shown in #33.5 of [4] that Z is a Banach lattice. Further, one readily proves that Z is saturated whenever X and Y are and that it has the Fatou property whenever X and Y do.

THEOREM 3. *Let $X_i = X_i(\Omega)$, $Y_i = Y_i(\Omega)$, $i = 1, 2$, be Banach lattices which have the Fatou property and let $X_\theta = X_1^{1-\theta}X_2^\theta$, $Y_\theta = Y_1^{1-\theta}Y_2^\theta$ for some fixed θ, $0 < \theta < 1$. Suppose T is a linear operator which satisfies*

$$0 \le g_n \uparrow g \in X_i \ \mu\text{-a.e.} \ \Rightarrow \ 0 \le Tg_n \uparrow Tg \in Y_i \ \mu\text{-a.e.}$$

with

$$\|Tf_i\|_{Y_i} \le M_i\|f_i\|_{X_i} \quad \text{for } f_i \in X_i,\ i = 1, 2.$$

Then

$$\|Tf\|_{Y_\theta} \le M_\theta\|f\|_{X_\theta} \quad \text{for } f \in X_\theta,\ \text{where } M_\theta \le M_1^{1-\theta}M_2^\theta.$$

Proof. Consider $f \in X_\theta$ with

$$|f| \le \lambda g^{1-\theta}h^\theta \ \mu\text{-a.e.}, \lambda > 0; \quad g, h \ge 0; \quad \|g\|_{X_1}, \|h\|_{X_2} \le 1.$$

Then, by the abstract Hölder inequality ([9], p. 143)

$$|Tf| \le T|f| \le \lambda T(g^{1-\theta}h^{\theta}) \le \lambda [Tg]^{1-\theta}[Th]^{\theta}$$
$$\le \lambda M_1^{1-\theta} M_2^{\theta} \left(\frac{Tg}{M_1}\right)^{1-\theta} \left(\frac{Th}{M_2}\right)^{\theta}.$$

Hence, $\|Th\|_{Y_\theta} \le \lambda M_1^{1-\theta} M_2^{\theta}$ and we are done.

THEOREM 4 (Lozanovskiĭ [11]). *Let $X = X(\Omega)$ be a Banach lattice with Köthe dual $X' = X'(\Omega)$. Suppose X (and hence X') is saturated and has the Fatou property. Set $Z = X(\Omega)^{1/2} X'(\Omega)^{1/2}$. Then $Z = L^2(\Omega)$ isometrically.*

Proof. We begin by observing that Z is a saturated Banach lattice which has the Fatou property.

Given $f \in Z$, let $\lambda > 0$ be such that $|f| \le \lambda g^{1/2} h^{1/2}$ for $0 \le g \in X$, $0 \le h \in X'$, with $\|g\|_X, \|h\|_{X'} \le 1$. Then

$$\|f\|_2 = \Big(\int_{\Omega} |f|^2 d\mu\Big)^{1/2} \le \lambda \Big(\int_{\Omega} gh\, d\mu\Big)^{1/2} \le \lambda (\|g\|_X \|h\|_{X'})^{1/2} \le \lambda\,,$$

and so $\|f\|_2 \le \|f\|_Z$. Here, we have used Hölder's inequality.

Suppose, next, that $f \in L^2(\Omega)$ and $\|f\|_2 = 1$. We have $|f| = \sqrt{|f|^2}$, where $|f|^2 \in L^1(\Omega)$ and $\||f|^2\|_1 = 1$. By Theorem 1 in [6], $|f|^2 = gh$, where $\|g\|_X \|h\|_{X'} = 1$; indeed, without loss of generality, $\|g\|_X = \|h\|_{X'} = 1$. It follows that $\|f\|_Z \le 1 = \|f\|_2$. The same is then clearly true of any $f \in L^2(\Omega)$. This completes the proof.

Given a Banach lattice $X = X(\Omega)$ and measurable $w : \Omega \to \mathbb{R}_+$, define

$$X(w) = \{F : \Omega \to \mathbb{C} : \infty > \|F\|_{X(w)} = \|Fw\|_X\}\,.$$

It is easily seen $X(w)$ is a Banach lattice which is saturated whenever X is and has the Fatou property whenever X does; further, $X(w)' = X'(w^{-1})$. We thus have

COROLLARY 5. *Let $X = X(\Omega)$ be a Banach lattice with Köthe dual $X' = X'(\Omega)$ and assume X (and hence X') is saturated and has the Fatou property. Suppose $w : \Omega \to \mathbb{R}_+$ is measurable. Then $X(w)^{1/2} X'(w^{-1})^{1/2} = L^2(\Omega)$ isometrically.*

We now record two additional results for r.i. spaces, the first of which characterizes one of the hypotheses in Theorem 1.

LEMMA 6 (D. Boyd [3]). *Suppose $X = X(\mathbb{R}^n)$ is an r.i. space of functions on $\mathbb{R}^n$. Then the mapping $f \to f^{++}$ is bounded on X if and only if $\lim_{s\to\infty} h(s) = 0$, where $h(s)$ is the (finite) operator norm of the dilation operator $(E_s f)(x) := f(sx)$ ($s > 0$, $x \in \mathbb{R}^n$) from X to itself.*

LEMMA 7. *Suppose $X = X(\mathbb{R}^n)$ is an r.i. space of functions on $\mathbb{R}^n$. Then there is a positive constant C such that for all $0 \le f, g \in \mathcal{R.D.}$*,

$$\| \|f(x-y)g(y)\|_{X(dy)}\|_{X(dx)} \le C\|f\|_X \|g\|_X \,. \tag{16}$$

Proof. Given $0 \le f, g \in \mathcal{R.D.}$ and $x, y \in \mathbb{R}^n$, we have

$$f(x-y)g(y) = f(|x-y|)g(|y|) \le f(|x|/2)g(|y|) + f(|x-y|)g(|x|/2)\,,$$

since $f(|x-y|) \le f(|x|/2)$ if $|x-y| \ge |x|/2$ while $g(|y|) \le g(|x|/2)$ if $|y| \ge |x|/2$ (one of these cases must hold as $|x| \le |x-y| + |y|$). Thus, the left side of (16) is at most

$$\begin{aligned} &\| \|f(|x|/2)g(|y|) + f(|x-y|)g(|x|/2)\|_{X(dy)}\|_{X(dx)} \\ &\quad \le \|f(|x|/2)\|_X \|g\|_X + \|f\|_X \|g(|x|/2)\|_X \le 2h(1/2)\|f\|_X\|g\|_X\,, \end{aligned}$$

where we have used the fact that X is translation-invariant.

Finally, we show how the sufficiency of (11) for $X(w)$ to be closed under convolution reduces to (10) (in Theorem 1) and (12) (in Theorem 2). Indeed, assuming first the hypotheses of Theorem 1 we have, by (15), with $F = f/w \ge 0$, $G = g/w \ge 0$,

$$\begin{aligned} \|F*G\|_{X(w)} &= \sup_{\substack{\|h\|_{X'}\le 1\\ h\ge 0}} \int_{\mathbb{R}^n} \left(\frac{f}{w} * \frac{g}{w}\right) hw \\ &\le \sup_{\substack{\|h\|_{X'}\le 1\\ h\ge 0}} \int_{\mathbb{R}^n} \left(\frac{f^+}{w} * \frac{g^+}{w}\right) h^{++} w \qquad \text{given (10)} \\ &\le C \sup_{\substack{\|h\|_{X'}\le 1\\ h\ge 0}} \left\| \left(\frac{f^+}{w} * \frac{g^+}{w}\right) w \right\|_X \|h^{++}\|_{X'} \qquad \text{by Hölder's inequality} \\ &\le C \left\| \left(\frac{f^+}{w} * \frac{g^+}{w}\right) w \right\|_X \qquad \text{since } h \to h^{++} \text{ is bounded on } X' \\ &\le C \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \left\| \frac{w(x)}{w(x-y)w(y)} \right\|_{X'(dy)} \| \|f^+(x-y)g^+(y)\|_{X(dy)}\|_{X(dx)} \\ &\qquad\qquad \text{as in (5)} \\ &\le C\|f^+\|_X\|g^+\|_X = C\|f\|_X\|g\|_X = \|F\|_X(w)\|G\|_X(w)\,, \end{aligned} \tag{17}$$

since, by Lemma 7, (9) holds for the nonnegative $\mathcal{R.D.}$ functions f^+, g^+.

Assuming the hypotheses of Theorem 2 instead, we again obtain (17), but this time with $\|h^+\|_{X'} = \|h\|_{X'}$ in place of $\|h^{++}\|_{X'}$ (by (12)) and since $\|h\|_{X'} \le 1$, no assumption on X' is needed now.

3. The weighted Riesz–Sobolev inequalities. As shown in the last section, the sufficiency of (11) for $X(w)$ to be closed under convolution depends, under varying assumptions on X and w, on the following two theorems:

THEOREM 8. *Suppose $w \in \mathcal{M}$. Then*

$$\int_{\mathbb{R}^n} \left(\frac{f}{w} * \frac{g}{w}\right) hw \le C \int_{\mathbb{R}^n} \left(\frac{f^+}{w} * \frac{g^+}{w}\right) h^{++} w \qquad \text{for } f, g, h \ge 0 . \tag{18}$$

THEOREM 9. *Suppose $w \in \mathcal{M}_\infty$. Then*

$$\int_{\mathbb{R}^n} \left(\frac{f}{w} * \frac{g}{w}\right) hw \le C \int_{\mathbb{R}^n} \left(\frac{f^+}{w} * \frac{g^+}{w}\right) h^{+} w \qquad \text{for } f, g, h \ge 0 \tag{19}$$

if and only if w satisfies (13).

The proofs of Theorems 8 and 9 require certain monotonicity properties of $w \in \mathcal{M}$. These are a consequence of the following general result.

LEMMA 10. *Suppose $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $\Phi(x) = \Phi(|x|)$, $x \in \mathbb{R}^n$. If there exists $C > 0$ such that*

$$\Phi(x) \le C\Phi(z) \qquad \text{for } |x| \ge |z| , \tag{20}$$

then

$$\int_{|x|\ge r} \chi_E(x)\Phi(x)\, dx \le C|E| \min\Big\{\Phi(r), |F|^{-1} \int_F \Phi(y)\, dy\Big\} \tag{21}$$

for all $E \subset \mathbb{R}^n$, $r > 0$ and $F \subset B_r(0)$. In particular,

$$\int_E \Phi(x)\, dx \le (C^2+1) \int_{|x|\le r_n} \Phi(x)\, dx \qquad \text{for } E \subset \mathbb{R}^n,\ r_n = C_n^{-1}|E|^{1/n} . \tag{22}$$

If there exists $C > 0$ such that

$$\Phi(x) \le C(C_n|x|)^{-n} \int_{|y|\le|x|} \Phi(y)\, dy \qquad \text{for } x \in \mathbb{R}^n , \tag{23}$$

then

$$\int_{|x|\le r} \chi_E(x)\Phi(x)\, dx \le (C+1) \int_{|x|\le r} \chi_E^{++}(x)\Phi(x)\, dx , \tag{24}$$

for all $E \subset \mathbb{R}^n$ and $r > 0$.

Proof. We obtain (21) from (20) since $\Phi(x) \le C\Phi(r)$ and $\Phi(x) \le C\Phi(y)$ whenever $|x| > r$ and $|y| < r$. Then (22) follows on writing

$$\int_E \Phi(x)\, dx = \int_{|x|<r_n} \chi_E(x)\Phi(x)\, dx + \int_{|x|\ge r_n} \chi_E(x)\Phi(x)\, dx$$

and applying (21) with $F = B_{r_n}(0)$.

If $r \le r_n$, then (24) is trivial, so we suppose $r > r_n$. We have

$$\begin{aligned}
&\int\limits_{|x|\le r} \chi_E(x)\Phi(x)\,dx \\
&\quad \le \int\limits_{|x|\le r_n} \Phi(x)\,dx + C \int\limits_{r_n\le|x|\le r} \chi_E(x)\Big\{(C_n|x|)^{-n} \int\limits_{|z|\le|x|} \Phi(z)\,dz\Big\}\,dx \\
&\quad \le \int\limits_{|x|\le r_n} \Phi(x)\,dx + C \int\limits_{|z|\le r} \Phi(z)\Big\{ \int\limits_{\max(|z|,r_n)}^{r} \chi_E(x)(C_n|x|)^{-n}\,dx\Big\}\,dz \\
&\quad \le \int\limits_{|x|\le r_n} \Phi(x)\,dx + C \int\limits_{|z|\le r} \Phi(z)\frac{|E|}{\max\{(C_n|x|)^n, |E|\}}\,dz \\
&\quad \le (C+1) \int\limits_{|x|\le r} \chi_E^{++}(x)\Phi(x)\,dx\,.
\end{aligned}$$

LEMMA 11. *Suppose* $w \in \mathcal{M}$. *Then*

(i) $B(|x|,\cdot)$ *and* $B(\cdot,|x|)$ *satisfy* (20) *with* $C > 0$ *independent of* $x \in \mathbb{R}^n$;

(ii) $W(x) = \int_0^{|x|} \frac{w(|x|)}{w(|x|-s)w(s)} s^{n-1}\,ds$ *satisfies* (23).

Proof. (i) is obvious. To prove (ii) we first show that if $3|x|/4 \le |y| \le |x|$, then $W(x) \le CW(y)$. Now,

$$W(x) = \Big(\int_0^{|x|/2} + \int_{|x|/2}^{|x|}\Big)\frac{w(|x|)}{w(|x|-s)w(s)} s^{n-1}\,ds = I + II\,.$$

Since $w \in \mathcal{M}$, we have

$$I \le \int_0^{|x|/2} \frac{w(|y|)}{w(|y|-s)w(s)} s^{n-1}\,ds \le W(|y|)\,, \tag{25}$$

and

$$\begin{aligned}
II &\le \int_{|x|/2}^{|x|} \frac{w(|y|)}{w(|x|-s)w(s+|y|-|x|)} s^{n-1}\,ds \\
&\le \int_{|y|-|x|/2}^{|y|} \frac{w(|y|)}{w(|y|-t)w(t)} (t+|x|-|y|)^{n-1}dt \le C_n W(|y|)\,,
\end{aligned} \tag{26}$$

since $t+|x|-|y| \le Ct$ for $|y|-|x|/2 \le t|y|$ and $3|x|/4 \le |y| \le |x|$. From (25) and (26) we obtain $W(|x|) \le C_n W(|y|)$ for $3|x|/4 \le |y| \le |x|$, as claimed.

Iterating this inequality yields the doubling condition

$$W(x) \le C\left(\frac{|x|}{|y|}\right)^{\beta} W(|y|) \qquad \text{for } 0 < |y| \le |x|\,,$$

where C and β are positive constants depending only on the dimension n. We now obtain that W satisfies (23) easily from

$$\begin{aligned} W(x) &= C_\beta (C_n|x|)^{-n} \int\limits_{|y|\le|x|} W(x)\frac{|y|^\beta}{|x|^\beta}\,dy \\ &\le C_\beta (C_n|x|)^{-n} \int\limits_{|y|\le|x|} W(|y|)\,dy\,. \end{aligned}$$

Let $\mathbb{R}^n_+ = \{(x_1,\ldots,x_n) : x_1 \ge 0,\ldots,x_n \ge 0\}$. Define the set E^+ by $\chi_{E^+} = (\chi_E)^+$; this will be a ball (recall $|x| = |x_1|+\ldots+|x_n|$) centred at the origin with, say, radius r_{E^+}. Lastly, denote by $\widetilde{E}$ the ball concentric with E^+ and with radius $r_{\widetilde{E}} = \frac{1}{2} r_{E^+}$.

Proof of Theorems 8 and 9. To begin, observe that it is enough to prove (18) and (19) for nonnegative simple functions f, g and h which are symmetric with respect to all 2^n-orthotants of $\mathbb{R}^n$. Furthermore, we claim that one need only consider $f = \chi_E$, $g = \chi_F$, $h = \chi_G$, where E, F and G are sets symmetric with respect to all 2^n-orthotants of $\mathbb{R}^n$. For, suppose the latter fact to be true. Then the simple functions f, g and h referred to above can be written as finite sums of the form $f = \sum_i f_i \chi_{E_i}$, $g = \sum_j g_j \chi_{F_j}$, $h = \sum_k h_k \chi_{G_k}$, where the sets E_i, F_j and G_k are symmetric, with $E_i \supset E_{i+1}$, $F_j \supset F_{j+1}$, $G_k \supset G_{k+1}$ and the constants f_i, g_j and h_k are nonnegative. Hence, we get (for example) (18) as follows:

$$\begin{aligned} \int\limits_{\mathbb{R}^n} \left(\frac{f}{w} * \frac{g}{w}\right) hw &= \sum_{i,j,k} f_i g_j h_k \int\limits_{\mathbb{R}^n} \left(\frac{\chi_{E_i}}{w} * \frac{\chi_{F_j}}{w}\right)\chi_{G_k} w \\ &\le C \sum_{i,j,k} f_i g_j h_k \int\limits_{\mathbb{R}^n} \left(\frac{\chi_{E_i^+}}{w} * \frac{\chi_{F_j^+}}{w}\right)\chi_{G_k}^{++} w = C \int\limits_{\mathbb{R}^n} \left(\frac{f^+}{w} * \frac{g^+}{w}\right) h^{++} w\,. \end{aligned}$$

Summarizing, we have shown that in order to prove (18) and (19), it is enough to establish, respectively,

$$\begin{aligned} &\iint\limits_{E\times F} \chi_G(x+y)\frac{w(x+y)}{w(x)w(y)}\,dx\,dy \\ &\qquad \le C \iint\limits_{E^+\times F^+} \chi_G^{++}(x+y)\frac{w(x+y)}{w(x)w(y)}\,dx\,dy\,, \end{aligned} \tag{27}$$

for symmetric sets E, F, $G \subset \mathbb{R}^n$, and

$$\iint\limits_{E\times F} \chi_G(x+y)\frac{w(x+y)}{w(x)w(y)}\,dx\,dy \le C \iint\limits_{E^+\times F^+} \chi_G^+(x+y)\frac{w(x+y)}{w(x)w(y)}\,dx\,dy\,, \tag{28}$$

for symmetric sets E, F, $G \subset \mathbb{R}^n$.

To prove (27) and (28) we distinguish three cases, in all of which it may be assumed without loss of generality that $|E| \le |F|$.

Case 1: $|E| \le |F| \le |G|$. In this case we actually have the stronger inequality (28) for $w \in \mathcal{M}$ without any additional assumptions. Indeed, since E and F are symmetric and $w(x+y) \le Cw(|x|+|y|)$, the left side of (28) is at most

$$C \int\limits_F \int\limits_E B(|x|,|y|)\,dx\,dy \le C \int\limits_F \int\limits_{\widetilde{E}} B(|x|,|y|)\,dx\,dy$$

by Lemma 11(i) and (21) of Lemma 10, and thus at most

$$\begin{aligned} &\le C \int\limits_{\widetilde{F}} \int\limits_{\widetilde{E}} B(|x|,|y|)\,dx\,dy \\ &= 2^n C \iint\limits_{\mathbb{R}^n_+\times\mathbb{R}^n_+} \chi_{\widetilde{E}}(x)\chi_{\widetilde{E}}(y) B(|x|,|y|)\,dx\,dy \end{aligned} \tag{29}$$

upon reversing the order of integration and applying (21) again. Since $\widetilde{E} + \widetilde{F} \subset G^+$ and $|x+y| = |x|+|y|$ for $x, y \in \mathbb{R}^n_+$, the last integral in (29) is at most

$$C \iint\limits_{\mathbb{R}^n_+\times\mathbb{R}^n_+} \chi_E^+(x)\chi_F^+(y)\chi_G^+(x+y)\frac{w(x+y)}{w(x)w(y)}\,dx\,dy \le C \int\limits_{\mathbb{R}^n} \left(\frac{\chi_E^+}{w} * \frac{\chi_F^+}{w}\right)\chi_G^+ w\,.$$

Case 2: $|E| \le |G| \le |F|$. Here again there holds the stronger inequality (28) assuming only $w \in \mathcal{M}$. We have

$$\begin{aligned} \int\limits_{\mathbb{R}^n} \left(\frac{\chi_E}{w} * \frac{\chi_F}{w}\right)\chi_G w &\le \iint\limits_{E\times\mathbb{R}^n} \chi_G(x+y)B(|x|,|y|)\,dx\,dy \\ &\le C \iint\limits_{E\cap\mathbb{R}^n_+\times\mathbb{R}^n_+} \chi_{\mathcal{G}}(x,y)B(|x|,|y|)\,dy\,dx\,, \end{aligned} \tag{30}$$

where $\chi_{\mathcal{G}}(x,y)=\sum\chi_G(x_1\pm y_1,\ldots,x_n\pm y_n)$ for $x,y\in\mathbb{R}^n_+$, the sum being extended over all choices of $\pm$. The last term in (30) equals

$$C\int_{E\cap\mathbb{R}^n_+}\int_{\mathcal{G}_x}B(|x|,|y|)\,dy\,dx\qquad\text{where } \chi_{\mathcal{G}_x}(y)=\chi_{\mathcal{G}}(x,y),\ x,y\in\mathbb{R}^n_+\,.$$

Arguing as in case 1 and observing that $|\mathcal{G}_x|\le 2^n|G|$ for all $x\in\mathbb{R}^n_+$, we obtain the upper bound

$$\begin{aligned}&C\iint_{\mathbb{R}^n_+\times\mathbb{R}^n_+}\chi_{\widetilde{E}}(x)\chi_{\widetilde{G}}(y)\frac{w(x+y)}{w(x)w(y)}\,dx\,dy\\ &\le C\iint_{\mathbb{R}^n_+\times\mathbb{R}^n_+}\chi_{\widetilde{E}}(x)\chi_{\widetilde{G}}(y)\chi_{G^+}(x+y)\frac{w(x+y)}{w(x)w(y)}\,dx\,dy\qquad\text{since } \widetilde{E}+\widetilde{G}\subset G^+\\ &\le C\int_{\mathbb{R}^n}\left(\frac{\chi_E^+}{w}*\frac{\chi_F^+}{w}\right)\chi_G^+w\,.\end{aligned}$$

Case 3: $|G|\le|E|\le|F|$. In this case we can only obtain (27) for $w\in\mathcal{M}$. We then prove (28) holds for $w\in\mathcal{M}_\infty$ if and only if (13) does. The left side of (27) is at most

$$(31)\qquad\Big(\iint_{\widetilde{E}\times\widetilde{E}}+\iint_{(\widetilde{E}\times\widetilde{E})^c}\Big)\chi_G(x+y)\frac{w(x+y)}{w(x)w(y)}\,dx\,dy=I+II\,.$$

Let $r_k=x_k+y_k$, $s_k=y_k$, $r=\sum_k|r_k|$ and $s=\sum_k|s_k|$. Since $|r-s|\le\sum_k|r_k-s_k|=|x|$, we have $w(|r-s|)\le Cw(|x|)$, and we may bound I by

$$(32)\qquad\int_{E^+}\chi_G(r_1,\ldots,r_n)\int_0^{r_{\tilde{E}}}\frac{w(r)}{w(|r-s|)w(s)}s^{n-1}\,ds\,dr_1\ldots dr_n\,.$$

We now show that the inner integral in (32) satisfies

$$(33)\qquad\int_0^{r_{\tilde{E}}}\frac{w(r)}{w(|r-s|)w(s)}s^{n-1}\,ds\le C[W(r)+W(r_{\widetilde{E}})]\,.$$

Indeed, when $r_{\widetilde{E}}\le r\le r_{E^+}$, the left side of (33) is at most $CW(r_{\widetilde{E}})$ since $w\in\mathcal{M}$; while, for $0\le r\le r_{\widetilde{E}}$, we have, letting $r_0=\min\{r,r_{\widetilde{E}}-r\}$ and observing that $w(r)/w(r+s)\le C$ and $w(r_{\widetilde{E}})/w(r_{\widetilde{E}}-s)\ge c$,

$$\int_0^{r_{\tilde{E}}}\frac{w(r)}{w(|r-s|)w(s)}s^{n-1}\,ds=W(r)+\int_0^{r_{\tilde{E}}-r}\frac{w(r)}{w(r+s)w(s)}(r+s)^{n-1}\,ds$$

and

$$\int_0^{r_{\tilde E}-r} \frac{w(r)}{w(r+s)w(s)}(r+s)^{n-1}\,ds$$
$$\le C\Big[r^{n-1}\int_0^{r_0} B(r_{\widetilde E}-s,s)\,ds + \int_r^{r_{\tilde E}} B(r_{\widetilde E}-s,s)s^{n-1}\,ds\Big]$$
$$\le C\int_0^{r_{\tilde E}} B(r_{\widetilde E}-s,s)s^{n-1}\,ds = CW(r_{\widetilde E})\,.$$

Thus,

$$I \le C\int_{E^+} \chi_G(r_1,\dots,r_n)W(r)\,dr_1\dots dr_n + C|G|W(r_{\widetilde E})\,. \tag{34}$$

Both terms on the right side of (34) are no larger than

$$C\int_{E^+} \chi_G^{++}(r)W(r)\,dr_1\dots dr_n\,;$$

this is true for the first term by Lemma 11(ii) and (24) of Lemma 10, while for the second term we have, by Lemma 11(ii) again,

$$|G|W(r_{\widetilde E}) \le C\frac{|G|}{(r_{\widetilde E})^n}\int_{\widetilde E} W(r)\,dr_1\dots dr_n \le C\int_{E^+} \chi_G^{++}(r)W(r)\,dr_1\dots dr_n\,.$$

Since

$$\int\int_{E\times F} \chi_G^{++}(x+y)\frac{w(x+y)}{w(x)w(y)}\,dx\,dy$$
$$\ge \int\int_{(E^+\cap\mathbb{R}^n_+)\times(E^+\cap\mathbb{R}^n_+)} \chi_G^{++}(|x|+|y|)B(|x|,|y|)\,dx\,dy$$
$$\ge c\int_{E^+} \chi_G^{++}(r)W(r)\,dr_1\dots dr_n\,,$$

we get I dominated by the right side of (27).

Using the notations $\mathcal{G}$ and $\mathcal{G}_x$ as in case 2 above, term II in (31) is seen to be at most

$$\Big(\int\int_{\widetilde E\times\widetilde E^{c}} + \int\int_{\widetilde E^{c}\times\widetilde E} + \int\int_{\widetilde E^{c}\times\widetilde E^{c}}\Big)\chi_{\mathcal{G}_x}(y)B(|x|,|y|)\,dx\,dy = II_1+II_2+II_3\,. \tag{35}$$

Now,

$$(36)\quad II_1 = \int_{\widetilde{E}} \Big\{ \int_{\widetilde{E}^c} \chi_{\mathcal{G}_x}(y) B(|x|,|y|)\,dy \Big\}\,dx$$
$$\le C \int_{E^+} \Big\{ \frac{|\mathcal{G}_x|}{|E|} \int_{E^+} B(|x|,|y|)\,dy \Big\}\,dx \qquad \text{by (21)}$$
$$\le C \iint_{E^+\times E^+} \frac{|G|}{|E|} B(|x|,|y|)\,dx\,dy \qquad \text{since } |\mathcal{G}_x| \le 2^n|G|\,.$$

Similarly,

$$(37)\qquad II_2 \le C \iint_{E^+\times E^+} \frac{|G|}{|E|} B(|x|,|y|)\,dx\,dy\,.$$

Again,

$$(38)\qquad II_3 \le C|E||G|B(r_{\widetilde{E}}, r_{\widetilde{E}}) \qquad \text{by Lemma 11(i)}$$
$$\le C \iint_{E^+\times E^+} \frac{|G|}{|E|} B(|x|,|y|)\,dx\,dy\,.$$

But the common right side of (36), (37) and (38) is no bigger than

$$\iint_{E^+\times E^+} \chi_G^{++}(x+y)B(x,y)\,dx\,dy\,,$$

since $\chi_G^{++}(2r_{E^+}) = C|G|/|E|$, which is dominated, in turn, by the right side of (18).

Next, we show that when $w \in \mathcal{M}_\infty$, (28) and (13) are equivalent. Suppose (28) holds. Taking $E = F = B_r(0)$ and $G = B_{r+\delta}(0) - B_{r-\delta}(0)$, $0 < \delta \le r/2$, in (28) yields

$$(39)\qquad \int_{|x|<r} \int_{|y|<r} \chi_G(x+y) \frac{w(x+y)}{w(x)w(y)}\,dy\,dx$$
$$\le C \int_{|x|<r} \int_{|y|<r} \chi_G^+(x+y) \frac{w(x+y)}{w(x)w(y)}\,dy\,dx\,.$$

On the left side of (39) restrict attention to x and y in the first orthotant and make the substitution $t_k = x_k + y_k$, $s_k = x_k$, $t = \sum_k t_k$, $s = \sum_k s_k$ to get the lower bound

$$C \int_0^{r-\delta} s^{n-1} \int_{r-\delta}^{r+\delta} B(s,t-s)t^{n-1}\,dt\,ds$$
$$\geq c \int_0^{r-\delta} s^{n-1} B(s, r+\delta-s) \int_{r-\delta}^{r+\delta} t^{n-1}\,dt\,ds$$
$$\geq c\delta r^{n-1} \int_0^{r-\delta} \frac{w(r+\delta)}{w(r+\delta-s)w(s)} s^{n-1}\,ds\,.$$

As for the right side of (39), with $\varepsilon^n = c\delta r^{n-1}$, it is dominated by

$$C \int_{|x|\leq r} \int_{B_\varepsilon(-x)} \frac{w(\varepsilon)}{w(x)w(y)}\,dy\,dx < \infty\,,$$

since $w \in \mathcal{M}_\infty$. We conclude

$$\varepsilon^n \int_0^{r/2} \frac{w(r+\delta)}{w(r+\delta-s)w(s)} s^{n-1}\,ds$$
$$\leq Cw(\varepsilon) \int_0^r s^{n-1} \frac{ds}{w(s)} \int_{B_\varepsilon(-x)} \frac{dy}{w(y)} < \infty\,.$$

Dividing by ε^n and letting $\varepsilon \to 0+$, we obtain (13).

Now suppose that $w \in \mathcal{M}_\infty$ and that (13) holds. With a view to bounding I in (31) by the right side of (28) we claim that, given (13),

$$(40)\qquad \int_0^{r_{\tilde{E}}} \frac{w(r)}{w(|r-s|)w(s)} s^{n-1}\,ds \leq C \int_0^{r_{E^+}} s^{n-1} \frac{ds}{w(s)^2}, \qquad 0 \leq r \leq r_{E^+}\,.$$

For $r_{\widetilde{E}} \leq r \leq r_{E^+}$, the left side of (40) is at most

$$\int_0^{r_{\tilde{E}}} \frac{w(r_{\widetilde{E}})}{w(r_{\widetilde{E}}-s)w(s)} s^{n-1}\,ds \qquad \text{since } w \in \mathcal{M}$$
$$\leq C \int_0^{r_{E^+}} s^{n-1} \frac{ds}{w(s)^2} \qquad \text{by (13).}$$

When $0 \leq r \leq r_{\widetilde{E}}$,

$$\int_0^r \frac{w(r)}{w(r-s)w(s)} s^{n-1}\,ds \leq C \int_0^{r_{E^+}} \frac{s^{n-1}}{w(s)^2}\,ds$$

by (13), while, arguing as for (33),

$$\int_r^{r_{\tilde E}} \frac{w(r)}{w(s-r)w(s)} s^{n-1} ds = \int_0^{r_{\tilde E}-r} \frac{w(r)}{w(r+s)w(s)} (r+s)^{n-1}\, ds$$

$$\le C \int_0^{r_{\tilde E}} \frac{w(r_{\widetilde E})}{w(r_{\widetilde E}-s)w(s)} s^{n-1}\, ds \le C \int_0^{r_{E^+}} \frac{s^{n-1}}{w(s)^2}\, ds\,,$$

by (13). This proves (40), so we have

$$I \le C \int_{E^+} \chi_G(r_1,\dots,r_n)\, dr_1 \dots dr_n \int_0^{r_{E^+}} \frac{s^{n-1}}{w(s)^2}\, ds \le C|G| \int_{E^+} \frac{dx}{w(x)^2}\,. \tag{41}$$

We now show (41) holds with I replaced by II. By symmetry, II_1 in (35) satisfies

$$II_1 \le C \int_{\widetilde E \cap \mathbb{R}^n_+} \Big\{ \int_{(E^+\cap\mathbb{R}^n_+)^c} \chi_{\mathcal{G}_x}(y) B(|x|,|y|)\, dy \Big\}\, dx$$

$$\le C \int_{\widetilde E \cap \mathbb{R}^n_+} |G| B(|x|, r_{\widetilde E} - |x|)\, dx = C|G| \int_0^{r_{\tilde E}} B(r_{\widetilde E} - s, s) s^{n-1}\, ds$$

$$\le C|G| \int_0^{r_{E^+}} \frac{s^{n-1}}{w(s)^2}\, ds\,,$$

by (13). The term II_2 in (35) is dealt with similarly. Again,

$$II_3 \le C|E||G|B(r_{\widetilde E}, r_{\widetilde E})$$

$$\le C|G| \int_0^{r_{\tilde E}} B(r_{\widetilde E} - s, s) s^{n-1}\, ds \le C|G| \int_0^{r_{E^+}} \frac{s^{n-1}}{w(s)^2}\, ds$$

by (13). Since $w \in \mathcal{M}_\infty$,

$$\frac{|G|}{w(x)^2} \le C \int_{-G^+\cap\mathbb{R}^n_+} \frac{w(-y)}{w(x-y)w(x)}\, dy\,,$$

whence, by (41) (for II as well as for I), the left side of (28) is at most

$$\int_{E^+\cap\mathbb{R}^n_+} \int_{-G^+\cap\mathbb{R}^n_+} \frac{w(-y)}{w(x-y)w(x)}\, dy\, dx \le C \int_{E^+\cap\mathbb{R}^n_+} \int_{(-G^+\cap\mathbb{R}^n_+)-x} \frac{w(x+y)}{w(x)w(y)}\, dy\, dx$$

$$\le C \iint_{E^+\times E^+} \chi_{G^+}(x+y) \frac{w(x+y)}{w(x)w(y)}\, dy\, dx\,,$$

which completes the proof.

4. Necessary conditions. In this section we prove the necessity half of Theorems 1 and 2. In fact, we show that, given $w \in \mathcal{M}$, $X(w)$ closed under convolution implies (11). But first we prove simpler necessary conditions which are valid in a wider context than that of Theorem 1 or 2.

LEMMA 12. *Suppose w is even on $\mathbb{R}^n$, i.e. $w(x) = w(-x)$ for all $x \in \mathbb{R}^n$. If $X = X(\mathbb{R}^n)$ is an r.i. space and $X(w)$ is closed under convolution, then $X(w) \subset L^1(\mathbb{R}^n)$ or, equivalently, $w^{-1} \in X'$. Moreover, if $C > 0$ is as in (8), then*

$$\|f\|_{L^1} \le C\|f\|_{X(w)} \quad \text{for } f \in X(w)\,.$$

Proof. Fix $f \in X(w)$ with $\|f\|_{X(w)} = 1$ and define $T : X(w) \to X(w)$ by $(Tg)(x) = (|f| * g)(x)$, $x \in \mathbb{R}^n$. By (8), T is bounded on $X(w)$ with norm at most C and, by duality, T' is bounded on $X(w)' = X'(w^{-1})$ with norm at most C. But, since w is even, $T' = T$, so, by Theorem 3 and Corollary 5, $|f| * L^2 \subset L^2$ with norm at most C and it follows that

$$\begin{aligned}\int_{\mathbb{R}^n} ||f|^\wedge(\zeta)|^2 |g(\zeta)|^2\, d\zeta &= \int_{\mathbb{R}^n} |(|f| * \widehat{g})(x)|^2\, dx \\ &\le C \int_{\mathbb{R}^n} |\widehat{g}(x)|^2\, dx = C \int_{\mathbb{R}^n} |g(\zeta)|^2\, d\zeta\end{aligned}$$

for all $g \in L^2$. Thus $||f|^\wedge(\zeta)|$ is bounded by C and, in particular,

$$\|f\|_1 = \int_{\mathbb{R}^n} |f| = |f|^\wedge(0) \le C = C\|f\|_{X(w)}\,.$$

LEMMA 13. *Suppose w is radial, finite a.e. and satisfies*

$$B(r_1, s) \le C B(r_2, s) \quad \text{for } s > 0,\ r_1 \ge r_2 > 0\,, \tag{42}$$

yet fails to satisfy

$$w(y) \le C w(z) \quad \text{for } 0 < y < z \tag{43}$$

for the same constant C; that is,

$$w(y) > C w(z) \quad \text{for some } 0 < y < z\,. \tag{44}$$

Then $(w^{-1})^+(x) = \infty$ for all $x \in \mathbb{R}^n$.

Proof. w radial and finite a.e. implies there exists $M > 0$ and a set $E \subset \{x \in \mathbb{R}^n_+ : y \le |x| \le z\}$, $|E| > 0$, with $w(x) \le M$ for all x with $|x| \in E$. We will be done if we can show that for each $k = 1, 2, \ldots$,

$$w(x) \le M r^k \quad \text{for } |x| \in E + k(z - y)\,,$$

where $r = Cw(z)/w(y) < 1$ by (44). But, for $|x| \in E + k(z - y)$, say $|x| = u + k(z - y)$, $u \in E$, we have

$$
\begin{aligned}
w(x) = w(|x|) = w(u) \prod_{j=0}^{k-1} \frac{w(u+(j+1)(z-y))}{w(u+j(z-y))} \\
\le M \prod_{j=0}^{k-1} C\frac{w(z)}{w(y)} \qquad \text{by (42)} \\
\le Mr^k .
\end{aligned}
$$

COROLLARY 14. *If w is radial and satisfies* (42) *and $X(w)$ is closed under convolution*, *then* (43) *holds.*

Proof. By Lemma 12, $w^{-1} \in X'$, which means $(w^{-1})^+(x) < \infty$ for all $0 \neq x \in \mathbb{R}^n$. We suppose now that w satisfies (42), $X = X(\mathbb{R}^n)$ is an r.i. space and $X(w)$ is closed under convolution (i.e. (8) holds) and prove that (11) holds. Begin by fixing $r > 0$. For $g(x) = g(|x|) \ge 0$ we have

$$
\begin{aligned}
(45) \qquad & \int\limits_{|x|<r} ((\chi_{B_r(0)} w^{-1}) * (gw^{-1}))(x) w(x)\, dx \\
& \le \|(\chi_{B_r(0)} w^{-1}) * (gw^{-1})\|_{X(w)} \|\chi_{B_r(0)}\|_{X'} \\
& \le C\|gw^{-1}\|_{X(w)} \|\chi_{B_r(0)} w^{-1}\|_{X(w)} \|\chi_{B_r(0)}\|_{X'} \qquad \text{by (8)} \\
& \le C\|\chi_{B_r(0)}\|_X \|\chi_{B_r(0)}\|_{X'} \|g\|_X \le Cr^n \|g\|_X
\end{aligned}
$$

by (15). Now, the left side of (45) is

$$
\begin{aligned}
(46) \qquad & \int\limits_{|x|<r} \int\limits_{|y|<r} \frac{w(x)}{w(x-y)w(y)} g(y)\, dy\, dx \\
& \ge cr^n \int\limits_{|y|<r/2} g(y) \frac{1}{r^n - |y|^n} \int\limits_{|y|\le|x|\le r} \frac{w(x)}{w(x-y)w(y)}\, dx\, dy \\
& \ge cr^n \int\limits_{\substack{|y|<r/2 \\ y\in\mathbb{R}^n_+}} g(y) \frac{1}{r^n - |y|^n} \int\limits_{\substack{|y|\le|x|\le r \\ x-y\in\mathbb{R}^n_+}} \frac{w(x)}{w(x-y)w(y)}\, dx\, dy \\
& \ge cr^n \int\limits_{\substack{|y|<r/2 \\ y\in\mathbb{R}^n_+}} g(y) \frac{1}{r^n - |y|^n} \int\limits_{|y|}^{r} B(s-|y|, |y|) s^{n-1}\, ds\, dy \\
& \ge cr^n \int\limits_{\substack{|y|<r/2 \\ y\in\mathbb{R}^n_+}} g(y) B(r-|y|, |y|)\, dy \qquad \text{since } w \in \mathcal{M} \\
& \ge cr^n \int\limits_{\mathbb{R}^n} g(y) \chi_{B_{r/2}(0)}(y) B(r-|y|, |y|)\, dy .
\end{aligned}
$$

Combining (45) and (46) yields

$$\int_{\mathbb{R}^n} \chi_{B_{r/2}(0)}(y) B(r-|y|,|y|) g(y)\,dy \le C\|g\|_X ,$$

which, by duality, implies

$$\|\chi_{B_{r/2}(0)}(y) B(r-|y|,|y|)\|_{X'(dy)} \le C .$$

Thus, given $x \in \mathbb{R}^n$, we have, by (43),

$$\left\|\chi_{B_{|x|/2}(0)}(y)\frac{w(x)}{w(x-y)w(y)}\right\|_{X'(dy)} \le C\|\chi_{B_{|x|/2}(0)}(y)B(|x|-|y|,|y|)\|_{X'(dy)} \le C .$$

From (43) and the rearrangement-invariance of X' we further obtain for all $z \in \mathbb{R}^n$, $|z| = |x|$,

$$\left\|\chi_{B_{|x|/2}(z)}(y)\frac{w(x)}{w(x-y)w(y)}\right\|_{X'(dy)} \le \left\|\chi_{B_{|x|/2}(x)}(y)\frac{w(x)}{w(x-y)w(y)}\right\|_{X'(dy)} \le \left\|\chi_{B_{|x|/2}(0)}(y)\frac{w(x)}{w(x-y)w(y)}\right\|_{X'(dy)} .$$

As $B_{|x|}(0)$ is covered by $B_{|x|/2}(0)$, together with a finite number (independent of x) of $B_{|x|/2}(z)$, $|z| = |x|$, we conclude

$$\left\|\chi_{B_{|x|}(0)}(y)\frac{w(x)}{w(x-y)w(y)}\right\|_{X'(dy)} \le C . \tag{47}$$

By (43) again,

$$\left\|\chi_{\mathbb{R}^n-B_{|x|}(0)}(y)\frac{w(x)}{w(x-y)w(y)}\right\|_{X'(dy)} \le C\left\|\chi_{\mathbb{R}^n-B_{|x|}(0)}(y)\frac{1}{w(x-y)}\right\|_{X'(dy)} \le C\|w^{-1}\|_{X'} \le C , \tag{48}$$

in view of Lemma 12, and, together, (47) and (48) yield (11).

5. Examples. Let

$$w(x) = \begin{cases} 1, & -3 < x < 3, \\ 9^k[3^k - (1-3^{-k})||x| - 2\cdot 3^k|], & 3^k < |x| < 3^{k+1},\ k = 1,2,\ldots \end{cases} \tag{49}$$

We will prove that w satisfies (11) for all r.i. spaces X, yet $L^{pq}(w)$ is not an algebra when $1 < p < q \le \infty$.

The assertion concerning (11) is an immediate consequence of the fact that $L^1 \cap L^\infty$ is the smallest r.i. space and

LEMMA 15. *Let w be defined on $\mathbb{R}$ by* (49). *Then*

$$\left\| \frac{w(x)}{w(x-y)w(y)} \right\|_{L^1\cap L^\infty(dy)} \le C \qquad \text{for a.e. } x \in \mathbb{R}^n .$$

Proof. It is sufficient to consider $x > 0$, indeed $x > 3$. Let j and k be integers, $k \ge 1$ and $0 \le j \le k-1$, such that

$$3^k - 3^{j+1} < |x - 2\cdot 3^k| < 3^k - 3^j + 1 .$$

We show

$$\frac{w(x)}{w(x-y)w(y)} \le 324\Big(\frac{1}{w(x-y)} + \sum_{i=-1}^{1} W_{k+i}(y)\Big) , \tag{50}$$

where

$$W_l(y) = \frac{1 + 9^l H(|y| - 3^l)}{w(y)}, \qquad l = 0, 1, 2, \ldots$$

($H = \chi_{\mathbb{R}_+}$ being the Heaviside function) is readily seen to be in $L^1 \cap L^\infty(dy)$ uniformly in l. Observe that $w(x) \le 4 \cdot 9^{k+j+1}$ and consider the following cases for y, assuming $j \ge 1$.

Case 1: $|y - 2\cdot 3^k| < 3^k - 3^{j-1}$. Here, $w(y) \ge 9^{k+j-1}$, so $w(x)/w(y) \le 324$ and

$$\frac{w(x)}{w(x-y)w(y)} \le \frac{324}{w(x-y)} .$$

Case 2: $3^k - 3^{j-1} < |y - 2\cdot 3^k| < 3^k + 2\cdot 3^{k-1}$. We have $y > 3^{k-1}$ and

$$|x-y| \ge |y - 2\cdot 3^k| - |x - 2\cdot 3^k| > 3^j - 3^{j-1} - 1 \ge 3^{j-1} ,$$

so $w(x-y) \ge 9^{j-1}$ and

$$\frac{w(x)}{w(x-y)w(y)} \le \frac{4\cdot 9^{k+j+1}}{9^{j-1}w(y)} \le 324 W_{k-1}(y) .$$

Case 3: $|y - 2\cdot 3^k| > 3^k + 2\cdot 3^{k-1}$, $y > 0$. Either $0 < y < 3^{k-1} \le x/2$ and we are done by symmetry, or $y \ge 3^{k+1} + 2\cdot 3^{k-1}$, which means $y - x \ge 3^{k-1}$, $w(x-y) \ge 9^{k-1}$ and

$$\frac{w(x)}{w(x-y)w(y)} \le 4W_{k+1}(y) .$$

Case 4: $y < 0$. If $-3^{j-1} < y < 0$, then $3^k < x - y < 3^{k+1}$ and

$$|x - y - 2\cdot 3^k| \le |x - 2\cdot 3^k| + |y| < 3^k - 3^j + 3^{j-1} + 1 ,$$

so $w(x-y) \ge 9^{k+j-1}$, $w(x)/w(x-y) \le 324$, whence

$$\frac{w(x)}{w(x-y)w(y)} \le \frac{324}{w(y)} .$$

If $y < -3^{j-1}$, then $w(y) \geq 9^{j-1}$, and

$$\frac{w(x)}{w(x-y)w(y)} \leq 324 W_k(y)\,.$$

Finally, when $j = 0$, one of y and $x-y$ is greater than 3^{j-1}. Therefore, (50) holds then also.

To see that $L^{pq}(w)$ is not an algebra when $1 < p < q \leq \infty$, let N be a large positive integer and set $f = \sum_{k=1}^{n} 3^{-k}\chi_{E_k}$, where $E_k = \bigcup I_j$ and $I_j = (3^j, 3^j + 3/2)$ for $3^{kp+1} \leq j \leq 3^{(k+1)p}$. We show that

$$\left\| w\left(\frac{f}{w} * \frac{f}{w}\right)\right\|_{L^{pq}} \leq C\|f\|_{L^{pq}}^2$$

implies

$$N^{1/p+1/q} \leq C N^{2/q}$$

with $C > 0$ independent of N, and hence that $q \leq p$.

Since $\mu_f(t) \leq (3^{p+1}/2)t^{-p}\chi_{(3^{-N},3^{-1})}(t)$, we have

$$\|f\|_{L^{pq}}^2 \leq 3^{2(p+1)} N^{2/q}\,.$$

Next, $(f*f)(x) \neq 0$ only when $(f*f)(x) = \int_{I_j} f(x-y)f(y)\,dy$ and $x \in I_j + I_{j'}$ for some j and j'; moreover, for $x \in I_j + I_j'$ and $y \in I_j$,

$$\frac{w(x)}{w(x-y)w(y)} \geq \frac{1}{1000}\,.$$

Thus,

$$\begin{aligned} w(x)\left(\frac{f}{w} * \frac{f}{w}\right)(x) &= \int\limits_{\mathbb{R}} \frac{w(x)}{w(x-y)w(y)} f(x-y)f(y)\,dy \\ &\geq \frac{1}{1000}(f*f)(x) = \frac{1}{1000}\sum_{k=1}^{n} 3^{-k} \int\limits_{E_k} f(x-y)\,dy\,. \end{aligned}$$

Suppose, now, that $3^{-N} < t \leq 3^{-1}$ and that the positive integer l satisfies $3^{-l-1} < t \leq 3^{-l}$. Then

$$\begin{aligned} \left|\left\{x : w(x)\left(\frac{f}{w} * \frac{f}{w}\right)(x) > \frac{t}{1000}\right\}\right| &\geq |\{x : (f*f)(x) > 3^{-l}\}| \\ &\geq \sum_{k=1}^{l} \left|\left\{x : \int\limits_{E_k} f(x-y)\,dy > 3^{-(l-k)}\right\}\right| \\ &\geq \sum_{k=1}^{l} 3^{kp} 3^{(l-k)p} = l3^{lp} \geq \frac{1}{\log 3}\frac{\log\frac{1}{3t}}{(3t)^p}\,. \end{aligned}$$

It follows that

$$\left\| w\left(\frac{f}{w} * \frac{f}{w}\right)\right\|_{L^{pq}} \geq CN^{1/p+1/q} > 0\,,$$

$c > 0$ independent of N, and so we are done.

In the case $p > q$ we are unable to construct a weight w satisfying

$$\left\| \frac{w(x)}{w(x-y)w(y)} \right\|_{L^{p'q'}(dy)} \leq C, \qquad x \in \mathbb{R}^n$$

($(L^{pq})' = L^{p'q'}$) for which $L^{pq}(w)$, $p > q$, is not an algebra, though we believe such a w exists. In any event, we can show Nikol'skiĭ's proof will not work in this case, since (9) does not hold for $X = L^{pq}$ when $p > q$. (Of course, what we just proved implies (9) does not hold for L^{pq} when $q > p$.)

Indeed, as we now prove, (9) with $X = L^{pq}$ implies $p \leq q$. For, take $f = g = \chi_{E_N}$, where $E_N = \bigcup_{k=1}^{N} I_k$, with $I_k = [4^k, 4^k + 1/k]$, $k = 1, \ldots, N$. Then,

$$|E_N| = \sum_{k=1}^{N} \frac{1}{k} \leq C \log N\,,$$

whence

$$\|\chi_{E_N}\|^2_{L^{pq}} \leq C|E_N|^{2/p} \leq C(\log N)^{2/p}.$$

We claim

(51) $$\| \, \|\chi_{E_N}(x-y)\chi_{E_N}(y)\|_{L^{pq}(dy)}\|_{L^{pq}(dx)} \geq c(\log N)^{1/p+1/q}\,,$$

so that (9) entails $(\log N)^{1/p+1/q} \leq C(\log N)^{2/p}$ and so $p \leq q$. Observe that the left side of (51) equals

$$\|(\chi_{E_N} * \chi_{E_N})^{1/p}\|_{L^{pq}} \geq C\Big\{ \int_{N^{-2/p}}^{N^{-1/p}} |\{\chi_{E_N} * \chi_{E_N} > 2t^p\}|^{q/p} t^{q-1}\, dt \Big\}^{1/q}.$$

Now,

$$\chi_{E_N} * \chi_{E_N} \geq 2 \sum_{j=1}^{N} \sum_{k=j}^{N} \frac{1}{k}\; \chi_{I_{j,k}}\,,$$

where the $I_{j,k} = [4^j + 4^k + 1/k,\ 4^j + 4^k + 1/j]$ are pairwise disjoint. So, when $N^{-2/p} < t < N^{-1/p}$,

$$\begin{aligned} |\{\chi_{E_N} * \chi_{E_N} > 2t^p\}| &\geq \sum_{j=1}^{T} \sum_{k=j}^{T} \left(\frac{1}{j} - \frac{1}{k}\right), \qquad T = \left[\frac{1}{t^p}\right], \\ &\geq \frac{1}{2} \sum_{j=1}^{[T/3]} \sum_{k=2j}^{T} \frac{1}{j} \geq \frac{1}{2}(T - 2[T/3]) \sum_{j=1}^{[T/3]} \frac{1}{j} \geq c\frac{\log N}{t^p}\,. \end{aligned}$$

Thus,

$$\|(\chi_{E_N} * \chi_{E_N})^{1/p}\|_{L^{pq}} \ge c\Big\{\int\limits_{N^{-2/p}}^{N^{-1/p}} \Big(\frac{\log N}{t^p}\Big)^{q/p} t^{q-1}\,dt\Big\}^{1/q}$$

$$\ge c(\log N)^{1/p}\Big\{\int\limits_{N^{-2/p}}^{N^{-1/p}} \frac{dt}{t}\Big\}^{1/q} \ge c(\log N)^{1/p+1/q}\,.$$

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