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Operators on spaces of analytic functions

by

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Abstract. Let M_z be the operator of multiplication by z on a Banach space of functions analytic on a plane domain G. We say that M_z is *polynomially bounded* if $\|M_p\| \le C\|p\|_G$ for every polynomial p. We give necessary and sufficient conditions for M_z to be polynomially bounded. We also characterize the finite-codimensional invariant subspaces and derive some spectral properties of the multiplication operator in case the underlying space is Hilbert.

Introduction. Consider a Banach space $\mathcal{E}$ of functions analytic on a plane domain G, such that for each $\lambda \in G$ the linear functional e_λ of evaluation at λ is bounded on $\mathcal{E}$. Assume further that $\mathcal{E}$ contains the constant functions and that multiplication by the independent variable z defines a bounded linear operator M_z on $\mathcal{E}$. In case $\mathcal{E} = \mathcal{H}$ is a Hilbert space the continuity of point evaluations along with the Riesz representation theorem imply that for each $\lambda \in G$ there is a unique function $k_\lambda \in \mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda\rangle, f \in \mathcal{H}$. The function k_λ is the *reproducing kernel* for the point λ.

A complex-valued function φ on G for which $\varphi f \in \mathcal{E}$ for every $f \in \mathcal{E}$ is called a *multiplier* of $\mathcal{E}$ and the collection of all multipliers is denoted by $\mathcal{M}(\mathcal{E})$. Each multiplier φ of $\mathcal{E}$ determines a multiplication operator M_φ on $\mathcal{E}$ by $M_\varphi f = \varphi f$, $f \in \mathcal{E}$. Each multiplier is a bounded analytic function on G. In fact $\|\varphi\|_G \le \|M_\varphi\|$. A good source on this topic is [7].

Twenty years after the appearance of [7] it is reasonable to expect some words explaining the motivation of such a study and of any developments in the area. The description of invariant subspaces in abstract spaces has in fact appeared under some additional hypotheses and one of the first results (for simply connected domains) seems to be [6]. This kind of Beurling's theorem

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requires some knowledge of exceptional sets, as the more recent paper [2] seems to confirm. A good source on this topic is [3].

For G an open connected (not necessarily simply connected) subset of the complex plane and α an ordinal number, the set G_α is defined as in Sarason [4, p. 525].

In this article we give necessary and sufficient conditions for M_z to be polynomially bounded, characterize the finite-codimensional subspaces and give some spectral properties.

Polynomial boundedness of the multiplication operator. Let $\mathcal{E}$ be a Banach space of analytic functions on G such that $1 \in \mathcal{E}$, point evaluations are bounded linear functionals and $M_z \in \mathcal{B}(\mathcal{E})$. Recall that M_z is called *polynomially bounded* if $\|M_p\| \le C\|p\|_G$ for every polynomial p. In this section we give necessary and sufficient conditions for M_z to be polynomially bounded.

THEOREM 1. *Let $\mathcal{E}$ be a Banach space of functions analytic on a plane domain G such that $1 \in \mathcal{E}$, and for every λ in G the functional e_λ is bounded and $z\mathcal{E} \subset \mathcal{E}$. If $H^\infty(G_\alpha) \subset \mathcal{M}(\mathcal{E})$ then the map $W_\alpha : H^\infty(G_\alpha) \to \mathcal{B}(\mathcal{E})$ given by $W_\alpha(\varphi) = M_\varphi$ is bounded. Conversely, if $\|M_p\| \le C\|p\|_G$ for every polynomial p then $H^\infty(G_\alpha) \subset \mathcal{M}(\mathcal{E})$. Hence W_α is well defined and bounded.*

Proof. Assume $H^\infty(G_\alpha) \subset \mathcal{M}(\mathcal{E})$. Then $W_\alpha : H^\infty(G_\alpha) \to \mathcal{B}(\mathcal{E})$ is bounded by the closed graph theorem. To see this let $f_n \to f$ in $H^\infty(G_\alpha)$ and $M_{f_n} \to A$. Then $Ag = \lim M_{f_n} g = \lim f_n g$. Since $f_n g \to fg$ pointwise we have $Ag = fg$. Therefore $A = M_f$ and consequently W_α is bounded.

If M_z is polynomially bounded we prove by induction that $H^\infty(G_\alpha) \subset \mathcal{M}(\mathcal{E})$. If $\alpha = 1$ and $\varphi \in H^\infty(G_1)$, then there is a sequence $\{p_n\}$ of polynomials such that $p_n \to \varphi$ pointwise boundedly. Since $\|M_{p_n}\| \le C\|p_n\|_G \le C_0$, by passing to a subsequence we may assume that $M_{p_n} \to A$ (WOT). Therefore φ is a multiplier and $A = M_\varphi$.

Now assume M_z is polynomially bounded and $H^\infty(G_{\alpha-1}) \subset \mathcal{M}(\mathcal{E})$. Let $f \in H^\infty(G_\alpha)$. Then there is a sequence $\{f_n\}$ in $H^\infty(G_{\alpha-1})$ which is uniformly bounded on G and converges to f at each point of G [4, Theorem 1, p. 523]. Because $H^\infty(G_{\alpha-1}) \subset \mathcal{M}(\mathcal{E})$ we conclude that $W_{\alpha-1}$ is bounded. Hence $\|M_{f_n}\| \le C\|f_n\|_{G_{\alpha-1}} \le C_1$. By passing to a subsequence we may assume that $M_{f_n} \to A$ (WOT). Therefore $Ag = fg$ for all g and f is a multiplier.

Suppose α is a limit ordinal. Let $\{\beta(n)\}$ be an increasing sequence of ordinals such that α is the least ordinal exceeding every $\beta(n)$. Let $f \in H^\infty(G_\alpha)$. Invoking the proof of Theorem 2 of Sarason [4, p. 525] there is a sequence $\{f_n\}$ in $H^\infty(G_{\beta(n)})$ with $f_n \to f$ pointwise on G_1 and $\sup_n \|f_n\|_{G_{\beta(n)}} \le M$ for some $M > 0$. Since $\|f_n\|_G = \|f_n\|_{G_{\beta(n)}}$ we can show that $H^\infty(G_\alpha) \subseteq \mathcal{M}(\mathcal{E})$.

Proposition 4.9 of [1] along with Theorem 1 yields the following

COROLLARY. *M_z is polynomially bounded if and only if $H^\infty(G_\alpha) \subset \mathcal{M}(\mathcal{E})$.*

In fact, if $H^\infty(G_1) \subset \mathcal{M}(\mathcal{E})$ then $H^\infty(G_\alpha) \subset \mathcal{M}(\mathcal{E})$ for every α and M_z is polynomially bounded. Even the inclusion of the uniform (on $\overline{G}$) limits of the polynomials, $P(\overline{G})$, in $\mathcal{M}(\mathcal{E})$ implies that $H^\infty(G_\alpha) \subset \mathcal{M}(\mathcal{E})$ for every α.

Next we point out the significance of Theorem 1 with an eye towards its applications. In [5] we have shown that if G is finitely connected and M_z is polynomially bounded then it is reflexive. Hence for such a domain G if a knowledge of multipliers is available so that $H^\infty(G_\alpha) \subseteq \mathcal{M}(\mathcal{E})$ then M_z is reflexive. In a sense the invariant subspaces of M_z are linked to the way $H^\infty(G_\alpha)$ sits in the space of multipliers. In particular, there are quite a number of spaces for which $H^\infty(G_1) = \mathcal{M}(\mathcal{E})$, hence $(M_z, \mathcal{E})$ is reflexive.

Finite-codimensional invariant subspaces. Let G be a bounded domain in the complex plane. Let $\mathcal{E}$ be a Banach space of functions analytic on G satisfying the same conditions as before. We further assume that for every λ in G, $\operatorname{ran}(M_z - \lambda) = \ker e_\lambda$.

Note that the continuity of e_λ ($\lambda \in G$) implies that the point evaluations of derivatives of all orders are continuous with respect to the norm of $\mathcal{E}$. This is a consequence of an easy automatic continuity result: If a finite-codimensional linear subspace Y of a Banach space X is the range of a continuous linear mapping N on X, then it must be closed [8, Lemma 3.3]. Apply this to $X = \ker e_\lambda$, $Y = (M_z - \lambda)X = \ker(\delta_\lambda|_X)$, where $\delta_\lambda(f) = f'(\lambda)$, to get the continuity of δ_λ (first on X, then on the whole space).

A few comments are in order. Note that if $\lambda \in G$ then $\operatorname{ran}(M_z - \lambda) \subset \ker e_\lambda$. Therefore we only assume that if $f \in \mathcal{E}$ and $f(\lambda) = 0$ then $f/(z - \lambda)$ is in $\mathcal{E}$. A space $\mathcal{E}$ satisfying the above conditions is called a *Banach space of analytic functions on G*.

In order to characterize the finite-codimensional invariant subspaces for Banach spaces of analytic functions, we need the following lemma.

LEMMA 1. *Let $\mathcal{E}$ be a Banach space of analytic functions on G. Let p be a polynomial that has all its roots in G. Then $p\mathcal{E}$ is a subspace of $\mathcal{E}$ invariant under M_z. Moreover, $\dim \mathcal{E}/p\mathcal{E} = \deg p$.*

Proof. Let $\deg p = n$. Suppose $\lambda_1, \ldots, \lambda_n$ denote the zeros of p, repeated according to multiplicity. One can easily see that

$$p\mathcal{E} = \{f \in \mathcal{E} : f(\lambda_1) = \ldots = f(\lambda_n) = 0\},$$

in case of a multiple zero we require derivatives to equal zero appropriately.

In fact, if $f \in \mathcal{E}$ satisfies $f(\lambda_1) = \ldots = f(\lambda_n) = 0$ then repeated application of our assumption shows that $f/(z-\lambda_1)\ldots(z-\lambda_n) \in \mathcal{E}$ or equivalently $f \in p\mathcal{E}$.

The fact that point evaluations of derivatives of all orders are continuous shows that $p\mathcal{E}$ is closed. Since $p\mathcal{E}$ is the intersection of the kernels of n linearly independent linear functionals, we conclude that the codimension of $p\mathcal{E}$ is $\deg p$. ∎

THEOREM 2. *Let $\mathcal{E}$ be a Banach space of analytic functions on a plane domain G such that $\sigma(M_z) \subset \overline{G}$ and $\operatorname{ran}(M_z - \lambda)$ is dense in $\mathcal{E}$ for every $\lambda \in \partial G$. Let $\mathcal{F}$ be a closed finite-codimensional subspace of $\mathcal{E}$ that is invariant under multiplication by z. Then $\mathcal{F} = p\mathcal{E}$ for some polynomial p all of whose roots lie in G.*

Proof. Let $A : \mathcal{E}/\mathcal{F} \to \mathcal{E}/\mathcal{F}$ be the linear transformation defined by $A(g+\mathcal{F}) = zg + \mathcal{F}$. Since $\mathcal{F}$ is invariant under M_z we see that A is well-defined. If h is a polynomial then

$$h(A)(g+\mathcal{F}) = hg + \mathcal{F} \quad \text{for every } g \in \mathcal{E}.$$

Since A acts on a finite-dimensional space, there is a nonzero polynomial h, with $\deg h \le \dim \mathcal{E}/\mathcal{F}$, such that $h(A) = 0$. This shows that $h\mathcal{E} \subset \mathcal{F}$.

Write $h = pq$ where p is a polynomial all of whose roots lie in G and q is a polynomial whose roots lie in $\mathbb{C} \setminus G$. Now let λ be a root of q. If $\lambda \notin \overline{G}$ then $M_z - \lambda$ is invertible, so $(z-\lambda)\mathcal{E} = \mathcal{E}$. If $\lambda \in \partial G$ then by hypothesis, $(z-\lambda)\mathcal{E}$ is dense in $\mathcal{E}$. Therefore $q\mathcal{E}$ is dense in $\mathcal{E}$, hence $p\mathcal{E} \subseteq (h\mathcal{E})^- \subseteq \mathcal{F}$. We also have

$$\dim \mathcal{E}/\mathcal{F} \le \dim \mathcal{E}/p\mathcal{E} = \deg p \le \deg h \le \dim \mathcal{E}/\mathcal{F},$$

where the middle equality follows from Lemma 1. From the above inequality it follows that $\mathcal{E}/p\mathcal{E} = \mathcal{E}/\mathcal{F}$ so $p\mathcal{E} = \mathcal{F}$. ∎

Remark. The technical assumption on the range density in Theorem 2 deserves some discussion providing us with an idea in which spaces it is satisfied. For example, it holds for H^p on the unit disc, but only for $p < \infty$ (z being inner, $1-z$ is an outer function, hence the claim for $\lambda = 1$ holds, and so is the case for other λ, $|\lambda| = 1$, by rotation).

Spectral properties. Let $\mathcal{H}$ be a Hilbert space of functions analytic on a plane domain G as in the previous section, i.e. $\operatorname{ran}(M_z - \lambda) = \ker e_\lambda$. Our aim is to discuss the spectral properties of A, where $A = M_z$. For a treatment of such operators on Banach spaces of analytic functions see [3]. However, in the Hilbert space setting the proofs turn out to be simpler.

PROPOSITION 1. *Let $\mathcal{M}$ be an invariant subspace of A^*. Then*

$$\sigma_{\mathrm{ap}}(A^*|_{\mathcal{M}}) \cap G^* = \sigma_{\mathrm{p}}(A^*|_{\mathcal{M}}).$$

Proof. Let $\lambda \in G$ such that $\overline{\lambda} \in \sigma_{\mathrm{ap}}(A^*|_{\mathcal{M}})$. Let $\{f_n\}$ be a sequence of unit vectors in $\mathcal{M}$ such that $\|(A^*-\overline{\lambda})f_n\| \to 0$. Write $f_n = a_n k_\lambda + g_n$ where $g_n \perp k_\lambda$. Clearly $\|(A^*-\overline{\lambda})g_n\| \to 0$. Since $\operatorname{ran}(A^*-\overline{\lambda})$ is closed it follows that $A^*-\overline{\lambda}$ is bounded below on $\{k_\lambda\}^\perp$. Hence $\|g_n\| \to 0$. The sequence $\{a_n\}$ is clearly bounded. By passing to a subsequence assume that $a_n \to a$. Hence $f_n \to a k_\lambda$, so $k_\lambda \in \mathcal{M}$. ∎

PROPOSITION 2. *If $\mathcal{M}$ is a cyclic invariant subspace for A then*

$$\sigma(A^*|_{\mathcal{M}^\perp}) \cap G^* = \sigma_{\mathrm{p}}(A^*|_{\mathcal{M}^\perp}).$$

Proof. Let $\mathcal{M} = [f]$. Using Proposition 1 we only need to show that

$$\sigma_{\mathrm{c}}(A^*|_{\mathcal{M}^\perp}) \cap G^* \subset \sigma_{\mathrm{p}}(A^*|_{\mathcal{M}^\perp}) = \{\overline{\lambda} \in G^* : f(\lambda) = 0\}$$

where $\sigma_{\mathrm{c}}(T)$ denotes the compression spectrum of T. Choose $\overline{\lambda} \in G^*$ such that $f(\lambda) \neq 0$. If $h \perp ((A^*-\overline{\lambda})\mathcal{M}^\perp)$ then for every g in $\mathcal{M}^\perp$ we have $0 = \langle h, (A^*-\overline{\lambda})g\rangle = \langle (A-\lambda)h, g\rangle$ so $(A-\lambda)h \in (\mathcal{M}^\perp)^\perp = \mathcal{M}$. Choose a sequence $\{p_n\}$ of polynomials such that $p_n f \to (A-\lambda)h$. Because

$$\langle p_n f, k_\lambda\rangle \to \langle (A-\lambda)h, k_\lambda\rangle = \langle h, (A^*-\overline{\lambda})k_\lambda\rangle = 0$$

and $f(\lambda) \neq 0$ we have $p_n(\lambda) \to 0$. Let

$$q_n(z) = (p_n(z) - p_n(\lambda))/(z-\lambda).$$

Then

$$(A-\lambda)q_n(A)f = p_n(A)f - p_n(\lambda)f \to (A-\lambda)h.$$

Since $\operatorname{ran}(A-\lambda)$ is closed, $A-\lambda$ is bounded below on $(\ker(A-\lambda))^\perp = \operatorname{ran}(A^*-\overline{\lambda}) = \mathcal{H}$. Thus $q_n f \to h$, which gives $h \in \mathcal{M}$. Hence

$$((A^*-\overline{\lambda})\mathcal{M}^\perp)^\perp \subseteq (\mathcal{M}^\perp)^\perp.$$

Therefore $\overline{\lambda}$ does not belong to the compression spectrum of $A^*|_{\mathcal{M}^\perp}$. ∎

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Extension of multilinear mappings on Banach spaces

by

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Dedicated to the memory of Leopoldo Nachbin (1922–1993)

Abstract. By following an idea of Nicodemi we study certain sequences of extension operators for multilinear mappings on Banach spaces starting from any given extension operator for linear mappings. In this way we obtain several new properties of the extension operators previously studied by Aron, Berner, Cole, Davie and Gamelin. As an application of our methods we show the existence of plenty of unbounded scalar-valued homomorphisms on the locally convex algebra of all continuous polynomials on each infinite-dimensional Banach space. This improves a result of Dixon.

Introduction. The problem of extending holomorphic functions from a Banach space E to a larger Banach space F was first studied by Aron and Berner [3]. They showed that the holomorphic functions of bounded type on E extend in a natural way to E'', yielding an extension operator from $\mathcal{H}_b(E)$ into $\mathcal{H}_b(E'')$. To achieve their goal they constructed extension operators for the spaces of multilinear forms and then used Taylor series expansions to extend holomorphic functions.

It is in general possible to extend multilinear forms on E to E'' in many different ways. Davie and Gamelin [6], and Aron, Cole and Gamelin [4], have established important properties of the extension operators of Aron and Berner, and have given a different, much simpler, description of those operators. Very recently Lindström and Ryan [13] have constructed other extension operators for multilinear forms by using ultrapowers of Banach

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