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Spectrum of multidimensional dynamical systems with positive entropy

by

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Abstract. Applying methods of harmonic analysis we give a simple proof of the multidimensional version of the Rokhlin–Sinaĭ theorem which states that a Kolmogorov $\mathbb{Z}^d$-action on a Lebesgue space has a countable Lebesgue spectrum. At the same time we extend this theorem to $\mathbb{Z}^\infty$-actions. Next, using its relative version, we extend to $\mathbb{Z}^\infty$-actions some other general results connecting spectrum and entropy.

1. Introduction. One of the important results in classical ergodic theory is the Rokhlin–Sinaĭ theorem which states that every Kolmogorov automorphism ($\mathbb{Z}$-action) of a Lebesgue space has a countable Lebesgue spectrum (cf. [RS]). This theorem has been extended to measure-preserving $\mathbb{Z}^d$-actions in [Ka]. The main tool used in the proofs of these theorems are perfect σ-algebras. The proof of their existence is complicated and it seems that it is very difficult to extend it to measure-preserving actions of general groups. It is worth mentioning that it is still an open question, asked by Thouvenot, whether Kolmogorov actions of any countable abelian group have a countable Haar spectrum.

In this paper we give a simple proof of the above mentioned multidimensional version of the Rokhlin–Sinaĭ theorem by a construction of two groups of unitary operators satisfying a commutation relation of the Weyl type. This method allows us also to extend this theorem to the case $d = \infty$.

Our method is similar to that used by Helson in the investigation of invariant subspaces (cf. [H]) and by Mandrekar and Nadkarni (cf. [MN]) to simplify the proof of the generalized F. and M. Riesz theorem concerning the quasi-invariance of analytic measures on compact groups.

The idea of our proof may be used without major changes to prove the following relative version of the result mentioned above. Every ergodic and relatively Kolmogorov $\mathbb{Z}^d$-action T ($1 \le d \le \infty$) on a Lebesgue space

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$(X,\mathcal{B},\mu)$ of positive entropy with respect to a factor σ-algebra $\mathcal{H}$ has a countable Lebesgue spectrum on the subspace $L^2(X,\mu)\ominus L^2(\mathcal{H})$. From this result it easily follows that $\mathbb{Z}^d$-actions with singular spectrum or with spectrum of finite multiplicity have zero entropy.

2. Preliminaries. For $d\in\mathbb{N}\cup\{\infty\}$ we denote by $\mathbb{Z}^d$ the d-dimensional integers if $d\in\mathbb{N}$ and the countable direct sum of copies of $\mathbb{Z}$ if $d=\infty$.

If $A\subset\mathbb{Z}^d$ is a finite set then $|A|$ denotes the number of elements of A.

A sequence (A_n) of finite subsets of $\mathbb{Z}^d$ is said to be a *Følner sequence* if for every $g\in\mathbb{Z}^d$,

$$\lim_{n\to\infty}|A_n|^{-1}|(g+A_n)\Delta A_n|=0.$$

Let $\Omega(\mathbb{Z}^d)$ be the set of all linear orders of $\mathbb{Z}^d$ compatible with the group operation. For $\omega\in\Omega(\mathbb{Z}^d)$ we denote by N_ω the set of all negative elements of $\mathbb{Z}^d$ with respect to ω.

For a given finite measure σ on the dual group $\widehat{\mathbb{Z}}^d$ we denote by $\widehat{\sigma}$ the Fourier transform of σ, i.e.

$$\widehat{\sigma}(g)=\int_{\widehat{\mathbb{Z}}^d}\chi(g)\,\sigma(d\chi),\qquad g\in\mathbb{Z}^d.$$

It is well known that the group $\widehat{\mathbb{Z}}^d$ is isomorphic to the d-dimensional torus $\mathbb{T}^d$. We denote by λ the Lebesgue measure on $\mathbb{T}^d$ in the case $d\in\mathbb{N}$ and the Haar measure in the case $d=\infty$.

Let $(X,\mathcal{B},\mu)$ be a Lebesgue probability space. We denote by $\mathcal{N}$ the trivial sub-σ-algebra of $\mathcal{B}$. For a given sub-σ-algebra $\mathcal{A}\subset\mathcal{B}$ the symbol $L^2(\mathcal{A})$ stands for the subspace of $L^2(X,\mu)$ consisting of all $\mathcal{A}$-measurable functions. We put

$$L_0^2(\mathcal{A})=\Big\{f\in L^2(\mathcal{A}):\int_X f\,d\mu=0\Big\}.$$

Let T be an action of $\mathbb{Z}^d$ (briefly, a $\mathbb{Z}^d$-*action*) on $(X,\mathcal{B},\mu)$, i.e. a homomorphism of $\mathbb{Z}^d$ into the group of all measure-preserving automorphisms of $(X,\mathcal{B},\mu)$. We denote by T^g the automorphism which is the image of g under T, for $g\in\mathbb{Z}^d$.

Let $\mathcal{H}\subset\mathcal{B}$ be a *factor σ-algebra*, i.e. $T^g\mathcal{H}=\mathcal{H}$ for all $g\in\mathbb{Z}^d$. The factor of T determined by $\mathcal{H}$ is denoted by $T_{\mathcal{H}}$.

A $\mathbb{Z}^d$-action T induces a unitary representation $U=U_T$ of $\mathbb{Z}^d$ in $L^2(X,\mu)$ defined by

$$U^g f=f\circ T^g,\qquad f\in L^2(X,\mu),\ g\in\mathbb{Z}^d.$$

For $f\in L^2(X,\mu)$ we denote by σ_f the *spectral measure* of f, i.e. the measure on $\widehat{\mathbb{Z}}^d$ such that

$$(U^g f,f)=\int_{\widehat{\mathbb{Z}}^d}\chi(g)\,\sigma_f(d\chi),\qquad g\in\mathbb{Z}^d.$$

We say that T *has a Lebesgue spectrum* if the maximal spectral type of U_T is equal to the type of the measure λ.

If, in addition, the multiplicity function of U_T is infinite we say that T *has a countable Lebesgue spectrum*.

For a countable measurable partition P of X and a factor σ-algebra $\mathcal{H}\subset\mathcal{B}$ we denote by $H(P)$ and $H(P\mid\mathcal{H})$ the entropy of P and the conditional entropy of P under $\mathcal{H}$ respectively. For the definitions and basic properties of the above entropies the reader is referred to [Pa]. Let $\mathcal{Z}$ denote the set of all countable measurable partitions P with $H(P)<\infty$.

For $P\in\mathcal{Z}$ and $\omega\in\Omega(\mathbb{Z}^d)$ we put

$$P_T=\bigvee_{g\in\mathbb{Z}^d}T^gP,\qquad P^-=P_\omega^-=\bigvee_{g\in N_\omega}T^gP.$$

Arguing similarly to [Ki1] one shows that for every Følner sequence (A_n), $P\in\mathcal{Z}$ and a factor σ-algebra $\mathcal{H}$ the limit

$$\lim_{n\to\infty}|A_n|^{-1}H\Big(\bigvee_{g\in A_n}T^gP\,\Big|\,\mathcal{H}\Big)=h(P,T\mid\mathcal{H})$$

exists and does not depend on the choice of (A_n). It can be shown in the same way as in [Pi] that for every $P\in\mathcal{Z}$ and $\omega\in\Omega(\mathbb{Z}^d)$,

$$h(P,T\mid\mathcal{H})=H(P\mid P_\omega^-\vee\mathcal{H}).$$

We define the *relative entropy* $h(T\mid\mathcal{H})$ of T with respect to $\mathcal{H}$ by

$$h(T\mid\mathcal{H})=\sup\{h(P,T\mid\mathcal{H}):P\in\mathcal{Z}\},$$

and the *relative Pinsker σ-algebra* $\pi(T\mid\mathcal{H})$ of T with respect to $\mathcal{H}$ as the smallest σ-algebra containing all $P\in\mathcal{Z}$ with $h(P,T\mid\mathcal{H})=0$.

It is clear that $h(T\mid\mathcal{N})$ is equal to the entropy $h(T)$ defined in [Ki1]. The σ-algebra $\pi(T\mid\mathcal{N})$ is called the *Pinsker σ-algebra* of T.

3. Main result. A $\mathbb{Z}^d$-action T is said to be a *relative Kolmogorov action* (briefly *K-action*) with respect to $\mathcal{H}$ if $\pi(T\mid\mathcal{H})=\mathcal{H}$, and a *K-action* if that equality holds for $\mathcal{H}=\mathcal{N}$.

GENERALIZED PINSKER FORMULA. *For every $P,Q\in\mathcal{Z}$ and $\omega\in\Omega(\mathbb{Z}^d)$,*

$$h(P\vee Q,T)=h(P,T)+H(Q\mid Q_\omega^-\vee P_T).$$

Proof. Let (A_n) be a Følner sequence in $\mathbb{Z}^d$ such that $A_n\subset A_{n+1}$, $n\ge 1$, and $\bigcup_{n=1}^\infty A_n=\mathbb{Z}^d$. It is well known that such a sequence exists (cf. [G]). For $P \in \mathcal{Z}$ we put

$$P_n = \bigvee_{g \in A_n} T^g P, \quad n \geq 1.$$

Let P, $Q \in \mathcal{Z}$ and $\omega \in \Omega(\mathbb{Z}^d)$. Applying elementary properties of the conditional entropy we have

$$H(P_n \vee Q_n) = H(P_n) + H(Q_n \mid P_n) \geq H(P_n) + H(Q_n \mid P_T), \quad n \geq 1.$$

Dividing both sides by $|A_n|$ and taking the limit as $n \to \infty$ we get

$$h(P \vee Q, T) \geq h(P,T) + H(Q \mid Q_\omega^- \vee P_T).$$

In order to show the opposite inequality we take $n_0 \geq 1$ such that $\mathcal{O} \in A_{n_0}$, where $\mathcal{O}$ denotes the zero element of $\mathbb{Z}^d$. It is easy to check that $h(P_n, T) = h(P,T)$, $n \geq 1$, and therefore, using simple properties of the conditional entropy, we have

$$\begin{aligned} h(P \vee Q, T) &\leq h(P_n \vee Q, T) = H(P_n \vee Q \mid (P_n)_\omega^- \vee Q_\omega^-) \\ &= H(P_n \mid (P_n)_\omega^- \vee Q_\omega^-) + H(Q \mid (P_n)_\omega^- \vee P_n \vee Q_\omega^-) \\ &\leq h(P,T) + H(Q \mid Q_\omega^- \vee P_n), \quad n \geq n_0. \end{aligned}$$

Letting $n \to \infty$ we get the desired result.

COROLLARY 1. *For every P, $Q \in \mathcal{Z}$ and $\omega \in \Omega(\mathbb{Z}^d)$ we have*

$$\lim_{g \in N_\omega} H(P \mid P_\omega^- \vee T^g Q_\omega^-) = h(P,T).$$

Proof. Since the proof is similar to that in the classical case we give only a sketch.

Let $g \in N_\omega$. From the Generalized Pinsker Formula and simple properties of the conditional entropy we have

$$\begin{aligned} h(P,T) + H(Q \mid Q_\omega^- \vee P_T) &= h(P \vee T^g Q, T) \\ &= H(P \mid P_\omega^- \vee T^g Q_\omega^-) + H(Q \mid Q_\omega^- \vee T^{-g}(P \vee P_\omega^-)). \end{aligned}$$

Hence

$$\begin{aligned} (1) \qquad H(P \mid P_\omega^- \vee T^g Q_\omega^-) &= h(P,T) + H(Q \mid Q_\omega^- \vee P_T) \\ &\quad - H(Q \mid Q_\omega^- \vee T^{-g}(P \vee P_\omega^-)). \end{aligned}$$

Since $\bigvee_{g \in N_\omega} T^{-g}(P \vee P_\omega^-) = P_T$, we get the desired result by passing to the limit as $g \in N_\omega$.

COROLLARY 2. *If T is a K-action then for every $P \in \mathcal{Z}$ and $\omega \in \Omega(\mathbb{Z}^d)$ we have*

$$\bigcap_{g \in N_\omega} T^g \mathcal{P}_\omega = \mathcal{N}$$

where $\mathcal{P}_\omega = P \vee P_\omega^-$.

Proof. For $A \in \bigcap_{g \in N_\omega} T^g \mathcal{P}_\omega$ and the partition $Q = \{A, X \setminus A\}$ we have

$$0 = H(Q \mid T^g \mathcal{P}_\omega) \geq H(Q \mid Q_\omega^- \vee T^g \mathcal{P}_\omega), \quad g \in N_\omega.$$

Therefore it follows from Corollary 1 that $h(Q,T) = 0$. By our assumption Q is trivial and so $A \in \mathcal{N}$.

In the proof of the main result we shall use a property of $\mathbb{Z}^d$ called rigidity. We formulate it for more general groups.

Let G be an abelian, countable and torsion-free group written additively. We denote by $J(G)$ the set of all monomorphisms $J : G \to \mathbb{R}$. It is known ([Ru]) that $J(G) \neq \emptyset$.

For $g \in G$ and $\chi \in \widehat{G}$ we put $\langle \chi, g \rangle = \chi(g)$ and we consider the dual homomorphism $\widehat{\jmath} : \mathbb{R} \to \widehat{G}$ defined by the formula

$$\langle \widehat{\jmath}(t), g \rangle = e^{2\pi i t j(g)}, \quad t \in \mathbb{R}, \ g \in G.$$

Since the equality $\langle \widehat{\jmath}(t), g \rangle = 1$ for all $t \in \mathbb{R}$ implies $g = \mathcal{O}$ we see that $\widehat{\jmath}(\mathbb{R})$ is a dense subgroup of $\widehat{G}$. We denote by $K(G)$ the subgroup of $\widehat{G}$ generated by the set $\bigcup_{j \in J(G)} \widehat{\jmath}(\mathbb{R})$.

We say that G is *rigid* if $K(G) = \widehat{G}$.

It is easy to show that every nontrivial subgroup of a rigid group is also rigid. This follows at once from the fact that characters of a subgroup extend to characters of the group (cf. [HR]).

One can also show that the class of rigid groups is closed under taking direct products. We omit the proof because we do not need this property.

LEMMA. *The group $\mathbb{Z}^d$ is rigid.*

Proof. It is enough to consider the case $d = \infty$. Let χ be a character of $\mathbb{Z}^\infty$. It is well known that there exists a uniquely determined element $\underline{u} = (u_n) \in \mathbb{T}^\infty$ such that

$$\langle \chi, g \rangle = \exp(2\pi i(n_1 u_1 + n_2 u_2 + \ldots)), \quad g = (n_1, n_2, \ldots) \in \mathbb{Z}^\infty.$$

Let $\underline{a} = (a_n) \in \mathbb{T}^\infty$ be independent over $\mathbb{Z}$ and such that $\underline{a} + \underline{u}$ is also independent over $\mathbb{Z}$. The possibility of the choice of $\underline{a}$ with the above properties may be shown as follows. The group $\mathbb{T}^\infty$ is compact, connected and satisfies the second countability axiom; therefore it is monothetic and $\lambda(S) = 1$ where S denotes the set of generators of $\mathbb{T}^\infty$ (cf. [HR]). Hence $\lambda(S \cap (S + \underline{u})) = 1$ and so it is enough to take as $\underline{a}$ an arbitrary element of $S \cap (S - \underline{u})$. For $\underline{x} = (x_n) \in \mathbb{T}^\infty$ we put

$$j_{\underline{x}}(g) = n_1 x_1 + n_2 x_2 + \ldots$$

It follows from the choice of $\underline{a}$ that $j_{\underline{a}}$ and $j_{\underline{a}+\underline{u}}$ belong to $J(\mathbb{Z}^\infty)$. Since

$$\widehat{\jmath}_{\underline{a}}(-1) \cdot \widehat{\jmath}_{\underline{a}+\underline{u}}(1) = \chi$$

we have $\chi \in K(\mathbb{Z}^\infty)$, i.e. $\mathbb{Z}^\infty$ is rigid.

Remark. *The additive group $\mathbb{Q}$ of rationals is not rigid.*

Proof. Let $\mathbb{Z}^{(p)}$ be the group of all rationals of the form m/p^n where $m \in \mathbb{Z}$, $n \in \mathbb{N} \setminus \{0\}$ and p is a prime number. We shall show that $\mathbb{Z}^{(p)}$, and hence $\mathbb{Q}$, is not rigid.

It is easy to see that for every $j \in J(\mathbb{Z}^{(p)})$ we have $\widehat{\jmath}(\mathbb{R}) = \widehat{\jmath}_0(\mathbb{R})$ where $j_0(g) = g$, $g \in \mathbb{Z}^{(p)}$. Therefore $K(\mathbb{Z}^{(p)}) = \widehat{\jmath}_0(\mathbb{R})$. It is known ([HR]) that the dual group of $\mathbb{Z}^{(p)}$ is isomorphic to the group $[0,1) \times \Delta_p$ where Δ_p denotes the group of all p-adic integers and the group law is defined as follows:

$$(\alpha, x) + (\beta, y) = (\alpha + \beta - [\alpha + \beta],\ x + y - [\alpha + \beta]u)$$

where $[t]$ means the integer part of $t \in \mathbb{R}$ and $u = (1, 0, 0, \ldots)$. The character on $\mathbb{Z}^{(p)}$ corresponding to (α, x) is

$$\chi_{(\alpha,x)}(g) = \exp(2\pi i g(\alpha - (x_0 + px_1 + \ldots + p^{n+1}x_{n-2}))), \quad g \in \mathbb{Z}^{(p)},\ x = (x_n) \in \Delta_p .$$

For $t \in \mathbb{R}$ and $g = m/p^n$ we have

$$\begin{aligned}\langle \widehat{\jmath}_0(t), g\rangle &= \exp(2\pi i g t) = \exp(2\pi i g(t - [t] - (-[t])))\\ &= \exp(2\pi i g(t - [t] - (-[t]) \cdot u)) .\end{aligned}$$

Hence it is clear that $\widehat{\jmath}_0(\mathbb{R})$ is different from the dual group of $\mathbb{Z}^{(p)}$, i.e. $\mathbb{Z}^{(p)}$ is not rigid.

Arguing similarly, one can show that if G is rigid then every nonzero element of G has a finite number of divisors.

In fact, if there exists $g_0 \in G \setminus \{0\}$ such that the equation $ng = g_0$ has an infinite number of solutions $(n, x) \in \mathbb{N} \times G$ then G contains a subgroup isomorphic to a group of the form $\bigcup_{n=1}^{\infty} p_n^{-1}\mathbb{Z}$ where (p_n) is an increasing sequence of integers such that $p_n \mid p_{n+1}$, $n \geq 1$. Proceeding as in the above proof one shows that this group is not rigid and so G is not rigid.

THEOREM 1. *Every Kolmogorov $\mathbb{Z}^d$-action has a countable Lebesgue spectrum.*

Proof. First we consider the case $h(T) < \infty$. Since T is ergodic there exists a finite partition P with $P_T = \mathcal{B}$ (cf. [Ro]).

Fix $j \in J(\mathbb{Z}^d)$ and let $\omega \in \Omega(\mathbb{Z}^d)$ be the order induced by the usual order of $\mathbb{R}$ via j. Since $j(G)$ is cofinal with $\mathbb{R}$ it follows from the assumption and Corollary 2 that

$$(2) \qquad \bigcap_{t \in \mathbb{R}} \bigvee_{j(g) < t} T^g P = \bigcap_{g \in N_\omega} T^g \mathcal{P}_\omega = \mathcal{N} .$$

We put

$$H_t = L_0^2\Big(\bigvee_{j(g) < t} T^g P\Big), \quad t \in \mathbb{R} .$$

It is easy to check that (H_t) is an increasing family of linear subspaces of $L_0^2(X, \mu)$ and

$$(3) \qquad \overline{\bigcup_{t \in \mathbb{R}} H_t} = L_0^2(X, \mu) ,$$

$$(4) \qquad \text{if } t_n \nearrow t \text{ then } H_t = \overline{\bigcup_{n=1}^{\infty} H_{t_n}} ,$$

$$(5) \qquad H_{t + j(g)} = U^{-g} H_t, \quad g \in \mathbb{Z}^d,\ t \in \mathbb{R} .$$

It follows at once from (2) that

$$(6) \qquad \bigcap_{t \in \mathbb{R}} H_t = \{0\} .$$

Let E_t be the orthogonal projection on the subspace H_t, $t \in \mathbb{R}$. It follows from (3)–(6) that the family (E_t) is a decomposition of unity in $L_0^2(X, \mu)$ with

$$(7) \qquad E_{t + j(g)} = U^{-g} E_t U^g, \quad g \in \mathbb{Z}^d,\ t \in \mathbb{R} .$$

We put

$$V^t = \int_{\mathbb{R}} \exp(2\pi i t s)\, dE_s .$$

The family $(V^t, t \in \mathbb{R})$ is a one-parameter group of unitary operators in $L_0^2(X, \mu)$ and, by (7), we have

$$U^g V^t = \exp(2\pi i t j(g)) V^t U^g, \quad g \in \mathbb{Z}^d,\ t \in \mathbb{R} .$$

Hence, for every $g \in \mathbb{Z}^d$, $t \in \mathbb{R}$ and $f \in L_0^2(X, \mu)$ we get

$$\begin{aligned}\widehat{\sigma}_{V^t f}(g) &= (U^g V^t f, V^t f) = \exp(2\pi i t j(g)) \widehat{\sigma}_f(g)\\ &= \langle \widehat{\jmath}(t), g\rangle \widehat{\sigma}_f(g) = \widehat{\delta}_{\widehat{\jmath}(t)}(g) \cdot \widehat{\sigma}_f(g)\end{aligned}$$

and so

$$(8) \qquad \sigma_{V^t f} = \delta_{\widehat{\jmath}(t)} * \sigma_f \quad \text{and} \quad \sigma_{V^{-t} f} = \delta_{\overline{\widehat{\jmath}(t)}} * \sigma_f .$$

Now, suppose the measure σ_f is of the maximal spectral type. It follows from (8) that

$$\delta_{\widehat{\jmath}(t)} * \sigma_f \ll \sigma_f, \quad \delta_{\overline{\widehat{\jmath}(t)}} * \sigma_f \ll \sigma_f$$

and so the measures $\delta_{\widehat{\jmath}(t)} * \sigma_f$ and σ_f are equivalent.

Since $\mathbb{Z}^d$ is rigid and the set of characters $\chi \in \widehat{\mathbb{Z}}^d$ for which the measures $\delta_\chi * \sigma_f$ and σ_f are equivalent is a group, we see that these measures are equivalent for every $\chi \in \widehat{\mathbb{Z}}^d$. Hence σ_f is equivalent to λ, i.e. the action T has a Lebesgue spectrum.

Now, suppose $h(T) = \infty$. There exists a sequence $(\mathcal{B}_n)$ of factor σ-algebras of T with $\mathcal{B}_n \subset \mathcal{B}_{n+1}$, $h(T_{\mathcal{B}_n}) < \infty$, $n \geq 1$, and $\bigvee_{n=1}^{\infty} \mathcal{B}_n = \mathcal{B}$. It follows from the assumption that $T_{\mathcal{B}_n}$ is a K-action and therefore, by the first part of the proof, it has a Lebesgue spectrum in $L_0^2(\mathcal{B}_n)$, $n \geq 1$.

Let $f \in L_0^2(X,\mu)$ be such that σ_f is of the maximal spectral type. There exists an increasing sequence $(m_n) \subset \mathcal{N}$ and a sequence $(f_n) \subset L_0^2(X,\mu)$ with $f_n \in L_0^2(\mathcal{B}_{m_n})$, $n \geq 1$, and $\|F_n - f\| \to 0$ as $n \to \infty$. Therefore we have

$$\sigma_{f_n} \ll \lambda \ll \sigma_f, \quad n \geq 1.$$

Passing to the limit as $n \to \infty$ we see that σ_f is equivalent to λ, i.e. T has a Lebesgue spectrum in $L_0^2(X,\mu)$.

In order to show that the multiplicity of the spectrum of T is infinite it is enough to show that there exists a factor σ-algebra $\mathcal{A}$ such that T has a countable Lebesgue spectrum in $L_0^2(\mathcal{A})$. This follows from the assumption that $h(T) > 0$. By the generalized Sinaĭ theorem (cf. [Ki2]) there exists a partition P such that $\mathcal{A} = P_T$ is a Bernoulli factor σ-algebra. It follows from [Kir] that T has a countable Lebesgue spectrum in $L_0^2(\mathcal{A})$, which completes the proof.

Now, let $\mathcal{H}$ be a fixed factor σ-algebra of T. Proceeding in the same way as above one can show the following relative version of Theorem 1.

THEOREM 2. *Every ergodic and positive entropy Kolmogorov $\mathbb{Z}^d$-action T with respect to $\mathcal{H}$ has a countable Lebesgue spectrum in the subspace $L_0^2(X,\mu) \ominus L_0^2(\mathcal{H})$.*

The results needed to prove Theorem 2 are the following.

Let $P, Q \in \mathcal{Z}$ and $\omega \in \Omega(\mathbb{Z}^d)$. We have:

(9) $h(P \vee Q, T \mid \mathcal{H}) = h(P,T \mid \mathcal{H}) + H(Q \mid Q_\omega^- \vee P_T \vee \mathcal{H})$.

(10) $\lim_{g \in N_\omega} H(P \mid P_\omega^- \vee T^g Q_\omega^- \vee \mathcal{H}) = h(P,T \mid \mathcal{H})$.

(11) If T is a relative K-action of $\mathbb{Z}^d$ with respect to $\mathcal{H}$ then for every $P \in \mathcal{Z}$ and $\omega \in \Omega(\mathbb{Z}^d)$ we have

$$\bigcap_{g \in N_\omega} (T^g \mathcal{P}_\omega \vee \mathcal{H}) = \mathcal{H}.$$

(12) If T is ergodic with $h(T \mid \mathcal{H}) < \infty$ then there exists a finite partition P with $P_T \vee \mathcal{H} = \mathcal{B}$ ([Ro]).

(13) If T is ergodic with $h(T \mid \mathcal{H}) > 0$ then there exists a partition P of X such that P_T is a Bernoulli factor σ-algebra which is independent of $\mathcal{H}$ ([T]).

The family of subspaces used in the relative case is defined as follows:

$$H_t = L_0^2\Big(\bigvee_{j(g)<t} T^g P \vee \mathcal{H}\Big) \ominus L^2(\mathcal{H})$$

where $j \in J(\mathbb{Z}^d)$ and P is a partition such that $P_T \vee \mathcal{H} = \mathcal{B}$.

One easily shows, using (10), that every $\mathbb{Z}^d$-action is a relative K-action with respect to $\pi(T)$.

COROLLARY 3. *Every ergodic $\mathbb{Z}^d$-action with positive entropy has a countable Lebesgue spectrum in the subspace $L^2(X,\mu) \ominus L^2(\pi(T))$.*

Hence we have at once, as in the classical case, the following

COROLLARY 4. *The $\mathbb{Z}^d$-actions with a singular spectrum or with a spectrum of finite multiplicity have zero entropies.*

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