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Semigroups affiliated with algebras of operators

by

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Abstract. If $\mathcal{B}$ is a Banach algebra of bounded linear operators on a Banach space X, and if A is a closed operator on X such that $(\lambda - A)^{-1}$ is in $\mathcal{B}$ for some $\lambda \in \mathbb{C}$, then A is said to be affiliated with $\mathcal{B}$. This paper examines when such an operator generates a C_0 or an integrated semigroup $\{T(t)\}_{t\geq 0}$ in $\mathcal{B}$. The spectral and essential spectral properties of A and $\{T(t)\}_{t\geq 0}$ relative to $\mathcal{B}$ are also studied. A number of consequences involving specific algebras are included.

Introduction. The concept of a closed operator affiliated with a Banach algebra $\mathcal{B}$ of operators contained in $B(X)$ is defined in [6] where the spectral and Fredholm theories for such operators are developed. Algebras of operators for which this concept is of particular interest can be found in the study of linear integral operators and differential equations ([10] and [6]), in the study of interpolated operators on Lebesgue spaces [5], and in the study of regular operators on a Banach lattice ([2] and [16]).

The notion of a semigroup of operators, and its generator, being affiliated with $\mathcal{B}$ is defined and considered briefly in [6]. All semigroups appearing in [6] are contraction semigroups and little is said about the related spectral properties for such. In this paper, we consider general strongly continuous semigroups and integrated semigroups and look more closely at spectral relationships, relative to $\mathcal{B}$, of a semigroup and its generator.

The basic observations involving generation of semigroups affiliated with $\mathcal{B}$ are contained in Section 1. Section 2 consists of spectral and essential-spectral properties relative to $\mathcal{B}$ while Section 3 exhibits some specific algebras of operators. The following briefly motivates interest in one such algebra.

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Consider the abstract Cauchy problem on a Banach space X:

$$\text{(ACP)}\qquad \begin{aligned} &\dot{x}(t) = Ax(t), \quad t \ge 0, \\ &x(0) = x_0, \quad x_0 \in D(A), \text{ the domain of } A. \end{aligned}$$

If A is a closed, densely defined operator with nonempty resolvent, then (ACP) has a unique solution for every x_0 in the domain of A if, and only if, A generates a strongly continuous semigroup. However, many Cauchy problems have solutions even when A does not generate a strongly continuous semigroup; in particular, it is not always natural to restrict the problem to the case in which A is densely defined. The theory of integrated semigroups—as recently developed in [1] and [12], among others—can be applied to this class of problems arising when A is not densely defined. In order to consider the adjoint Cauchy problem it is, of course, necessary to assume that the domain of A is dense in X so that the adjoint operator A^* is well defined. A dual system defined by a bilinear form on $X \times Y$ (as in [10, p. 43]) allows one to consider a dual problem to (ACP) in instances when the domain of A is not dense, and to retain some connection between the two problems. The Banach algebra, $\mathcal{A}(X, Y)$, connected with this setup consists of all bounded operators T on X which have a bounded "adjoint" operator $T^\dagger$ defined on the Banach space Y via a bilinear form on $X \times Y$: $\langle Tx, y\rangle = \langle x, T^\dagger y\rangle$ ($x \in X$, $y \in Y$). Even in situations where the domain of A is dense in X, the adjoint operator defined on the dual space X^* may be difficult to handle or unknown. The algebra $\mathcal{A}(X,Y)$ again allows one to consider a related adjoint equation involving an operator $A^\dagger$ on a Banach space Y which may be more accessible.

1. Affiliated semigroups. Let X be a Banach space and let $B(X)$ denote the Banach algebra of bounded linear operators on X with the operator norm, $\|\cdot\|$, and $F(X)$ the ideal of finite rank operators. Throughout the paper, $\mathcal{B} \equiv (\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will denote a Banach algebra of operators contained in $B(X)$ with $I \in \mathcal{B}$ and, for some $C \ge 0$, $C\|T\|_{\mathcal{B}} \ge \|T\|$ for $T \in \mathcal{B}$. A closed operator A on X is said to be *affiliated* with $\mathcal{B}$ if there exists a $\lambda \in \mathbb{C}$ such that $R(\lambda, A) \equiv (\lambda - A)^{-1}$ is in $\mathcal{B}$. For such an operator, the resolvent of A relative to $\mathcal{B}$ is a nonempty set denoted by $\varrho_{\mathcal{B}}(A) \equiv \{\lambda \in \mathbb{C} : R(\lambda, A) \in \mathcal{B}\}$. The spectral and Fredholm properties of a bounded operator in $\mathcal{B}$ have been studied for a variety of specific algebras ([2], [4], [5], and [10]) and so useful information concerning the spectral and Fredholm properties of A can be derived from corresponding properties of the bounded operator $R(\lambda, A)$ relative to $\mathcal{B}$. Our interest is with closed operators that are affiliated with $\mathcal{B}$ and generate strongly continuous (C_0) semigroups or integrated semigroups.

Because of the role that strong convergence plays in the theory of C_0 semigroups, we make the following definition [6]: a subset $\mathcal{C} \subseteq \mathcal{B}$ has the

strong convergence property (or SCP) if whenever $\{T_n\} \subset \mathcal{C}$, $\|T_n\|_{\mathcal{B}} \le M$ for $n \ge 1$, and $T_n x \to Tx$ for all $x \in X$, it follows that $T \in \mathcal{C}$ and $\|T\|_{\mathcal{B}} \le M$.

The following technical lemma will be used several times.

LEMMA 1.1. *Assume $\mathcal{B}$, as above, has the strong convergence property. Let $S : [0, \infty) \to \mathcal{B}$ be a strongly continuous function and let $g : [0,\infty) \to \mathbb{R}^+$ be a continuous function, with $\|S(t)\|_{\mathcal{B}} \le g(t)$ for all $t \in [0,\infty)$. If $f : [0,\infty) \to \mathbb{C}$ is any continuous function such that $\int_0^\infty |f(t)|\, g(t)\, dt$ exists, then $\int_0^\infty f(t)S(t)\, dt \in \mathcal{B}$, where the operator $\int_0^\infty f(t)S(t)\, dt$ is defined pointwise. Also, $\| \int_0^\infty f(t)S(t)\, dt\|_{\mathcal{B}} \le \int_0^\infty |f(t)|\, g(t)\, dt$.*

Proof. For each $m \in \mathbb{N}$, the strong continuity of S and the conditions on S, g and f imply that for each $x \in X$, the integral $\int_0^m f(t)S(t)x\, dt$ exists as a Riemann integral [9, Theorem 3.3.2]. Since $\int_0^\infty |f(t)|\, g(t)\, dt < \infty$ and $C\|\cdot\|_{\mathcal{B}} \ge \|\cdot\|$, we see that $\lim_{m\to\infty} \int_0^m f(t)S(t)x\, dt = \int_0^\infty f(t)S(t)x\, dt$ exists for each $x \in X$. These define the following operators in $B(X)$: $\int_0^m f(t)S(t)\, dt$ and $\int_0^\infty f(t)S(t)\, dt$ (each denoting $x \mapsto \int_0^a f(t)S(t)x\, dt$, $0 \le a \le \infty$). To prove $\int_0^\infty f(t)S(t)\, dt$ is in $\mathcal{B}$ with $\| \int_0^\infty f(t)S(t)\, dt\|_{\mathcal{B}} \le \int_0^\infty |f(t)|\, g(t)\, dt$, it suffices—by the strong convergence property—to show that

$$\Big\| \int_0^m f(t)S(t)\, dt \Big\|_{\mathcal{B}} \le \int_0^m |f(t)|\, g(t)\, dt \quad \text{for each } m \in \mathbb{N}.$$

To show this, fix $m \in \mathbb{N}$. For $n \in \mathbb{N}$ define $V_n^m \in \mathcal{B}$ to be $\sum_{i=1}^n \frac{m}{n} f(\tau_i)S(\tau_i)$, where $|f(\tau_i)|\, g(\tau_i)$ is the minimum of $|f(t)|\, g(t)$ on $[(i-1)m/n, im/n]$. For each $x \in X$, $V_n^m(x)$ is a Riemann sum; thus $\lim_{n\to\infty} V_n^m(x) = \int_0^m f(t)S(t)x\, dt$. We also have $\|V_n^m\|_{\mathcal{B}} \le \sum_{i=1}^n \frac{m}{n} |f(\tau_i)|\, g(\tau_i) \le \int_0^m |f(t)|\, g(t)\, dt$. These two facts, and the strong convergence property on $\mathcal{B}$, imply $\| \int_0^m f(t)S(t)\, dt\|_{\mathcal{B}} \le \int_0^m |f(t)|\, g(t)\, dt$.

The concept of an integrated semigroup was introduced to provide a framework for addressing Cauchy problems not governed by C_0 semigroups and it has since been developed in a number of articles. For relevant information on this subject, see [1] and [12].

Let A be a linear operator on X. If for some $n \in \mathbb{N}$, and constants M, ω there is a strongly continuous family $\{S(t)\}_{t\ge 0}$ in $B(X)$ with $\|S(t)\| \le Me^{\omega t}$ for all $t \ge 0$ such that $R(\lambda, A)$ exists for $\lambda > \omega$ and is given by

$$R(\lambda, A)x = \lambda^n \int_0^\infty e^{-\lambda t} S(t)x\, dt \quad \text{for } x \in X,$$

then A is called the generator of the *n-times integrated semigroup* $\{S(t)\}_{t\ge 0}$.

We proceed with a generation theorem for integrated semigroups affiliated with $\mathcal{B}$. In the case where $\mathcal{B} = B(X)$ it is due to W. Arendt [1] and extends the Hille–Yosida Theorem. For more general algebras, it identifies generators, A, of semigroups in $\mathcal{B}$ with the condition that $R(\lambda, A)/\lambda^n$ is a Laplace transform (in the sense of [1]) in $\mathcal{B}$.

THEOREM 1.2. *Assume $\mathcal{B}$ has the SCP relative to X. Let $n \in \mathbb{N}\cup\{0\}$, $\omega \in \mathbb{R}$ and $M \geq 0$. For a closed linear operator A, the following are equivalent:*

(1) *A generates an $(n+1)$-times integrated semigroup $\{S(t)\}_{t\geq 0} \subseteq \mathcal{B}$ which is exponentially bounded in the $\mathcal{B}$-norm and satisfies*

$$\limsup_{h\downarrow 0}(1/h)\|S(t+h)-S(t)\|_{\mathcal{B}} \leq Me^{\omega t} \quad (t\geq 0);$$

(2) *there exists $a \geq \max\{\omega, 0\}$ such that $(a,\infty) \subseteq \varrho_{\mathcal{B}}(A)$ and*

$$\|(\lambda-\omega)^{k+1}[R(\lambda,A)/\lambda^n]^{(k)}/k!\|_{\mathcal{B}} \leq M \quad \text{for all } \lambda > a,\ k = 0,1,\ldots$$

Proof. (2)⇒(1). Using Widder's classical theorem on Laplace transforms of real-valued functions, Corollary 1.2 of [1] defines a map $S : [0,\infty) \to \mathcal{B}$ that will satisfy the inequality in (1). The map S is defined by observing that for each $\varphi \in \mathcal{B}^*$, there exists $f(\cdot,\varphi) \in L^\infty[0,\infty)$ with

$$\langle R(\lambda,A)/\lambda^n, \varphi\rangle = \int_0^\infty e^{-\lambda t} f(t,\varphi)\,dt \quad \text{and} \quad |f(t,\varphi)| \leq Me^{\omega t}\|\varphi\|;$$

hence the equation $\langle S(t),\varphi\rangle = \int_0^t f(s,\varphi)\,ds$ defines a map $S : [0,\infty) \to \mathcal{B}$ with

$$\langle R(\lambda,A)/\lambda^{n+1},\varphi\rangle = \lambda\int_0^\infty e^{-\lambda t}\langle S(t),\varphi\rangle\,dt \quad \text{for all } \varphi\in\mathcal{B}^*.$$

Using the definition of $S(t)$, a series of calculations shows: $\|S(t+h)-S(t)\|_{\mathcal{B}} \leq M\max\{e^{\omega(t+h)}, e^{\omega t}\}\cdot h$ $(t,h\geq 0)$ (showing the inequality holds); S is continuous in the $\mathcal{B}$-norm and hence is strongly continuous; and, there exists an $M' \geq 0$ such that $\|S(t)\|_{\mathcal{B}} \leq M'e^{\omega' t}$, where $\omega' = \max\{0,\omega\}$. Thus $R(\lambda,A)x = \lambda^{n+1}\int_0^\infty e^{-\lambda t}S(t)x\,dt$ exists for all $x\in X$, and hence A is the generator of the $(n+1)$-times integrated semigroup $\{S(t)\}_{t\geq 0}$.

Note: The map S can be defined even if $\mathcal{B}$ is merely a Banach space (see [1]). This fact will be used in the proof of Theorem 3.1.

(1)⇒(2). Let $\omega' = \max\{0,\omega\}$. By hypothesis, for $x\in X$ and $\lambda > \omega'$ we have

$$\frac{R(\lambda,A)}{\lambda^{n+1}}x = \int_0^\infty e^{-\lambda t}S(t)x\,dt,$$

where $S(t)$ is strongly continuous and $\|S(t)\|_{\mathcal{B}} \leq M'e^{\omega' t}$ for some $M'\geq 0$. By Lemma 1.1 with $f(t) = \lambda e^{-\lambda t}$ and $g(t) = M'e^{\omega' t}$, the operator $R(\lambda,A)/\lambda^n = \int_0^\infty \lambda e^{-\lambda t}S(t)\,dt$ is an element of $\mathcal{B}$ for $\lambda\geq\omega'$. If $\lambda>\omega'$, then $(R(\lambda,A)/\lambda^n)x$ is infinitely differentiable as a map from $\mathbb{R}$ into X. We need to show $R(\lambda,A)/\lambda^n$ is infinitely differentiable as a map from $\mathbb{R}$ into $\mathcal{B}$ and that the derivatives satisfy the inequality in (2). Let $h(\lambda,t) = \lambda e^{-\lambda t}$. Lemma 1.1 with

$$f(t) = \frac{\partial^{k-1}h(\lambda+1/n,t)}{\partial\lambda^{k-1}} - \frac{\partial^{k-1}h(\lambda,t)}{\partial\lambda^{k-1}}$$

and $g(t) = M'e^{\omega' t}$ implies

$$w_n = \frac{R(\lambda+1/n,A)-R(\lambda,A)}{1/n}$$

is an element of $\mathcal{B}$. An argument using the SCP shows w_n converges to

$$\left[\frac{R(\lambda,A)}{\lambda^n}\right]^{(k)} = \int_0^\infty \frac{\partial^k h(\lambda,t)}{\partial\lambda^k}S(t)\,dt \in \mathcal{B}.$$

Lastly, applying Lemma 1.1 with $f(t) = \partial^k h(\lambda,t)/\partial\lambda^k$ and $g(t) = Me^{\omega t}t$ shows that

$$\left\|\left[\frac{R(\lambda,A)}{\lambda^n}\right]^{(k)}\right\|_{\mathcal{B}} \leq \frac{Mk!}{(\lambda-\omega)^{k+1}}.$$

If the closed operator A is densely defined, we have the following theorem (cf. [1, Theorem 4.3]). In the case $n=0$, it is a generation theorem for C_0 semigroups affiliated with $\mathcal{B}$, and in that case is a stronger version of Theorems 33 and 34 in [6] (which address contraction semigroups). [Note: the results just mentioned include the concept of a cone C of operators in $B(X)$ in which the semigroup and the resolvent operators reside. A version of Lemma 1.1, where the function S maps from $[0,\infty)$ into C, can be used to prove the analogous results here. However, the only examples that the authors are aware of are cones contained in the set of positive operators on a Banach lattice X. Since a semigroup is positive if and only if its generator is resolvent positive (i.e., $R(\lambda,A)\geq 0$ for all sufficiently large λ), the added hypothesis seems unnecessary.]

THEOREM 1.3. *Assume $\mathcal{B}$ has the SCP relative to X. Let A be a densely defined operator such that $(a,\infty)\subseteq\varrho_{\mathcal{B}}(A)$ for some $a\geq 0$. For $n\in\mathbb{N}\cup\{0\}$, $\omega\leq a$ and $M\geq 0$, the following are equivalent:*

(1) *A generates an n-times integrated semigroup $\{T(t)\}_{t\geq 0}$ satisfying*

$$\|T(t)\|_{\mathcal{B}} \leq Me^{\omega t} \quad (t\geq 0);$$

(2) *$\|(\lambda-\omega)^{k+1}[R(\lambda,A)/\lambda^n]^{(k)}/k!\|_{\mathcal{B}} \leq M$ for all $\lambda > a$, $k=0,1,\ldots$*

Proof. (1)⇒(2). This is the same as the proof of (1)⇒(2) in Theorem 1.2 except for letting $h(\lambda,t) = e^{-\lambda t}$ and $g(t) = Me^{\omega t}$.

(2)⇒(1). By Theorem 1.2, A generates an $(n+1)$-times integrated semigroup $\{S(t)\}_{t\geq 0}$. Proposition 3.3 and Corollary 3.4 of [1] imply $S(\cdot)x$ is in $C([0,\infty),X)$ for all $x\in X$, and that $T(t)x = \frac{d}{dt}S(t)x$ defines a strongly continuous family, $\{T(t)\}_{t\geq 0}$, of linear operators. Part (1) of Theorem 1.2 and the SCP imply that $\{T(t)\}_{t\geq 0} \subseteq \mathcal{B}$ and $\|T(t)\|_{\mathcal{B}} \leq Me^{\omega t}$. Finally, $R(\lambda, A)x = \lambda^{n+1}\int_0^\infty e^{-\lambda t}S(t)x\,dt = \lambda^n \int_0^\infty e^{-\lambda t}T(t)x\,dt$, for $\lambda > a$.

2. Spectral theory. Spectral and Fredholm theory of operators relative to some specific algebras have been studied in [2], [4], [10], and [15]. The object of [6] is the study of spectral and Fredholm theory relative to $\mathcal{B}$ of an operator that is affiliated with $\mathcal{B}$. In this section, we briefly present the basic facts involving the spectral and essential-spectral properties of a C_0 semigroup and its generator relative to $\mathcal{B}$.

If A is a closed operator affiliated with $\mathcal{B}$, denote the spectrum of A relative to $\mathcal{B}$ by $\sigma_{\mathcal{B}}(A) \equiv \mathbb{C} \setminus \varrho_{\mathcal{B}}(A)$. It is clear that $\sigma(A) \subseteq \sigma_{\mathcal{B}}(A)$. Also note that if $R(\lambda, A) \in \mathcal{B}$, then

$$\sigma_{\mathcal{B}}(A) = \{(\lambda-\mu)^{-1} : \mu \in \sigma_{\mathcal{B}}(R(\lambda, A)),\ \lambda \neq \mu\}, \tag{2.1}$$

and so $\varrho_{\mathcal{B}}(A)$ is open and $\sigma_{\mathcal{B}}(A)$ is closed [6, Theorem 2].

In general, the spectral properties relative to $\mathcal{B}$ can be quite different from those relative to $B(X)$. A useful relationship between the spectral properties relative to $\mathcal{B}$ and those relative to $B(X)$ is the fact that if $T \in \mathcal{B}$, then every component of $\sigma_{\mathcal{B}}(T)$ intersects $\sigma(T)$ [5,Theorem 4.5]. This fact combined with (2.1) implies that the analogous property holds for the $\mathcal{B}$-spectrum of a closed operator on X affiliated with $\mathcal{B}$.

In studying relationships between the spectrum of $T(t)$ and that of the generator A, the following identities are useful [11, p. 14]:

$$(2.2)\quad \begin{aligned} (e^{t\lambda} - T(t))x &= (\lambda - A)\int_0^t e^{(t-s)\lambda}T(s)x\,ds, \quad \lambda \in \mathbb{C},\ x \in X;\\ (e^{t\lambda} - T(t))x &= \int_0^t e^{(t-s)\lambda}T(s)(\lambda - A)x\,ds, \quad \lambda \in \mathbb{C},\ x \in D(A). \end{aligned}$$

The first theorem is a familiar spectral mapping inclusion for $\mathcal{B}$.

THEOREM 2.1. *Assume $\mathcal{B}$ has the SCP relative to X. If A is the generator of a C_0 semigroup $\{T(t)\}_{t\geq 0}$ in $\mathcal{B}$ with $\|T(t)\|_{\mathcal{B}} \leq Me^{\omega t}$ for some $M \geq 0$, $\omega \in \mathbb{R}$, then*

$$e^{t\sigma_{\mathcal{B}}(A)} \subseteq \sigma_{\mathcal{B}}(T(t)) \quad \textit{for } t \geq 0\,.$$

Proof. The theorem is proved for the case $\mathcal{B} = B(X)$ in [14, Chapter 2, Theorem 2.3]. The only addition to the proof that is needed is to verify that $B_\lambda(t)$ is an element of $\mathcal{B}$ $(t \geq 0)$ where $B_\lambda(t)x = \int_0^t e^{\lambda(t-s)}T(s)x\,ds$ for all $x \in X$. This can be shown by applying Lemma 1.1 with the interval $[0,\infty)$ replaced by $[0,t)$ and with $g(s) = Me^{\omega s}$ and $f(s) = e^{\lambda(t-s)}$.

In the study of semigroups the Browder essential spectrum is commonly considered. For a detailed discussion on the role of essential spectra in semigroup theory and applications to population dynamics, see [3] and [18]. We start by looking at the relevant Fredholm theory. General Fredholm theory in a primitive Banach algebra is defined and studied in [7] where the theory is developed relative to the socle; see also [6] for details concerning the definitions that follow. As pointed out in [6], $\mathcal{B}$ must contain sufficiently many operators of finite rank in order to have a useful Fredholm theory. Therefore, from now on we assume that $\mathcal{B}$ has the following property:

(#) there exists a total subspace Y in X^* such that

$$\alpha \otimes x \in \mathcal{B} \text{ for all } x \in X,\ \alpha \in Y$$

(where $\alpha \otimes x(z) = \alpha(z)x$ $(z \in X)$). Under this assumption, $\mathcal{B}$ is a primitive Banach algebra and the socle of $\mathcal{B}$ can be identified as $\mathcal{F}_{\mathcal{B}} \equiv F(X) \cap \mathcal{B}$ [6, Propositions 3 and 4].

Now let $\mathcal{K}_{\mathcal{B}}$ be any closed nonzero inessential ideal of $\mathcal{B}$; for example, $\mathcal{K}_{\mathcal{B}}$ may be the closure of $\mathcal{F}_{\mathcal{B}}$ in $\mathcal{B}$. In the primitive Banach algebra $\mathcal{B}$, Fredholm theory relative to $\mathcal{F}_{\mathcal{B}}$ is equivalent to Fredholm theory relative to $\mathcal{K}_{\mathcal{B}}$. Define the set of *Fredholm elements* in $\mathcal{B}$ as $\Phi(\mathcal{B}) = \{T \in \mathcal{B} : T$ is invertible in $\mathcal{B}$ relative to $\mathcal{F}_{\mathcal{B}}$ (or $\mathcal{K}_{\mathcal{B}}$)$\}$, and the *$\mathcal{B}$-Fredholm spectrum* of $T \in \mathcal{B}$ as $\omega_{\mathcal{B}}(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \Phi(\mathcal{B})\}$. One can extend the $\mathcal{B}$-Fredholm theory to a closed operator, A, on X: A is *$\mathcal{B}$-Fredholm* (again, denoted $A \in \Phi(\mathcal{B})$) if there exist $R, S \in \mathcal{B}$ and $F, G \in \mathcal{K}_{\mathcal{B}}$ with

$$AR = I - F \text{ on } X, \qquad \text{and} \qquad SA = I - G \text{ on } D(A)\,. \tag{2.3}$$

Set $\omega_{\mathcal{B}}(A) = \{\lambda \in \mathbb{C} : \lambda - A \notin \Phi(\mathcal{B})\}$.

We extend the usual definition of the Browder spectrum in a Banach algebra (see [7]) to closed operators: A *$\mathcal{B}$-Riesz point* of a closed operator A is a point $\lambda \in \mathbb{C}$ satisfying either

(i) $\lambda \notin \sigma_{\mathcal{B}}(A)$; or
(ii) λ is isolated in $\sigma_{\mathcal{B}}(A)$ and $\lambda - A \in \Phi(\mathcal{B})$.

Define the *Browder spectrum* of A relative to $\mathcal{B}$ as $\beta_{\mathcal{B}}(A) = \{\lambda \in \mathbb{C} : \lambda$ is not a $\mathcal{B}$-Riesz point in $\sigma_{\mathcal{B}}(A)\}$. Then $\omega_{\mathcal{B}}(A) \subseteq \beta_{\mathcal{B}}(A) \subseteq \sigma_{\mathcal{B}}(A)$. We now show that spectral mapping inclusions hold, relative to $\mathcal{B}$, for the Browder and Fredholm spectra. The arguments here simplify somewhat the proofs of these properties in the case of $\mathcal{B} = B(X)$.

THEOREM 2.2. *Assume $\mathcal{B}$ has the SCP relative to X. If A is the generator of a C_0 semigroup $\{T(t)\}_{t\geq 0}$ in $\mathcal{B}$ that is exponentially bounded in the $\mathcal{B}$-norm, then for $t \geq 0$,*

(1) $e^{t\omega_{\mathcal{B}}(A)} \subseteq \omega_{\mathcal{B}}(T(t))$;
(2) $e^{t\beta_{\mathcal{B}}(A)} \subseteq \beta_{\mathcal{B}}(T(t))$.

Proof. (1) Assume $e^{t\lambda} \notin \omega_{\mathcal{B}}(T(t))$. Then there exist $W, V \in \mathcal{B}$ and $F, G \in \mathcal{K}_{\mathcal{B}}$ so that $(e^{t\lambda} - T(t))V = I - F$ and $W(e^{t\lambda} - T(t)) = I - G$. Combining these with (2.2) gives

$$(\lambda - A) \int_0^t e^{(t-s)\lambda} T(s) Vx\, ds = (I - F)x \quad (x \in X),$$

$$W \int_0^t e^{(t-s)\lambda} T(s)(\lambda - A)x\, ds = (I - G)x \quad (x \in D(A)).$$

Setting $S \equiv W(\int_0^t e^{(t-s)\lambda} T(s)\, ds)$ and $R \equiv (\int_0^t e^{(t-s)\lambda} T(s)\, ds)V$ gives operators R and S in $\mathcal{B}$ which satisfy (2.3) (with A replaced by $\lambda - A$). Therefore, $\lambda \notin \omega_{\mathcal{B}}(A)$.

(2) By Theorem 2.1, if $e^{t\lambda} \notin \sigma_{\mathcal{B}}(T(t))$, then $\lambda \notin \beta_{\mathcal{B}}(A)$. Assume $e^{t\lambda} \in \sigma_{\mathcal{B}}(T(t)) \setminus \beta_{\mathcal{B}}(T(t))$. Then $e^{t\lambda} \notin \omega_{\mathcal{B}}(T(t))$ and so by (1), $\lambda - A \in \Phi(\mathcal{B})$. By assumption, $e^{t\lambda}$ is isolated in $\sigma_{\mathcal{B}}(T(t))$. Properties of the exponential show that λ is then isolated in $\sigma_{\mathcal{B}}(A)$. Therefore, $\lambda \notin \beta_{\mathcal{B}}(A)$, and so (2) holds.

In $B(X)$, the spectral mapping property $e^{t\sigma(A)} = \sigma(T(t)) \setminus \{0\}$ fails to hold in general. However, such an equality does hold for the residual and point spectra [11, p. 85]; hence, the continuous spectrum, $\sigma_c(T(t))$, contains any points for which it fails. Since $\sigma_c(T(t)) \subseteq \omega(T(t))$, these points are in the Fredholm spectrum. By [15, Theorem 5], $\sigma_{\mathcal{B}}(T(t)) \setminus \omega_{\mathcal{B}}(T(t)) \subseteq \sigma(T(t)) \setminus \omega(T(t))$; hence, with the possible exception of zero, $\sigma_{\mathcal{B}}(T(t)) \setminus \omega_{\mathcal{B}}(T(t)) \subseteq \sigma(T(t)) \setminus \sigma_c(T(t)) \subseteq e^{t\sigma(A)} \subseteq e^{t\sigma_{\mathcal{B}}(A)}$. This combined with Theorem 2.2(1) gives

$$\sigma_{\mathcal{B}}(T(t)) \setminus \omega_{\mathcal{B}}(T(t)) \subseteq e^{t(\sigma_{\mathcal{B}}(A) \setminus \omega_{\mathcal{B}}(A))}. \tag{2.4}$$

A similar argument shows (cf. [3, Proposition 5])

$$\sigma_{\mathcal{B}}(T(t)) \setminus \beta_{\mathcal{B}}(T(t)) \subseteq e^{t(\sigma_{\mathcal{B}}(A) \setminus \beta_{\mathcal{B}}(A))}. \tag{2.5}$$

For completeness, we also observe that

$$\beta_{\mathcal{B}}(T(t)) \setminus \omega_{\mathcal{B}}(T(t)) \subseteq e^{t(\beta_{\mathcal{B}}(A) \setminus \omega_{\mathcal{B}}(A))}. \tag{2.6}$$

For, if $e^{t\lambda} \in \beta_{\mathcal{B}}(T(t)) \setminus \omega_{\mathcal{B}}(T(t))$, then $e^{t\lambda} - T(t) \in \Phi(\mathcal{B})$. The point $e^{t\lambda}$ is interior to $\beta_{\mathcal{B}}(T(t))$ since the set $\{\mu \in \mathbb{C} : \mu - T(t) \in \Phi(\mathcal{B})\}$ is open. Let V be an open neighborhood of $e^{t\lambda}$ contained in $\beta(T(t)) \setminus \omega(T(t))$. Then $V \subseteq \sigma_{\mathcal{B}}(T(t)) \setminus \omega_{\mathcal{B}}(T(t)) \subseteq e^{t\sigma_{\mathcal{B}}(A)}$. Hence V is an open neighborhood of $e^{t\lambda}$ in $e^{t\sigma_{\mathcal{B}}(A)}$, and so λ is a limit point of $\sigma_{\mathcal{B}}(A)$. Therefore $\lambda \in \beta_{\mathcal{B}}(A)$. This shows $\beta_{\mathcal{B}}(T(t)) \setminus \omega_{\mathcal{B}}(T(t)) \subseteq e^{t\beta_{\mathcal{B}}(A)}$. Inclusion (2.6) now follows from Theorem 2.2(1).

3. Some algebras of operators. One could consider closed operators affiliated with the algebra $B^r(X)$ of all regular operators on a Banach lattice X. Though this is not pursued here, $B^r(X)$ is an algebra in which the spectral and Fredholm theory is understood [2] and, like each of the algebras mentioned in this section, can be seen to satisfy the assumptions on $\mathcal{B}$ in Section 1 and the condition (#) in Section 2.

Jörgens algebras. Let X and Y be Banach spaces which, along with a nondegenerate bilinear form $\langle\cdot,\cdot\rangle$, comprise a dual system $\langle X, Y\rangle$ (as defined in [10, p. 43]). A set $S \subseteq X$ is said to be *Y-total* in X if $S^\perp \equiv \{y \in Y : \langle x, y\rangle = 0 \text{ for all } x \in S\} = \{0\}$. Let $\mathcal{A} \equiv \mathcal{A}(X, Y)$ be the set of all $T \in B(X)$ such that there exists a $T^\dagger \in B(Y)$ satisfying $\langle Tx, y\rangle = \langle x, T^\dagger y\rangle$ for all $x \in X$, $y \in Y$. When endowed with the norm $\|T\|_{\mathcal{A}} = \max\{\|T\|, \|T^\dagger\|\}$, $\mathcal{A}$ is a Banach algebra of operators; the spectral and Fredholm theory of an operator in $\mathcal{A}$ is developed in [4] and [10].

Now let A be a closed operator on X whose domain is Y-total in X. For such an operator, define the domain of $A^\dagger$ as the set, $D(A^\dagger)$, of all $y \in Y$ for which there exists $w \in Y$ satisfying $\langle Ax, y\rangle = \langle x, w\rangle$ for all $x \in D(A)$. For $y \in D(A^\dagger)$ and w as above, set $A^\dagger y = w$. Then $A^\dagger$ is well defined and, by definition, $\langle Ax, y\rangle = \langle x, A^\dagger y\rangle$ for $x \in D(A)$, $y \in D(A^\dagger)$. If in addition A is affiliated with $\mathcal{A}(X, Y)$, then $A^\dagger$ is a closed operator on Y and $D(A^\dagger)$ is X-total in Y [6]. Finally, note that the map $T \mapsto T \oplus T^\dagger$ defines a linear isometry of $\mathcal{A}(X, Y)$ into $(B(X \oplus Y), \|\cdot\|_{\max})$; denote the image of $\mathcal{A}$ under this map by $\widetilde{\mathcal{A}}$. A calculation shows that $\widetilde{\mathcal{A}}$ has the SCP on $X \oplus Y$ [6, Proposition 32].

As an example, if X is nonreflexive and if A is an operator on X^* with $D(A)$ X-total in X^*, then relative to the dual system $\langle X^*, X\rangle$ (where $\langle y, x\rangle = y(x)$, for $y \in X^*$, $x \in X$), $A^\dagger$ as defined above is called the *preconjugate* of A (see [8]).

The above set-up, along with a development of the spectral and Fredholm theory of a closed operator affiliated with $\mathcal{A}(X, Y)$, is given in [6, Section 3]. In particular, it is shown that if A is a closed operator such that $D(A)$ is Y-total in X and if A is affiliated with $\mathcal{A}(X, Y)$, then $\sigma_{\mathcal{A}}(A) = \sigma(A) \cup \sigma(A^\dagger)$, and $\omega_{\mathcal{A}}(A) = \omega(A) \cup \omega(A^\dagger) \cup \omega_1$ where $\omega_1 = \{\lambda \in \mathbb{C} : \lambda - A$ and $\lambda - A^\dagger$ are Fredholm, but [index of $\lambda - A$] + [index of $\lambda - A^\dagger$] $\neq 0\}$. Here, $\omega(A)$ denotes the Fredholm spectrum of an operator A (not necessarily densely defined) from $\overline{D(A)}$ into X.

A similar proof, along with [4, Theorem 2.5], can be used to check that $\beta_{\mathcal{A}}(A) = \beta(A) \cup \beta(A^\dagger)$. It can also be shown that $\omega_{\mathcal{A}}(A) = \omega(A) \Rightarrow \beta_{\mathcal{A}}(A) = \beta(A) \Leftrightarrow \sigma_{\mathcal{A}}(A) = \sigma(A)$ (a similar statement can be made involving $A^\dagger$; cf. [15, Corollary 7]).

The techniques used in the proof of Theorem 1.2 can be used again to prove the following theorem. If $Y = X^*$ in the statement below (so that A is densely defined), then $A^\dagger$ is the usual adjoint operator, A^*, which is known to generate an $(n+1)$-times integrated semigroup [1, Corollary 4.4].

THEOREM 3.1. *Let A be the generator of an n-times integrated semigroup $\{S(t)\}_{t\ge0}$ on X. Assume that $D(A)$ is Y-total in X and there exists some $a\in\mathbb{R}$ with $(a,\infty)\subseteq\varrho_{\mathcal{A}}(A)$. If*

$$\|(\lambda-\omega)^{k+1}[R(\lambda,A)/\lambda^n]^{(k)}/k!\|_{\mathcal{A}}\le M\qquad\text{for all } \lambda>a,\ k=0,1,\ldots,$$

then $A^\dagger$ generates an $(n+1)$-times integrated semigroup $\{E(t)\}_{t\ge0}$ on Y. Further, $E(t)=\int_0^t S(s)^\dagger\,ds$, where $\int_0^t S(s)^\dagger\,ds$ is defined by $\langle x,\int_0^t S(s)^\dagger y\,ds\rangle$ $=\int_0^t\langle x,S(s)^\dagger y\rangle\,ds$.

Proof. Since $\widetilde{\mathcal{A}}$ is isometrically linearly isomorphic to $\mathcal{A}$, $\widetilde{R}(\lambda,A)=R(\lambda,A)\oplus R(\lambda,A)^\dagger$ is differentiable and by hypothesis

$$\|(\lambda-\omega)^{k+1}[\widetilde{R}(\lambda,A)/\lambda^n]^{(k)}/k!\|_{\max}\le M\qquad\text{for all } \lambda\ge a,\ k=0,1,\ldots$$

As pointed out at the end of the proof of (2)⇒(1) in Theorem 1.2, since $(\widetilde{\mathcal{A}},\|\cdot\|_{\max})$ is a Banach space, there exists $\widetilde{V}:[0,\infty)\to\widetilde{\mathcal{A}}\subset B(X\oplus Y)$ with

$$\widetilde{R}(\lambda,A)(x\oplus y)=\lambda^{n+1}\int_0^\infty e^{-\lambda t}\widetilde{V}(t)(x\oplus y)\,dt\qquad\text{for all } x\oplus y\in X\oplus Y$$

and $\widetilde{V}$ is exponentially bounded in the $\|\cdot\|_{\max}$ norm. Since $\widetilde{\mathcal{A}}$ is the image of $\mathcal{A}$, there exists $\{V(t)\}_{t\ge0}$ in $B(X)$ with $\widetilde{V}(t)=V(t)\oplus V(t)^\dagger$, and both $V(t)$ and $V(t)^\dagger$ are exponentially bounded in their respective norms. Thus,

$$(R(\lambda,A)\oplus R(\lambda,A)^\dagger)(x\oplus y)=\Big(\lambda^{n+1}\int_0^\infty e^{-\lambda t}V(t)x\,dt\Big)\oplus\Big(\lambda^{n+1}\int_0^\infty e^{-\lambda t}V(t)^\dagger y\,dt\Big).$$

By [6], $R(\lambda,A^\dagger)=R(\lambda,A)^\dagger$, so the previous sentence implies that A is the generator of the $(n+1)$-times integrated semigroup $\{V(t)\}_{t\ge0}$ and $A^\dagger$ is the generator of the $(n+1)$-times integrated semigroup $\{V(t)^\dagger\}_{t\ge0}$. Since A generates $S(t)$, it follows that

$$R(\lambda,A)x=\lambda^n\int_0^\infty e^{-\lambda t}S(t)x\,dt=\lambda^{n+1}\int_0^\infty e^{-\lambda t}\int_0^t S(s)x\,ds\,dt.$$

The uniqueness of the Laplace transform implies $\int_0^t S(s)\,ds=V(t)$. To prove the last statement of the theorem, first note that $\langle x,V(t)^\dagger y\rangle=\langle V(t)x,y\rangle=\langle\int_0^t S(s)x\,ds,y\rangle$. Since $s\mapsto\langle S(s)x,y\rangle$ is integrable, [9, 3.7.12] implies $\langle\int_0^t S(s)x\,ds,y\rangle=\int_0^t\langle S(s)x,y\rangle\,ds=\int_0^t\langle x,S(s)^\dagger y\rangle\,ds$.

As a special case of a general dual system, consider the situation where A is a closed, densely defined operator on X. Let Y be a Banach space continuously embedded in X^* (e.g., Y is a closed subspace in X^*). If $D(A^*)\subseteq Y\subseteq X^*$ for some Banach space Y, then one can form the dual system $\langle X,Y\rangle$ which inherits the obvious bilinear form from $\langle X,X^*\rangle$: $\langle x,y\rangle=y(x)$ for $x\in X$, $y\in Y$. Since A is densely defined, $D(A^*)$ is X-total in X^* [17, p. 177], and hence this form is nondegenerate.

LEMMA 3.2. *Let A be a closed, densely defined operator on X and let Y be a closed subspace in X^* satisfying $D(A^*)\subseteq Y\subseteq X^*$. Then $A^\dagger$ is defined relative to the dual system $\langle X,Y\rangle$ and if $R(\lambda,A)\in B(X)$, then $R(\lambda,A^\dagger)\in B(Y)$. Further, $\|R(\lambda,A^\dagger)\|\le\|R(\lambda,A^*)\|$.*

Proof. We argue as in [6, Proposition 37] where $Y=\overline{D(A^*)}$. First note that for all y in Y, $A^*R(\lambda,A^*)y$ is in Y. Indeed, for $\alpha\in X^*$, $A^*R(\lambda,A^*)\alpha=-\lambda R(\lambda,A^*)\alpha-\alpha$. Since $R(\lambda,A^*)\alpha\in D(A^*)$, it follows that for y in Y, $A^*R(\lambda,A^*)y=A^*R(\lambda,A^*)\alpha=-\lambda R(\lambda,A^*)y-y$, which is an element of Y.

Now let $x\in D(A)$ and $y\in Y$. Then $\langle Ax,R(\lambda,A^*)y\rangle=\langle x,A^*R(\lambda,A^*)y\rangle$ $=\langle x,z\rangle$ for some $z\in Y$. Therefore, if $R(\lambda,A)\in B(X)$, then $R(\lambda,A^*)y\in D(A^\dagger)$ for all $y\in Y$. Now, $A^\dagger R(\lambda,A^*)y=A^*R(\lambda,A^*)y$, and so

$$(\lambda-A^\dagger)R(\lambda,A^*)y=(\lambda-A^*)R(\lambda,A^*)y=y\qquad(y\in Y).$$

The final inequality is clear.

We point out that if $Y=\overline{D(A^*)}$ (this is the usual "sun dual," $X^\odot$), then $A^\dagger$ is densely defined and $A^\dagger$ (which in this case is often denoted by $A^\odot$) generates a C_0 semigroup on Y.

COROLLARY 3.3. *Let A generate a C_0 semigroup $\{T(t)\}_{t\ge0}$ on X. If Y is a closed subspace of X^* satisfying $D(A^*)\subseteq Y\subseteq X^*$, then $A^\dagger$ generates an integrated semigroup $\{S(t)\}_{t\ge0}$ on Y where $S(t)=\int_0^t T(s)^\dagger\,ds$. Moreover, $\sigma(A^\dagger)\subseteq\sigma(A)$ and $\beta(A^\dagger)\subseteq\beta(A)$.*

Proof. If A generates a C_0 semigroup $\{T(t)\}_{t\ge0}$ on X, then there exist $M>0$ and $\omega\in\mathbb{R}$ such that $\|R(\lambda,A)^k\|\le M/(\lambda-\omega)^k$ for $k=1,2,\ldots,$ for all $\lambda>\omega$. By the preceding lemma, $\varrho(A)\subseteq\varrho(A^\dagger)$ and $\|R(\lambda,A^\dagger)\|\le\|R(\lambda,A^*)\|=\|R(\lambda,A)\|$. By Theorem 3.1, $A^\dagger$ generates the integrated semigroup $\{S(t)\}_{t\ge0}$ as indicated.

Finally, as noted prior to the theorem, $\sigma_{\mathcal{A}}(A)=\sigma(A)\cup\sigma(A^\dagger)$, $\beta_{\mathcal{A}}(A)=\beta(A)\cup\beta(A^\dagger)$, and $\beta_{\mathcal{A}}(A)=\beta(A)\Leftrightarrow\sigma_{\mathcal{A}}(A)=\sigma(A)$. These combine to show $\beta(A^\dagger)\subseteq\beta(A)$.

We mention a few situations to which this corollary applies.

EXAMPLE 3.4. Let (Ω,μ) be a measure space and let $BC \equiv BC(\Omega)$ denote the space of bounded continuous functions on Ω. The condition that $\mu(U) > 0$ for any nonempty open set U in Ω implies that the embedding of BC into $L^\infty \equiv L^\infty(\Omega)$ is one-to-one. In this case, BC and $L^1 \equiv L^1(\Omega)$ form a dual system via the form

$$\langle f, g\rangle = \int_\Omega fg\,d\mu \quad (f \in BC,\ g \in L^1).$$

The above theorem shows that if A generates a C_0 semigroup $T(t)$ on BC with the property that $D(A^*) \subseteq L^1$, then $A^\dagger$ generates an integrated semigroup $\{S(t)\}_{t\ge 0}$ on L^1, and $S(t) = \int_0^t T(s)^\dagger\,ds$. Also, $\sigma(A^\dagger) \subseteq \sigma(A)$ and $\beta(A^\dagger) \subseteq \beta(A)$.

On the other hand, suppose A is a closed operator on BC such that $D(A)$ is L^1-total in BC. If $A^\dagger$ generates a C_0 semigroup, $\{T(t)\}_{t\ge 0}$, on L^1 and has the property that $R(\lambda, A^\dagger)^*(L^\infty) \subseteq BC$, then $D(A^{\dagger *}) \subseteq BC \subseteq L^\infty$. For $A^{\dagger\dagger}$ defined relative to the dual system $\langle L^1, BC\rangle$, then $D(A) \subseteq D(A^{\dagger\dagger})$. Also, by Lemma 3.2, $A^\dagger$ is affiliated with the algebra $\mathcal{A}(L^1, BC)$ and hence, by [6, Proposition 12], $A^{\dagger\dagger}$ is a closed operator. Thus $A^{\dagger\dagger}$ is a closed extension of A which, by Corollary 3.3, generates an integrated semigroup $\{S(t)\}_{t\ge 0}$ on BC, with $S(t) = \int_0^t T(s)^\dagger\,ds$. Theorem 19 of [6] shows that the spectral theories of $A^{\dagger\dagger}$ and $A^\dagger$ are related in the same strong way as those of A and A^*; i.e., $\sigma(A^\dagger) = \sigma(A^{\dagger\dagger})$, with similar equalities for the Browder and Fredholm spectra.

EXAMPLE 3.5. Semigroups on the Banach space $L^1(\mu, X)$ of Bochner integrable functions from a measure space (Ω,μ) into a Banach space X are considered in [13]. In this setting involving vector-valued functions, the set $L^\infty(\mu, X^*)$ is generally a proper subspace of $L^1(\mu, X)^*$. The main idea in [13] is that the translation semigroup generated by A on $E = L^1([0,1], X)$ has the property that $\overline{D(A^*)} \subseteq L^\infty([0,1], X^*)$ (this then allows a straightforward identification of the semigroup dual, $E^\odot$). The point we wish to make here is that the pair $\langle L^1(\mu, X), L^\infty(\mu, X^*)\rangle$ forms a dual system, as described above, defined via the form $\langle f, g\rangle = \int_\Omega \langle f(\omega), g(\omega)\rangle\,d\mu(\omega)$ (for $f \in L^1(\mu, X)$, $g \in L^\infty(\mu, X^*)$). So in general, if A is the generator of a C_0 semigroup $\{T(t)\}_{t\ge 0}$ on $L^1(\mu, X)$ with the property that $D(A^*) \subseteq L^\infty(\mu, X^*)$, then $A^\dagger$ generates an integrated semigroup $\{S(t)\}_{t\ge 0}$ on $L^\infty(\mu, X^*)$ given by $S(t) = \int_0^t T(s)^\dagger\,ds$. Moreover, $\sigma(A^\dagger) \subseteq \sigma(A)$ and $\beta(A^\dagger) \subseteq \beta(A)$.

Extensions of operators. Let X be a dense subset of a Banach space $(Y, \|\cdot\|_Y)$ and assume that $(X, \|\cdot\|_X)$ is a Banach space continuously embedded in Y. Let $\mathcal{E} \equiv \mathcal{E}(X, Y)$ be the set of all $T \in B(X)$ which have a continuous extension, $\overline{T} \in B(Y)$. With the norm $\|T\|_{\mathcal{E}} = \max(\|T\|, \|\overline{T}\|)$, $\mathcal{E}$ is a Banach algebra of operators. A straightforward calculation shows that $\mathcal{E}$ also has the SCP relative to X.

Applying Theorem 1.3 to $\mathcal{E}$ gives the following theorem which provides a sufficient condition for the existence of a minimal closed extension of a closed densely defined operator on X. The proof is similar to that of [6, Theorem 39].

THEOREM 3.6. *Let X and Y be as above and let A be a closed densely defined operator on X. If there exists an $\omega \in \mathbb{R}$ with $(\omega, \infty) \subseteq \varrho_{\mathcal{E}}(A)$ and*

$$\|(\lambda - \omega)^{k+1}[R(\lambda, A)/\lambda^n]^{(k)}/k!\|_{\mathcal{E}} \le M \quad \text{for all } \lambda > \omega,\ k = 0, 1, \ldots,$$

then A is the generator of an n-times integrated semigroup, $\{S(t)\}_{t\ge 0}$, and A has a minimal extension $\overline{A}$ which is the generator of the n-times integrated semigroup $\{\overline{S}(t)\}_{t\ge 0}$.

Proof. By Theorem 1.3 there exists an n-times integrated semigroup $\{S(t)\}_{t\ge 0}$ in $\mathcal{E}$. Hence the one-parameter family of operators, $\{\overline{S}(t)\}_{t\ge 0}$, exists. A calculation using the density of X shows that $\{\overline{S}(t)\}_{t\ge 0}$ is strongly continuous on Y. To show that $\{\overline{S}(t)\}_{t\ge 0}$ is an n-times integrated semigroup, we check that for all $y \in Y$ and $s, t \ge 0$,

$$\text{(1)}\quad \overline{S}(t)\overline{S}(s)y = \frac{1}{n-1}\Big[\int_t^{s+t} (s+t-r)^{n-1}\overline{S}(r)y\,dr - \int_0^s (s+t-r)^{n-1}\overline{S}(r)y\,dr\Big].$$

The strong continuity of $\{\overline{S}(t)\}_{t\ge 0}$ implies the integrals in (1) exist. Since $\{S(t)\}_{t\ge 0}$ is an n-times integrated semigroup on X, (1) holds for $x \in X$; the dominance of the X-norm implies (1) holds for $x \in X \subseteq Y$. The density of X in Y implies (1) holds on all of Y. Finally, the dominance of the X-norm implies that the generator, B, of $\{\overline{S}(t)\}_{t\ge 0}$ is a closed extension of A. Hence, $\overline{A}$ exists and by [6, Proposition 20], $(\lambda - \overline{A})^{-1}$ exists for $\lambda > \omega$. So for λ sufficiently large, both $(\lambda - \overline{A})^{-1}$ and $(\lambda - B)^{-1}$ exist. Thus $D(\overline{A})$ is equal to $D(B)$.

The following notation will be used in the discussion below (see [11]):

$$\gamma \equiv \gamma(T(t)) = \lim_{t\to\infty} \frac{1}{t}\log\|T(t)\|$$

is the *growth bound* of a semigroup $\{T(t)\}_{t\ge 0}$; the *essential growth bound* is

$$\gamma_e \equiv \gamma_e(T(t)) = \lim_{t\to\infty} \frac{1}{t}\log|||\pi T(t)|||,$$

where $\pi : B(X) \to B(X)/K(X)$ is the canonical projection and $|||\cdot|||$ denotes the norm on the quotient algebra; the *spectral bound* of the generator A is $S(A) = \sup\{\mathrm{Re}\,\lambda : \lambda \in \sigma(A)\}$. Finally, define the *essential-spectral bound* as $S_e(A) = \sup\{\mathrm{Re}\,\lambda : \lambda \in \beta(a)\}$.

In the study of asymptotics for problems in population dynamics, a commonly considered property of semigroups is that of asynchronous exponential growth. For definitions, background discussion, and equivalent characterizations of this property see [18] and [3]. A necessary and sufficient condition for a semigroup to have asynchronous exponential growth is for γ_e to be strictly less than γ and the boundary spectrum, $\sigma_0(A) \equiv \{\lambda \in \sigma(A) : \mathrm{Re}\,\lambda = S(A)\}$, to consist of a single eigenvalue of multiplicity one.

Now suppose X is densely embedded in a Hilbert space Y (as above), and A generates a C_0 semigroup $\{T(t)\}_{t\geq 0}$ on X which extends to a C_0 semigroup $\{\overline{T}(t)\}_{t\geq 0}$ on Y generated by the minimal closed extension, $\overline{A}$, of A. The following relates a spectral bound property of $\overline{A}$, on Y, to a growth bound property on $\{T(t)\}_{t\geq 0}$, on X. In particular, the condition on $\overline{A}$ implies the semigroup on X will not have asynchronous exponential growth.

PROPOSITION 3.7. *If $\overline{A}$ is self-adjoint on Y and if $S_e(\overline{A}) = S(\overline{A})$, then $\gamma_e(T(t)) = \gamma(T(t))$.*

Proof. Since $\overline{A}$ is self-adjoint, each $\overline{T}(t)$ is self-adjoint; hence $\sigma(\overline{T}(t)) \subseteq \sigma(T(t))$ and $\beta(\overline{T}(t)) \subseteq \beta(T(t))$ (see, e.g., [15, Section 3B]), and so $\sigma_{\mathcal{E}}(T(t)) = \sigma(T(t))$ and $\beta_{\mathcal{E}}(T(t)) = \beta(T(t))$. As in Theorem 5 of [15], one can check that $\sigma_{\mathcal{E}}(A) \setminus \beta_{\mathcal{E}}(A) \subseteq \sigma(\overline{A}) \setminus \beta(\overline{A})$. Suppose $\gamma_e(T(t)) < \gamma(T(t))$. Then there exists a Riesz point, λ, in $\sigma(T(t))$. By (2.5),

$$\lambda \in \sigma(T(t)) \setminus \beta(T(t)) = \sigma_{\mathcal{E}}(T(t)) \setminus \beta_{\mathcal{E}}(T(t)) \subseteq e^{t(\sigma_{\mathcal{E}}(A)\setminus\beta_{\mathcal{E}}(A))} \subseteq e^{t(\sigma(\overline{A})\setminus\beta(\overline{A}))}.$$

Since $\overline{A}$ is self-adjoint, λ is real. This would imply that $S_e(\overline{A}) < S(\overline{A})$.

We conclude by mentioning an algebra of interpolated operators on Lebesgue spaces. Let L^p denote the Banach space of p-integrable functions on a σ-finite measure space (Ω, μ). Fix $1 \leq p < s \leq \infty$. In the case $s = \infty$, L^s denotes the sup-norm closure of $L^p \cap L^s$. Let $\mathcal{B}_{ps}$ denote the algebra of all bounded operators T on $L^p \cap L^s$ (a Banach space with norm $\|\cdot\|_{ps} = \max(\|\cdot\|_p, \|\cdot\|_s)$) that have bounded extensions T_p and T_s to L^p and L^s respectively. By the Riesz Convexity Theorem, T has a bounded extension $T_r \in B(L^r)$ for all $r \in [p,s]$. With respect to the norm $\|T\|_{\mathcal{B}_{ps}} = \max\{\|T_p\|, \|T_s\|\}$, $\mathcal{B}_{ps}$ is a Banach algebra. The spectral and Fredholm theories of an operator in this algebra are developed in [10] and [5]. Properties of a closed operator on $L^p \cap L^s$ affiliated with $\mathcal{B}_{ps}$ are investigated in [6].

For each $r \in [p,s]$, $L^p \cap L^s$ is a dense subspace of L^r and so a result similar to Theorem 3.6 applies to this setting—here, A is a closed and densely defined operator on $L^p \cap L^s$. Let $\{T_r(t)\}_{t\geq 0}$ denote the semigroup generated by A_r, the minimal closed extension of A to L^r. In case $\mu(\Omega) < \infty$ and A_2 is self-adjoint ($p \leq 2 \leq s$), the previous discussion applies directly with Y replaced by L^2, X by L^r for $r > 2$, and $\mathcal{E}$ by $\mathcal{B}_{ps}$. If $1 \leq r \leq 2$, the fact that $\sigma(T_2(t)) \subseteq \sigma(T_r(t))$ allows the same argument to be used in this case as well (see [5]).

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