

**Boundary behavior of subharmonic functions
in nontangential accessible domains**

by

SHIYING ZHAO (St. Louis, Mo.)

Abstract. The following results concerning boundary behavior of subharmonic functions in the unit ball of \mathbb{R}^n are generalized to nontangential accessible domains in the sense of Jerison and Kenig [7]: (i) The classical theorem of Littlewood on the radial limits. (ii) Ziomek's theorem on the L^p -nontangential limits. (iii) The localized version of the above two results and nontangential limits of Green potentials under a certain nontangential condition.

1. Introduction. In [7], Jerison and Kenig defined a class of bounded domains, the so-called nontangential accessible (NTA) domains, in \mathbb{R}^n for which nontangential approach regions are meaningful, and extended certain classical results on the boundary behavior of harmonic functions in the upper half-space of \mathbb{R}^n to this type of domains. In this paper we consider subharmonic functions and generalize three types of classical results on boundary behavior of subharmonic functions in the unit ball or the upper half-space of \mathbb{R}^n to NTA domains.

First, a classical theorem of Littlewood [9] states that if u is subharmonic in the unit disc and has a nonnegative harmonic majorant then u has a finite radial limit almost everywhere on the unit circle. The generalization to balls or upper half-spaces in higher dimensions is standard. In [3], Dahlberg studied this problem in Lipschitz domains and proved that the same conclusion is true for "radials" consisting of nontangential straight lines so that the directions of these lines vary Lipschitz continuously. In order to generalize this result to more general nonsmooth domains, the first question one needs to answer is what is the appropriate analogue of radial approach. In Definition 3.1 we define nontangential radials in NTA domains. Radials of this type, in general, are twisting (but not too much) and are possibly disconnected.

1991 *Mathematics Subject Classification*: Primary 31B25.

Key words and phrases: subharmonic function, Green potential, boundary limit, NTA domain.

The generalized Littlewood theorem (Theorem 4.1) in our setting reads as follows: If a subharmonic function in an NTA domain D has a nonnegative harmonic majorant, then it has a finite radial limit almost everywhere with respect to the harmonic measure of D on the subset of ∂D on which a nontangential mapping is defined. The basic idea of the proof of this result goes back to Littlewood's original paper [9], that is, by use of the Riesz decomposition theorem and Fatou's theorem, one can reduce the matter to showing that every Green potential has radial limit zero. In doing this it is necessary to consider "nonsingular" and "singular" parts of a Green potential separately (see §4 for this terminology). For the "nonsingular" part, the argument of Dahlberg in [3] continues to work in our case. However, the unit normal vector field of the boundary of a Lipschitz domain was heavily used there in order to deal with the "singular" part, and thus the extension to NTA domains requires a new approach. The idea of handling this part comes from Ullrich's paper [14], where the radial limits of Möbius invariant subharmonic functions in the unit ball of \mathbb{C}^n are studied. On the other hand, it is Jerison and Kenig's work [7] on the estimates for the harmonic measure of an NTA domain that opens the way for us to carry out Ullrich's idea.

Secondly, it is well-known that subharmonic functions in a domain, in general, have no nontangential limits on the boundary. However, some types of limits between radial limit and nontangential limit exist. One of the examples is the so-called nontangential limit in the L^p metric (with respect to the Lebesgue measure), or, in short, L^p -nontangential limit (see Definition 6.1), which was first studied by Ziomek [18]. (For some other types of limits of subharmonic functions we refer to [17].) Our second result is an extension of Ziomek's theorem (Theorem 6.2) which states that if a subharmonic function in an NTA domain has a nonnegative harmonic majorant then it has L^p -nontangential limit almost everywhere with respect to the harmonic measure for the p in the range of $1 \leq p < n/(n-2)$.

Finally, a remarkable result of Carleson [1] is that the nontangential lower (or upper) boundedness of a harmonic function in the upper half-space implies that it has a nontangential limit almost everywhere on the boundary. The same result is true for NTA domains (see [7]). The key technique is to construct a so-called sawtooth region over the boundary. For a subharmonic function, the similar result holds if nontangential limits are replaced by radial limits as shown in [3] for Lipschitz domains. On the other hand, as we mentioned above, a Green potential (and hence a subharmonic function) has no nontangential boundary limit in general, because of the wild behavior of "singular" parts of Green potentials. But if some condition is imposed to control this kind of behavior then it will have nontangential boundary limits. Several authors have studied these problems in various domains (see [15] for

upper half-spaces and [3] for Lipschitz domains, for example). In the last part of this paper, we consider these two problems in NTA domains. The first one (Theorem 8.3) can be considered as a localized version of Theorems 4.1 and 6.2 and the second is given in Theorem 8.5.

This paper is organized as follows. In §2, we recall the definition of an NTA domain in \mathbb{R}^n and two important facts about the harmonic measure of an NTA domain, which we need in the sequel. In §3, we give the definition of nontangential radial mappings, and we show that the radials of straight lines in Lipschitz domains studied in [3] are nontangential radials in our sense. The generalized Littlewood theorem is proved in §4, where a key estimate (Lemma 4.4) is assumed. In §5, we prove Lemma 4.4, which states that the radial maximal function of the "singular" part of a Green potential is weak- L^1 with respect to the harmonic measure. §§6 and 7 are devoted to proving the generalization of Ziomek's theorem. Finally, in §8, we consider Carleson's type results, namely, boundary limits of subharmonic functions under nontangential conditions.

Acknowledgements. The author thanks Professors R. Howard and M. Stoll for their useful advice, and thanks Professors B. E. J. Dahlberg and C. E. Kenig for suggestive conversations.

2. Notation and preliminaries. The ball centered at $X \in \mathbb{R}^n$ with radius $r > 0$ will be denoted by $B(X, r) = \{Y \in \mathbb{R}^n : |X - Y| < r\}$. For a bounded domain D in \mathbb{R}^n , the surface ball centered at $Q \in \partial D$ with radius $r > 0$ is defined by $\Delta(Q, r) = B(Q, r) \cap \partial D$. For a point $X \in D$, we shall denote by $\delta(X) = d(X, \partial D)$, the Euclidean distance from X to the boundary ∂D of D .

We now recall the definition of NTA domains (see [7; 16]): A bounded domain D in \mathbb{R}^n is said to be *nontangential accessible* (or *NTA*) if there exist constants $C_D > 1$ and $r_D > 0$ such that the following three conditions are satisfied:

(1) *Corkscrew condition.* If $Q \in \partial D$ and $0 < r < r_D$, then there is a point $A_r(Q) \in D$ such that

$$(2.1) \quad |A_r(Q) - Q| < r, \quad \delta(A_r(Q)) > C_D^{-1}r.$$

(2) *Complement condition.* The complement of D satisfies (1).

(3) *Harnack chain condition.* If $X, Y \in D$ with

$$|X - Y| \leq 2^k \min\{\delta(X), \delta(Y)\},$$

for some positive integer k , then there is a positive integer $m \leq C_D k$ and a sequence $\{Z_j\}_{j=0}^m \subset D$ such that $Z_0 = X$, $Z_m = Y$ and $|Z_{j-1} - Z_j| < \frac{1}{2}\delta(Z_j)$.

For a fixed NTA domain D , let $G(X, Y)$ denote the Green function of D . We shall use the convention that the Green function is positive and hence superharmonic. For $X \in D$, we let ω^X be the harmonic measure of D at X , that is, the unique probability Borel measure on ∂D so that for each Borel subset E of ∂D , $\omega^X(E)$ is the value at X of the solution of the Dirichlet problem with the boundary data χ_E , where χ_E is the characteristic function of E . (It is known that NTA domains are regular for the Dirichlet problem, see [7].)

The following two estimates for the harmonic measure of an NTA domain D are extremely useful for us: The first one is known as Dahlberg's comparison theorem ([7, Lemma 4.8]), which states that if $X \in D \setminus B(Q, 2r)$, then

$$(2.2) \quad C^{-1}G(X, A_r(Q)) \leq r^{2-n}\omega^X(\Delta(Q, r)) \leq CG(X, A_r(Q)),$$

for some constant C depending only on C_D .

The second one is the doubling property ([7, Lemma 4.9]), that is, for all $Q \in \partial D$ and $r > 0$,

$$(2.3) \quad \omega^X(\Delta(Q, 2r)) \leq C\omega^X(\Delta(Q, r)),$$

for some constant C depending only on C_D and X .

In what follows we shall use C and c to denote positive constants which are not necessarily the same at any two occurrences, and the dependency of a constant will be specified in the appropriate context. For an NTA domain $D \subset \mathbb{R}^n$, by the phrase "a constant depends on D " we mean that the constant depends on the structure constant C_D in the definition of NTA domains, on the dimension n , and sometimes also on a fixed point $O \in D$. We also use the notation $q_1 \lesssim q_2$ to mean that there is a constant c , depending only on D (or the dependency of the constant is clear from the context), such that $q_1 \leq cq_2$, and the notation $q_1 \approx q_2$ to mean that $q_1 \lesssim q_2$ and $q_2 \lesssim q_1$. For instance, (2.2) and (2.3) could be rewritten as $r^{2-n}\omega^X(\Delta(Q, r)) \approx G(X, A_r(Q))$ and $\omega^X(\Delta(Q, 2r)) \lesssim \omega^X(\Delta(Q, r))$, respectively.

3. Nontangential radial mappings

DEFINITION 3.1. Let D be an NTA domain in \mathbb{R}^n with constants C_D and r_D . Let $E \subset \partial D$ be a Borel set, and ϱ be a number depending only on E such that $0 < \varrho \leq C_D^{-1}r_D$. A mapping

$$\gamma : E \times [0, \varrho] \rightarrow D$$

is called *nontangentially radial* (abbreviated *NTR*) if the following two conditions are fulfilled:

(1) *Corkscrew condition.* For any $Q \in E$, if we set $\gamma_Q(t) = \gamma(Q, t)$, then the locus of γ_Q , $\{\gamma_Q(t) : t \in [0, \varrho]\}$, is a corkscrew at Q , that is, $\gamma_Q(0) = Q$ and

$$(3.1) \quad |\gamma_Q(t) - Q| < t, \quad \delta(\gamma_Q(t)) > C_D^{-1}t$$

for all $t \in (0, \varrho)$.

(2) *Radial condition.* There exists a constant $L > 0$ such that

$$(3.2) \quad |\gamma_Q(t) - \gamma_P(s)| \geq L^{-1}|Q - P|$$

for all $Q, P \in E$ and $s, t \in [0, \varrho]$.

We can piece local NTR mappings together to obtain a global one on ∂D in the following way:

DEFINITION 3.2. We say that an NTA domain $D \subset \mathbb{R}^n$ is *nontangential radial* (abbreviated *NTR*) if there are countably many Borel sets $E_k \subset \partial D$, $k = 1, 2, \dots$, such that

(1) $\{E_k\}$ are mutually disjoint and the set $\partial D \setminus \bigcup E_k$ has ω -measure zero.

(2) On each E_k there is an NTR mapping $\gamma_k : E_k \times [0, \varrho_k] \rightarrow D$.

(3) The family of NTR mappings $\gamma = \{\gamma_k\}_{k=1}^{\infty}$ is *compatible* in the following sense: There exists a constant $L > 0$ such that if $Q \in E_j$, $t \in [0, \varrho_j]$ and $P \in E_k$, $s \in [0, \varrho_k]$ then

$$(3.3) \quad |\gamma_j(Q, t) - \gamma_k(P, s)| \geq L^{-1}|Q - P|$$

for any pair of indices j and k .

REMARKS 3.3. (a) The mapping γ_Q is not necessarily continuous and hence its locus is possibly disconnected.

(b) For technical reasons we shall assume that ϱ is small enough so that there is a point $O \in D$ such that $\delta(O) > 2C_D^2\varrho$.

(c) It is obvious that the class of NTR domains is invariant under bi-Lipschitz homeomorphisms of \mathbb{R}^n . This fact implies that every chord-arc domain in \mathbb{R}^2 is an NTR domain, since it has been proved in [8] that any chord-arc domain is the image of the unit disk (or the upper half-plane in the unbounded case) under a global bi-Lipschitz homeomorphism. However, chord-arc domains are not Lipschitz in general.

(d) The radial condition (2) in Definition 3.1 is equivalent to the condition (2') below, which is more geometrical and is also in the form we actually want to use in obtaining radial limits of subharmonic functions.

In order to introduce the condition (2') we need some notation: For a fixed NTR mapping γ , let R^γ be the image of γ , i.e.,

$$(3.4) \quad R^\gamma = \{X \in D : X = \gamma_Q(t) \text{ for some } Q \in E \text{ and } t \in (0, \varrho)\},$$

and let $\pi : R^\gamma \rightarrow E$ be the *projection* along γ , which is defined by

$$\pi(\gamma_Q(t)) = Q, \quad Q \in E, \quad t \in (0, \varrho).$$

For any subset $F \subset D$, we define $\pi F = \pi(F \cap R^\gamma)$.

It is easy to verify the following

LEMMA 3.4. *Let D , E and γ be as in Definition 3.1. Then the condition (2) is equivalent to the following condition:*

(2') *The projection $\pi : R^\gamma \rightarrow E$ is well-defined and there is a constant $L' > 0$ such that if $Q \in E$ then*

$$(3.5) \quad \pi B(\gamma_Q(t), r) \subset \Delta(Q, L'r)$$

for all $r > 0$.

Moreover, the two constants, L in (3.2) and L' in (3.5), coincide.

In [3], Dahlberg studied nontangential radials of straight lines on Lipschitz domains. The following proposition shows that these are indeed NTR mappings in our sense.

PROPOSITION 3.5. *Let D be a Lipschitz domain in \mathbb{R}^n , and $E \subset \partial D$ be Borel measurable. Suppose for some $\varrho > 0$, $\gamma : E \times [0, \varrho) \rightarrow D$ is given by*

$$(3.6) \quad \gamma(Q, t) = Q + te(Q),$$

where e is a Lipschitz continuous mapping from E to the unit sphere S^{n-1} in \mathbb{R}^n so that $e(Q)$ is the axis of symmetry of an open truncated circular cone $\Gamma_\alpha^e(Q)$ in D at $Q \in E$. Then γ is an NTR mapping on E , provided ϱ is small enough.

PROOF. The corkscrew condition (1) is obvious. We now verify the radial condition (2).

Let $Q, P \in E$ and $0 < s \leq t < \varrho$ be given. It follows from the interior and exterior conditions of Lipschitz domains that there is a number σ with $0 < \sigma < 1$ such that $|(e(P), Q - P)| < \sigma|Q - P|$. Writing $t = l|Q - P|$ for some $l > 0$, we have

$$\begin{aligned} |te(Q) + Q - P| &= (t^2 + 2t(e(Q), Q - P) + |Q - P|^2)^{1/2} \\ &\geq (1 + l^2 - 2l\sigma)^{1/2}|Q - P| \geq (1 - \sigma^2)|Q - P|. \end{aligned}$$

Next, using the fact that e is Lipschitz continuous, i.e., there is a constant $\kappa > 0$ such that $|e(Q) - e(P)| \leq \kappa|Q - P|$, we then get

$$\begin{aligned} |\gamma_Q(t) - \gamma_P(s)| &= |Q + te(Q) - P - se(P)| \\ &\geq |Q + (t - s)e(Q) - P| - s|e(Q) - e(P)| \\ &\geq (1 - \sigma^2 - \varrho\kappa)|Q - P|. \end{aligned}$$

This gives condition (2) with $L = 1 - \sigma^2 - \varrho\kappa$, if ϱ is sufficiently small so that $1 - \sigma^2 - \varrho\kappa > 0$. ■

4. Radial boundary limits

THEOREM 4.1. *Let D be an NTA domain in \mathbb{R}^n and u be a subharmonic function on D which has a nonnegative harmonic majorant. If $E \subset \partial D$ is Borel measurable such that there is an NTR mapping γ on E , then the γ -radial limit $\lim_{t \searrow 0} u(\gamma_Q(t))$ exists and is finite for ω -almost all $Q \in E$.*

In order to prove this theorem, we need to introduce a splitting of the Green function G of the NTA domain D : For a fixed number $0 < \varkappa \leq 1/2$, we define

$$(4.1) \quad G_{0,\varkappa}(X, Y) = G(X, Y)\chi_{B(X, \varkappa\delta(X))}(Y),$$

and

$$(4.2) \quad G_{1,\varkappa}(X, Y) = G(X, Y)\chi_{D \setminus B(X, \varkappa\delta(X))}(Y),$$

where $X, Y \in D$. If $G\mu$ is a Green potential in D , we set

$$(4.3) \quad G_{i,\varkappa}\mu(X) = \int_D G_{i,\varkappa}(X, Y) d\mu(Y), \quad i = 0, 1.$$

We then have $G\mu = G_{0,\varkappa}\mu + G_{1,\varkappa}\mu$. We shall call $G_{0,\varkappa}\mu$ the *singular part* and $G_{1,\varkappa}\mu$ the *nonsingular part* of the Green potential $G\mu$, respectively.

We next recall that for an NTA domain D in \mathbb{R}^n , a *nontangential region* at $Q \in \partial D$ of opening $\alpha > 1$ is defined by

$$(4.4) \quad \Gamma_\alpha(Q) = \{X \in D : |X - Q| < \alpha\delta(X)\}.$$

We say that a function u defined in D has *nontangential limit* ξ if for any $\alpha > 1$, $u(X)$ restricted to $\Gamma_\alpha(Q)$ converges to ξ as $X \rightarrow Q$. It was shown in [7] that every positive harmonic function in an NTA domain has finite nontangential limit ω -a.e. on ∂D .

We remark that our notation of $\Gamma_\alpha(Q)$ is slightly different from the one used in [7]. We also notice that $\Gamma_\alpha(Q)$ might be empty if α is close enough to 1. But, for large α , say $\alpha > C_D$, the nontangential region $\Gamma_\alpha(Q)$ is always nonempty for all $Q \in \partial D$.

As far as boundary limits are concerned, the nonsingular part of a Green potential behaves like a harmonic function, as is shown in the following lemma, the proof of which is taken from [3].

LEMMA 4.2. *Let D be an NTA domain and $G\mu$ be a Green potential on D . Then for any \varkappa with $0 < \varkappa \leq 1/2$, the nonsingular part $G_{1,\varkappa}\mu$ as defined above has nontangential limit 0 ω -a.e. on ∂D .*

PROOF. It was shown in [10] (see also [4, Theorem 1.XII.18]) that $G\mu$ has fine limit 0 ω -a.e. on the Martin boundary of D and hence on the

topological boundary ∂D of D , since it has been proved that these two boundaries coincide for the NTA domain D (see [7, Theorem 5.9]).

Let $X \in D$ and denote by ν the restriction of μ to the set $D \setminus B(X, \varkappa\delta(X))$. Then $G\nu$ is harmonic in $B(X, \varkappa\delta(X))$. It follows from Harnack's inequality that $G\nu \gtrsim G\nu(X) = G_{1, \varkappa}\mu(X)$ in $B(X, \frac{1}{2}\varkappa\delta(X))$. Since $G\mu \geq G\nu$ we have

$$(4.5) \quad G\mu(Y) \gtrsim G_{1, \varkappa}\mu(X) \quad \text{for all } Y \in B(X, \frac{1}{2}\varkappa\delta(X)).$$

Suppose now that there is a sequence $\{X_k\}_{k=1}^\infty$ of points in D such that $X_k \rightarrow Q \in \partial D$ nontangentially but $G_{1, \varkappa}\mu(X_k) \geq m > 0$ for all k . Then it follows from (4.5) that $G\mu \gtrsim m$ in $\bigcup B_k$, where $B_k = B(X_k, \frac{1}{2}\varkappa\delta(X_k))$. However, the "bubble set" $\bigcup B_k$ is not semi-thin at Q as shown in [13] for NTA domains, so $G\mu$ cannot have fine limit 0 at Q , which is a contradiction. ■

In the rest of the section, we shall fix $\varkappa = (2C_D)^{-1}$ and write $G_0 = G_{0, \varkappa}$ and $G_1 = G_{1, \varkappa}$.

To control the boundary behavior of the singular part of a Green potential we need the following:

DEFINITION 4.3. Let D, E, γ and ϱ be as in Definition 3.1. For a Green potential $G\mu$ in D , we define the (*nontangential*) *radial maximal function* $M_0\mu$ of the singular part $G_0\mu$ of $G\mu$ by

$$(4.6) \quad M_0\mu(Q) = \sup_{0 < t < \varrho} G_0\mu(\gamma_Q(t)), \quad Q \in E.$$

The major step of the proof of Theorem 4.1 is the following:

LEMMA 4.4. Let D, E, γ and ϱ be as in Definition 3.1 and let $O \in D$ be as in Remarks 3.3(b). Suppose that $G\mu$ is a Green potential on D . Then the radial maximal function $M_0\mu$ of $G_0\mu$ is weak- $L^1(\omega^O)$. More precisely, there exists a constant $C > 0$, depending only on D , such that

$$(4.7) \quad \omega^O(\{Q \in E : M_0\mu(Q) > \lambda\}) \leq \frac{C}{\lambda} G\mu(O)$$

for all $\lambda > 0$.

The proof of this lemma will be given in the next section. We now assume it to prove Theorem 4.1.

Proof of Theorem 4.1. Let u be a subharmonic function on D with a nonnegative harmonic majorant h . Then by the Riesz decomposition theorem for subharmonic functions ([6, Theorem 6.18]) there is a Green potential $G\mu$ such that $u = h_0 - G\mu$, where h_0 is the least harmonic majorant of u . Since $h_0 \leq h$, it follows from [7, Theorem 6.4] that h_0 has finite nontangential limit ω -a.e. on ∂D . Now, we split $G\mu = G_0\mu + G_1\mu$ as before, and Lemma 4.2 tells us that $G_1\mu$ has nontangential limit 0 ω -a.e. on ∂D .

Thus the theorem will follow if we can prove that $\lim_{t \searrow 0} G_0\mu(\gamma_Q(t)) = 0$ for ω -almost all $Q \in E$. To prove this we need to show that for any $\varepsilon > 0$,

$$(4.8) \quad \omega^O(\{Q \in E : \limsup_{t \searrow 0} G_0\mu(\gamma_Q(t)) > \varepsilon\}) < \varepsilon$$

for some point $O \in D$.

Now, let $O \in D$ be as in Remarks 3.3(b). Without loss of generality, we may assume that $G\mu(O) < \infty$, since the Green potential is finite in a dense subset of D . Let $\varepsilon > 0$ be given. Choose a relatively compact subdomain Ω of D so that

$$G\nu(O) < \frac{\varepsilon^2}{C},$$

where ν is the restriction of μ to the set $D \setminus \Omega$ and C is as in Lemma 4.4. Since the Green potential $G(\mu|_\Omega)$ has uniform boundary limit 0 by [6, Theorem 6.23], we have

$$\begin{aligned} \{Q \in E : \limsup_{t \searrow 0} G_0\mu(\gamma_Q(t)) > \varepsilon\} &= \{Q \in E : \limsup_{t \searrow 0} G_0\nu(\gamma_Q(t)) > \varepsilon\} \\ &\subset \{Q \in E : M_0\nu(Q) > \varepsilon\}. \end{aligned}$$

Therefore, applying Lemma 4.4 to the measure ν , we obtain

$$\begin{aligned} \omega^O(\{Q \in E : \limsup_{t \searrow 0} G_0\mu(\gamma_Q(t)) > \varepsilon\}) &\leq \omega^O(\{Q \in E : M_0\nu(Q) > \varepsilon\}) \\ &\leq \frac{C}{\varepsilon} G\nu(O) < \varepsilon. \end{aligned}$$

Thus, (4.8) is established and hence the theorem follows. ■

An immediate consequence of Theorem 4.1 and the definition of NTR domains (Definition 3.2) is the following:

COROLLARY 4.5. If D is an NTR domain with an NTR mapping γ on ∂D , then every subharmonic function on D which has a nonnegative harmonic majorant has a finite γ -radial limit ω -a.e. on ∂D .

5. Proof of Lemma 4.4. We shall consider the case of $n \geq 3$ only, but when $n = 2$ all of the results in this section remain true and the proofs are similar. We keep the notation of the last section.

We need several additional lemmas on which the final proof will be based. We always assume the hypothesis of Lemma 4.4 and recall that $\varkappa = (2C_D)^{-1}$.

LEMMA 5.1. For $\sigma > 0$, define

$$(5.1) \quad M_\sigma\mu(Q) = \sup_{0 < t < \varrho} t^{2-n} \mu(B(\gamma_Q(t), \sigma t)), \quad Q \in E.$$

Then

$$(5.2) \quad M_0\mu(Q) \lesssim \sum_{j=0}^{\infty} 2^j M_{\sigma_j}\mu(Q), \quad Q \in E,$$

where $\sigma_j = \varkappa 2^{j/(2-n)}$, $j = 0, 1, 2, \dots$

Proof. Let $Q \in E$ and $0 < t < \varrho$ be fixed. We notice

$$G(\gamma_Q(t), Y) \lesssim |\gamma_Q(t) - Y|^{2-n}$$

for all $Y \in B(\gamma_Q(t), \varkappa t)$. Therefore, a simple calculation gives

$$\begin{aligned} G_0\mu(\gamma_Q(t)) &\lesssim \int_{B(\gamma_Q(t), \varkappa t)} |\gamma_Q(t) - Y|^{2-n} d\mu(Y) \\ &\approx \int_{B(\gamma_Q(t), \varkappa t)} \int_{|\gamma_Q(t) - Y|}^{\infty} r^{1-n} dr d\mu(Y) \\ &= \int_0^{\infty} r^{1-n} \mu(B(\gamma_Q(t), \varkappa t) \cap B(\gamma_Q(t), r)) dr \\ &= \sum_{j=0}^{\infty} \int_{\sigma_{j+1}t}^{\sigma_j t} r^{1-n} \mu(B(\gamma_Q(t), r)) dr \\ &\quad + \mu(B(\gamma_Q(t), \varkappa t)) \int_{\varkappa t}^{\infty} r^{1-n} dr \\ &\approx \sum_{j=0}^{\infty} 2^j t^{2-n} \mu(B(\gamma_Q(t), \sigma_j t)), \end{aligned}$$

where Fubini's theorem was used to change the order of integration. (5.2) follows from this by taking the sup over $0 < t < \varrho$. ■

LEMMA 5.2. Let $\lambda > 0$, $0 < \sigma < 1$, and set

$$(5.3) \quad E_\sigma(\lambda) = \{Q \in E : M_\sigma\mu(Q) > \lambda\},$$

and

$$(5.4) \quad R_\sigma(\lambda) = \{\gamma_Q(t) \in R^\gamma : t^{2-n} \mu(B(\gamma_Q(t), \sigma t)) > \lambda\},$$

where R^γ is given by (3.4). Then

$$\pi(R_\sigma(\lambda)) = E_\sigma(\lambda).$$

Moreover, there are countably many points $\{A_i \in R_\sigma(\lambda) : A_i = \gamma_{Q_i}(t_i), i = 1, 2, \dots\}$ such that $\{B(A_i, \sigma t_i)\}$ are mutually disjoint and

$$R_\sigma(\lambda) \subset \bigcup_{i=1}^{\infty} B(A_i, 5\sigma t_i) \cap D.$$

Proof. The first statement is obvious and the second is the standard covering lemma of Vitali type (see [12], for example). ■

LEMMA 5.3. Let $\lambda > 0$ and $0 < \sigma < 1$. Then there is a constant $\beta > 0$, depending only on D , such that

$$(5.5) \quad \omega^O(E_\sigma(\lambda)) \lesssim \frac{1}{\lambda} \sigma^{n+\beta-2} G\mu(O),$$

where $O \in D$ is as in Remarks 3.3(b).

Proof. First we claim that there is a constant $\beta > 0$ such that if $Q \in E$ then

$$(5.6) \quad G(O, \gamma_Q(\sigma t)) \lesssim \sigma^\beta G(O, Y),$$

for all $Y \in B(\gamma_Q(t), \varkappa \sigma t)$.

To prove (5.6), we fix $Q \in E$ and let $0 < t < \varrho$. According to the localization theorem of NTA domains ([7, Theorem 3.11]), there is an NTA domain $\Omega \subset D$, whose constant C_Ω is independent of Q , such that

$$B(Q, 2\varrho) \cap D \subset \Omega \subset B(Q, 2C_\Omega^2\varrho) \cap D.$$

Then $G(O, \cdot)$ is a positive harmonic function in the NTA domain Ω by the choice of the point $O \in D$, and vanishes identically on $\partial\Omega \cap B(Q, 2\varrho)$ ($= \partial D \cap B(Q, 2\varrho)$). By Lemma 4.4 of [7] and the Harnack chain condition for the NTA domain Ω , we find, since $0 < t < \varrho$, that

$$G(O, X) \lesssim G(O, A_t(Q)) \lesssim G(O, Y)$$

for all $X \in \overline{B(Q, t)} \cap D$ and $Y \in B(\gamma_Q(t), \varkappa \sigma t)$, where $A_t(Q)$ is a t -nontangential point of Q in Ω as in the definition of NTA domains. However, Lemma 4.1 of [7] states that there exists $\beta > 0$ which depends only on D such that

$$G(O, \gamma_Q(\sigma t)) \lesssim \left(\frac{|\gamma_Q(\sigma t) - Q|}{t} \right)^\beta \sup\{G(O, X) : X \in \partial B(Q, t) \cap D\}.$$

Therefore, (5.6) follows from the above two estimates, since $|\gamma_Q(\sigma t) - Q| < \sigma t$.

Next, if $\gamma_Q(t) \in R_\sigma(\lambda)$, then

$$\mu(B(\gamma_Q(t), \sigma t)) > \lambda t^{n-2}.$$

This together with (5.6) implies that

$$\begin{aligned} \int_{B(\gamma_Q(t), \sigma t)} G(O, Y) d\mu(Y) &\gtrsim \frac{1}{\sigma^\beta} G(O, \gamma_Q(\sigma t)) \mu(B(\gamma_Q(t), \sigma t)) \\ &> \frac{\lambda}{\sigma^\beta} t^{n-2} G(O, \gamma_Q(\sigma t)). \end{aligned}$$

Thus, if $A_i \in R_\sigma(\lambda)$ is as in Lemma 5.2, we have

$$(5.7) \quad t_i^{n-2} G(O, A_i^\sigma) \lesssim \frac{\sigma^\beta}{\lambda} \int_{B(A_i, \sigma t_i)} G(O, Y) d\mu(Y),$$

where we have denoted $\gamma_{Q_i}(\sigma t_i)$ by A_i^σ .

Finally, the radial condition (2') of the NTR mapping γ implies that

$$\pi B(A_i, 5\sigma t_i) \subset \Delta(Q_i, 5L\sigma t_i).$$

The doubling property of ω^O and Dahlberg's comparison theorem give

$$\omega^O(\Delta(Q_i, 5L\sigma t_i)) \lesssim \omega^O(\Delta(Q_i, \sigma t_i)),$$

and

$$\omega^O(\Delta(Q_i, \sigma t_i)) \approx t_i^{n-2} G(O, A_i^\sigma).$$

Thus, by using Lemma 5.2 together with the above facts and (5.7), we obtain

$$\begin{aligned} \omega^O(E_\sigma(\lambda)) &\leq \sum_{i=1}^{\infty} \omega^O(\pi B(A_i, 5\sigma t_i)) \leq \sum_{i=1}^{\infty} \omega^O(\Delta(Q_i, 5L\sigma t_i)) \\ &\lesssim \sum_{i=1}^{\infty} \omega^O(\Delta(Q_i, \sigma t_i)) \lesssim \sum_{i=1}^{\infty} (\sigma t_i)^{n-2} G(O, A_i^\sigma) \\ &\lesssim \frac{\sigma^{n+\beta-2}}{\lambda} \sum_{i=1}^{\infty} \int_{B(A_i, \sigma t_i)} G(O, Y) d\mu(Y) \leq \frac{1}{\lambda} \sigma^{n+\beta-2} G\mu(O). \end{aligned}$$

The last inequality is because the family $\{B(A_i, \sigma t_i)\}$ is mutually disjoint. This concludes the proof. ■

Proof of Lemma 4.4. Pick a number ε with $0 < \varepsilon < \beta/(n-2)$. Then by Lemma 5.1, we have

$$\begin{aligned} M_0\mu &\leq C \sum_{j=0}^{\infty} 2^j M_{\sigma_j}\mu = C \sum_{j=0}^{\infty} 2^{-\varepsilon j} 2^{(1+\varepsilon)j} M_{\sigma_j}\mu \\ &\leq \frac{C}{1-2^{-\varepsilon}} \sup_{j \geq 0} \{2^{(1+\varepsilon)j} M_{\sigma_j}\mu\}, \end{aligned}$$

where the constant C depends only on D . We next observe that

$$\{Q \in E : M_0\mu(Q) > \lambda\} \subset \bigcup_{j=0}^{\infty} E_{\sigma_j}(C2^{-(1+\varepsilon)j\lambda}),$$

where $E_\sigma(\lambda)$ is given by (5.3). Thus, we are able to apply Lemma 5.3 and

obtain

$$\begin{aligned} \omega^O(\{Q \in E : M_0\mu(Q) > \lambda\}) &\leq \sum_{j=0}^{\infty} \omega^O(E_{\sigma_j}(C2^{-(1+\varepsilon)j\lambda})) \\ &\leq \frac{C}{\lambda} G\mu(O) \sum_{j=0}^{\infty} \sigma_j^{n+\beta-2} 2^{(1+\varepsilon)j} \\ &= \frac{C}{\lambda} G\mu(O) \sum_{j=0}^{\infty} 2^{-(\beta/(n-2)-\varepsilon)j} = \frac{C}{\lambda} G\mu(O). \end{aligned}$$

The proof is therefore complete. ■

6. L^p -nontangential boundary limits

DEFINITION 6.1. Let D be an NTA domain in \mathbb{R}^n , and $1 \leq p < \infty$. We say that a function u defined in D has an L^p -nontangential limit ζ at $Q \in \partial D$ if

$$(6.1) \quad \lim_{t \searrow 0} \frac{1}{|\Gamma_\alpha^t(Q)|} \int_{\Gamma_\alpha^t(Q)} |u(X) - \zeta|^p dX = 0$$

for all $\alpha > 1$, where $\Gamma_\alpha^t(Q) = \Gamma_\alpha(Q) \cap B(Q, t)$ and $|\Gamma_\alpha^t(Q)|$ denotes the Lebesgue measure of the set $\Gamma_\alpha^t(Q)$.

As was pointed out in [18], at least formally, the ordinary nontangential limit may be viewed as a limiting case of the L^p -nontangential limit as $p \nearrow \infty$. It is also clear that the ordinary nontangential limit is stronger than the L^p -nontangential limit, that is, if u converges nontangentially to a ζ at $Q \in \partial D$ then u has L^p -nontangential limit ζ there. Hence, every positive harmonic function on D has finite L^p -nontangential limit ω -a.e. on ∂D . The same is true for the nonsingular part of a Green potential (see Lemma 5.2).

The following theorem is an extension of Ziemek's result ([18]) to NTA domains. For the case of Lipschitz domains in the plane see [18] where conformal mappings were used. We remark that it has been shown in [18] that the range of the exponent p which appears in the statement of the theorem is sharp. We also notice that the L^p -nontangential limit of u necessarily coincides with the γ -radial limit of u ω -a.e. on the set on which the NTR mapping γ is defined.

THEOREM 6.2. *Let D be an NTA domain in \mathbb{R}^n . If u is subharmonic in D and has a nonnegative harmonic majorant, then u has finite L^p -nontangential limit ω -a.e. on ∂D , provided $1 \leq p < n/(n-2)$.*

In order to prove the theorem, we need to prepare some lemmas. The first lemma may have some independent interest.

LEMMA 6.3. Suppose $G\mu$ is a Green potential in D . Then for fixed $O \in D$ and $\alpha > C_D$,

$$(6.2) \quad \int_{\partial D} \int_{\Gamma_\alpha(Q)} \delta(X)^{2-n} d\mu(X) d\omega^O(Q) \lesssim G\mu(O).$$

In particular,

$$(6.3) \quad \int_{\Gamma_\alpha(Q)} \delta(X)^{2-n} d\mu(X) < \infty$$

ω -a.e. for $Q \in \partial D$.

PROOF. The second assertion follows from the first if we choose $O \in D$ so that $G\mu(O) < \infty$.

We now prove (6.2). We shall use the following notation: For $X \in D$ and $\alpha > C_D$, we let

$$(6.4) \quad \tilde{\Gamma}_\alpha(X) = \{Q \in \partial D : X \in \Gamma_\alpha(Q)\}.$$

We denote by \tilde{X} a point of ∂D closest to X , that is, $\tilde{X} \in \partial D$ with $|\tilde{X} - X| = \delta(X)$, it then follows from the triangle inequality that $\tilde{\Gamma}_\alpha(X) \subset B(\tilde{X}, (1+\alpha)\delta(X))$.

We next recall that Dahlberg's comparison theorem implies that

$$\delta(X)^{2-n} \omega^O(\Delta(\tilde{X}, \delta(X))) \lesssim G(O, X)$$

for all $X \in D$. We notice that the last inequality is trivial on the set $\{X \in D : \delta(X) \geq r_0\}$, where $r_0 = \frac{1}{2} \min\{r_D, \delta(O)\}$, since on this set $G(O, \cdot)$ is uniformly bounded away from 0 and $\delta(X)^{2-n} \omega^O(\Delta(\tilde{X}, \delta(X))) \leq r_0^{2-n}$.

Now the above facts together with the doubling property of ω^O show that

$$\delta(X)^{2-n} \omega^O(\tilde{\Gamma}_\alpha(X)) \lesssim \delta(X)^{2-n} \omega^O(\Delta(\tilde{X}, \delta(X))) \lesssim G(O, X).$$

Therefore, by using Fubini's theorem, we obtain

$$\begin{aligned} \int_{\partial D} \int_{\Gamma_\alpha(Q)} \delta(X)^{2-n} d\mu(X) d\omega^O(Q) &= \int_D \delta(X)^{2-n} \omega^O(\tilde{\Gamma}_\alpha(X)) d\mu(X) \\ &\lesssim \int_D G(O, X) d\mu(X) = G\mu(O), \end{aligned}$$

as we desired. ■

Once again, we shall split the subharmonic function u in the theorem into several parts, and, as before, the interesting part will be the singular part of the corresponding Green potential in the splitting. The main estimate we need in the proof of Theorem 6.2 is the following:

LEMMA 6.4. Given $1 \leq p < n/(n-2)$ and $\alpha > C_D$, there exist constants $C > 0$ and $\beta > \alpha$, depending only on α, p and D , such that, if $G\mu$ is a Green potential in D then

$$(6.5) \quad \frac{1}{|\Gamma_\alpha^t(Q)|} \int_{\Gamma_\alpha^t(Q)} (G_0\mu(X))^p dX \leq C \left(\int_{\Gamma_\beta(Q)} \delta(Y)^{2-n} d\mu(Y) \right)^p$$

for all $t > 0$ and $Q \in \partial D$, where $G_0 = G_{0,\varkappa}$ is defined by (4.1) with $\varkappa = (8C_D\alpha)^{-1}$.

We leave the proof of the lemma to the next section, and now we assume this to complete the proof of Theorem 6.2.

PROOF OF THEOREM 6.2. Let $\alpha > C_D$ be fixed. By the same consideration as in the proof of Theorem 4.1, it suffices to prove that every Green potential $G\mu$ has L^p -nontangential limit 0 ω -a.e. on ∂D . To this end, we use the splitting $G\mu = G_0\mu + G_1\mu$, where $G_0 = G_{0,\varkappa}$ and $G_1 = G_{1,\varkappa}$ are defined by (4.1) and (4.2), respectively, with the choice of $\varkappa = (8C_D\alpha)^{-1}$. By Lemma 4.2, $G_1\mu$ has nontangential limit 0 ω -a.e. and hence has L^p -nontangential limit 0 ω -a.e. We now look at $G_0\mu$. We need to show that there is $O \in D$ such that

$$\omega^O \left(\left\{ Q \in D : \limsup_{t \searrow 0} \frac{1}{|\Gamma_\alpha^t(Q)|} \int_{\Gamma_\alpha^t(Q)} (G_0\mu(X))^p dX > \varepsilon \right\} \right) < \varepsilon$$

for any $\varepsilon > 0$.

Now, let $O \in D$ be fixed such that $G\mu(O) < \infty$. For a given $\varepsilon > 0$, the same reason as in the proof of Theorem 4.1 allows us to assume that

$$G\mu(O) < \varepsilon \left(\frac{\varepsilon}{C} \right)^{1/p},$$

where the constant C is as in Lemma 6.4. We set

$$E_\varepsilon = \left\{ Q \in \partial D : \int_{\Gamma_\beta(Q)} \delta(Y)^{2-n} d\mu(Y) > \left(\frac{\varepsilon}{C} \right)^{1/p} \right\},$$

where $\beta > \alpha$ corresponds to the given α as in Lemma 6.4. Applying Lemmas 6.4 and 6.3, we finally obtain

$$\begin{aligned} &\omega^O \left(\left\{ Q \in D : \limsup_{t \searrow 0} \frac{1}{|\Gamma_\alpha^t(Q)|} \int_{\Gamma_\alpha^t(Q)} (G_0\mu(X))^p dX > \varepsilon \right\} \right) \\ &\leq \omega^O \left(\left\{ Q \in \partial D : C \left(\int_{\Gamma_\beta(Q)} \delta(Y)^{2-n} d\mu(Y) \right)^p > \varepsilon \right\} \right) \end{aligned}$$

$$\begin{aligned} &< \left(\frac{C}{\varepsilon}\right)^{1/p} \int_{B_\varepsilon} \int_{\Gamma_\beta(Q)} \delta(Y)^{2-n} d\mu(Y) d\omega^O(Q) \\ &\leq \left(\frac{C}{\varepsilon}\right)^{1/p} \int_{\partial D} \int_{\Gamma_\beta(Q)} \delta(Y)^{2-n} d\mu(Y) d\omega^O(Q) \leq \left(\frac{C}{\varepsilon}\right)^{1/p} G\mu(O) < \varepsilon. \end{aligned}$$

Therefore the theorem is established. \blacksquare

7. Proof of Lemma 6.4. The following covering lemma for a truncated nontangential region is needed in order to prove Lemma 6.4. The idea behind this is quite simple, that is, a nontangential region $\Gamma_\alpha^t(Q)$ is roughly the union of nontangential balls of the boundary point $Q \in \partial D$.

LEMMA 7.1. *Let $\alpha > C_D$, $t > 0$ and $Q \in \partial D$. Then there is a covering \mathcal{B} of $\Gamma_\alpha^t(Q)$ consisting of nontangential balls centered at points of $\Gamma_\alpha^t(Q)$ such that*

- (1) if $B(Z, s) \in \mathcal{B}$ then $B(X, (8C_D\alpha)^{-1}\delta(X)) \subset B(Z, \lambda s)$ for all $X \in B(Z, s)$;
- (2) if $B(Z, s) \in \mathcal{B}$ then $B(Z, 2\lambda s) \subset \Gamma_\beta(Q)$;
- (3) if $B(Z, s) \in \mathcal{B}$ then $c_1 s < \delta(X) < c_2 s$ for all $X \in B(Z, \lambda s)$; and
- (4) $\sum_{B \in \mathcal{B}} |B| \leq C |\Gamma_\alpha^t(Q)|$,

for some constants $\lambda > 1$, $\beta > \alpha$, and $c_1, c_2, C > 0$, which depend only on α and D .

Proof. We fix a number κ so that $0 < \kappa < (4\alpha(1 + (8C_D\alpha)^{-1}))^{-1}$. For each nonnegative integer j put $\sigma_j = (2C_D)^{-j}$, and define

$$(7.1) \quad R_j(Q) = \Gamma_\alpha^{\sigma_j^{-1}t}(Q) \setminus \Gamma_\alpha^{\sigma_j t}(Q), \quad j = 1, 2, \dots$$

We claim that there is a positive integer N , depending only on α and D , such that, for each j , there is a covering \mathcal{B}_j of $R_j(Q)$ consisting of at most N balls with radius $\kappa\sigma_j t$ and centered at points of $R_j(Q)$.

The last claim follows from the well-known argument of the proof of the Besicovitch covering theorem (see [5], for example), but is somewhat simpler. We let j be fixed. By induction, we can choose a finite family $\{X_k\}_{k=1}^M$ so that $X_1 \in R_j(Q)$ and $X_{k+1} \in R_j(Q) \setminus \bigcup_{i=1}^k B_i$, $1 \leq k < M$, and so that $R_j(Q) \subset \bigcup_{k=1}^M B_k$, where $B_k = B(X_k, \kappa\sigma_j t)$. We shall prove that the covering $\mathcal{B}_j = \{B_k\}_{k=1}^M$ of $R_j(Q)$ meets our requirement. To see this, we notice that the collection $\{\frac{1}{2}B_k\}_{k=1}^M$ is disjoint, where $\frac{1}{2}B_k = B(X_k, \frac{1}{2}\kappa\sigma_j t)$, and $\bigcup_{k=1}^M \frac{1}{2}B_k \subset B(Q, (\sigma_{j-1} + \frac{1}{2}\kappa\sigma_j)t)$. Therefore, the number M is less than or equal to the number

$$N = \left[\left(\frac{(\sigma_{j-1} + \frac{1}{2}\kappa\sigma_j)t}{\frac{1}{2}\kappa\sigma_j t} \right)^n \right] = \left[\left(\frac{4C_D}{\kappa} + 1 \right)^n \right],$$

where $[q]$ denotes the largest integer dominated by q .

We now verify the properties of $\mathcal{B} = \bigcup_{j=1}^\infty \mathcal{B}_j$ listed in the lemma:

(1) To choose λ , let j be fixed and let $B(Z, s) \in \mathcal{B}_j$, i.e., $Z \in R_j(Q)$ and $s = \kappa\sigma_j t$. If $X \in B(Z, s)$ then, since $\sigma_j t \leq |Z - Q| < \alpha\delta(Z)$ and $\kappa < \alpha^{-1}$, we have

$$\delta(X) \geq \delta(Z) - |X - Z| > (\alpha^{-1} - \kappa)\sigma_j t > 0.$$

On the other hand, we have

$$\delta(X) \leq |X - Q| \leq |X - Z| + |Z - Q| \leq (2C_D + \kappa)\sigma_j t,$$

as $|Z - Q| < \sigma_{j-1}t = 2C_D\sigma_j t$. Thus,

$$B(X, (8C_D\alpha)^{-1}\delta(X)) \subset B(X, (8C_D\alpha)^{-1}(2C_D + \kappa)\sigma_j t) \subset B(Z, \lambda s),$$

where we have chosen

$$(7.2) \quad \lambda = 1 + (4\kappa\alpha)^{-1} + (8C_D\alpha)^{-1}.$$

A similar estimate shows that (2) is valid if we choose

$$(7.3) \quad \beta = 2 \frac{\kappa + C_D + (8C_D\alpha)^{-1}(2C_D + \kappa)}{(2\alpha)^{-1} - 2(1 + (8C_D\alpha)^{-1}\kappa)}.$$

(3) follows from (1) and (2) immediately. One can easily check that, if $B(Z, s)$ is as above and $X \in B(Z, \lambda s)$, then

$$(7.4) \quad 2[1 + (8C_D\alpha)^{-1}]s < \delta(X) < [1 + 2C_D + (4\kappa\alpha)^{-1} + (8C_D\alpha)^{-1}]s.$$

To prove (4), it suffices to show that there is a constant C , depending only on α and D , such that

$$(7.5) \quad \sum_{B \in \mathcal{B}_j} |B| \leq C |R_j(Q)|$$

for all $j = 1, 2, \dots$. The last inequality will follow if we can show that $R_j(Q)$ contains a ball with radius comparable with $\sigma_j t$. But this fact is a consequence of the definition of NTA domains. To be precise, first we take an integer $k > 1$ large enough so that

$$\frac{C_D + 2^{-k}}{1 - 2^{-k}} < \alpha.$$

This is possible since $\alpha > C_D$ by our assumption. Next, we choose ε with $0 < \varepsilon < 1$ close enough to 1 so that

$$2^\varepsilon(1 - 2^{1-k}) > 1,$$

and finally we take k even larger to make

$$2^\varepsilon(C_D + 2^{-k}) < 2C_D.$$

Now, by the definition of NTA domains, for $Q \in \partial D$, and $r = 2^\varepsilon C_D \sigma_j t$, there is $A \in D$ with

$$|A - Q| < 2^\varepsilon C_D \sigma_j t, \quad \delta(A) > 2^\varepsilon \sigma_j t.$$

We claim that $B(A, 2^{\varepsilon-k} \sigma_j t) \subset R_j(Q)$. To see this, let $X \in B(A, 2^{\varepsilon-k} \sigma_j t)$. Then

$$\delta(X) \geq \delta(A) - |A - X| > 2^\varepsilon (1 - 2^{-k}) \sigma_j t.$$

Thus,

$$\begin{aligned} |X - Q| &\leq |X - A| + |A - Q| < 2^\varepsilon (C_D + 2^{-k}) \sigma_j t \\ &< \frac{C_D + 2^{-k}}{1 - 2^{-k}} \delta(X) < \alpha \delta(X). \end{aligned}$$

Moreover,

$$|X - Q| < 2^\varepsilon (C_D + 2^{-k}) \sigma_j t < 2C_D \sigma_j t = \sigma_{j-1} t.$$

Finally,

$$\begin{aligned} |X - Q| &\geq |A - Q| - |X - A| \geq \delta(X) - |A - X| \\ &> 2^\varepsilon (1 - 2^{1-k}) \sigma_j t > \sigma_j t. \end{aligned}$$

These show that $X \in R_j(Q)$. ■

Proof of Lemma 6.4. Let \mathcal{B} be a covering of $\Gamma_\alpha^t(Q)$ as in Lemma 7.1. We first claim that for each $B(Z, s) \in \mathcal{B}$ we have

$$(7.6) \quad \int_{B(Z, s)} (G_0 \mu(X))^p dX \lesssim |B(Z, s)| \left(\int_{\Gamma_\beta(Q)} \delta(Y)^{2-n} d\mu(Y) \right)^p.$$

To prove this, we first notice that if $X \in B(Z, s)$ then $G(X, Y) \approx |X - Y|^{2-n}$ for all $Y \in B(X, (8C_D \alpha)^{-1} \delta(X))$, and if $Y \in B(Z, \lambda s)$ then $B(Z, s) \subset B(Y, 2\lambda s)$. Thus, by using Fubini's theorem and Minkowski's inequality for integrals, together with the properties of \mathcal{B} in Lemma 7.1, we get

$$\begin{aligned} \int_{B(Z, s)} (G_0 \mu(X))^p dX &\lesssim \int_{B(Z, s)} \left(\int_{B(X, (8C_D \alpha)^{-1} \delta(X))} |X - Y|^{2-n} d\mu(Y) \right)^p dX \\ &\leq \int_{B(Z, s)} \left(\int_{B(Z, \lambda s)} |X - Y|^{2-n} d\mu(Y) \right)^p dX \\ &\leq \left\{ \int_{B(Z, \lambda s)} \left(\int_{B(Z, s)} |X - Y|^{(2-n)p} dX \right)^{1/p} d\mu(Y) \right\}^p \\ &\leq \left\{ \int_{B(Z, \lambda s)} \left(\int_{B(Y, 2\lambda s)} |X - Y|^{(2-n)p} dX \right)^{1/p} d\mu(Y) \right\}^p \end{aligned}$$

$$\begin{aligned} &\approx \left\{ \int_{B(Z, \lambda s)} \left(\int_0^{2\lambda s} \rho^{(2-n)p+n-1} d\rho \right)^{1/p} d\mu(Y) \right\}^p \\ &= \frac{(2\lambda s)^{n+(2-n)p}}{n+(2-n)p} \left(\int_{B(Z, \lambda s)} d\mu(Y) \right)^p \\ &\lesssim s^n \left(\int_{B(Z, \lambda s)} \delta(Y)^{2-n} d\mu(Y) \right)^p \\ &\lesssim |B(Z, s)| \left(\int_{\Gamma_\beta(Q)} \delta(Y)^{2-n} d\mu(Y) \right)^p. \end{aligned}$$

Now, by using (7.6) and property (4) of \mathcal{B} in Lemma 7.1, we obtain

$$\begin{aligned} \frac{1}{|\Gamma_\alpha^t(Q)|} \int_{\Gamma_\alpha^t(Q)} (G_0 \mu(X))^p dX &\leq \frac{1}{|\Gamma_\alpha^t(Q)|} \sum_{B \in \mathcal{B}} \int_B (G_0 \mu(X))^p dX \\ &\leq \frac{C}{|\Gamma_\alpha^t(Q)|} \left(\sum_{B \in \mathcal{B}} |B| \right) \left(\int_{\Gamma_\beta(Q)} \delta(Y)^{2-n} d\mu(Y) \right)^p \\ &\leq C \left(\int_{\Gamma_\beta(Q)} \delta(Y)^{2-n} d\mu(Y) \right)^p, \end{aligned}$$

where β is as in Lemma 7.1 and the constant C depends only on α, p and D as we can see in the proof. ■

8. Boundary limits under nontangential conditions. First we recall a notation: If D is an NTA domain in \mathbb{R}^n then for a set $E \subset \partial D$ and $\alpha > 1$, we denote the *sawtooth region* of opening α over E by

$$(8.1) \quad S_\alpha(E) = \bigcup_{Q \in E} \Gamma_\alpha(Q).$$

If more than one domain is under consideration, we shall write $\Gamma_{\alpha, D}(Q)$ and $S_{\alpha, D}(E)$ to indicate the domain D they belong to.

The following lemma is taken from [7] (Lemmas 6.1 and 6.3).

LEMMA 8.1. *Let D be an NTA domain in \mathbb{R}^n , and $O \in D$ be a fixed point.*

(i) *For any $E \subset \partial D$ and $t, \varepsilon > 0, \beta > \alpha > C_D$, there exists a closed set $F \subset E$ and a number $r > 0$ such that $\omega^O(E \setminus F) < \varepsilon$ and $S_\beta^r(F) \subset S_\alpha^t(E)$.*

(ii) *For any $\alpha > C_D$, there exist $\beta, \theta > C_D$ such that for any closed set $F \subset \partial D$ there exists an NTA domain $\Omega \subset D$ which has the following properties:*

- (1) $\partial D \cap \partial \Omega = F$;
- (2) $\Omega \subset S_{\beta, D}(F)$;

- (3) $\Gamma_{\alpha,D}(Q) \subset \Gamma_{\theta,\Omega}(Q)$ for every $Q \in F$; and
(4) the harmonic measures ω of D and ω_Ω of Ω are mutually absolutely continuous on F .

We now give the first result of this section; our proof is on the line of one for harmonic functions (see Theorem 6.4 of [7]). We start with a definition.

DEFINITION 8.2. Let D be an NTA domain in \mathbb{R}^n . We say that a function u defined on D is *nontangentially bounded from above at $Q \in \partial D$* if there exist $\alpha > C_D$, $r > 0$ and $C > 0$ such that $u(X) \leq C$ for all $X \in \Gamma_\alpha^r(Q)$.

If $E \subset \partial D$, we say that u is *nontangentially bounded from above on E* if u is nontangentially bounded from above at every point of E .

THEOREM 8.3. Let D be an NTA domain in \mathbb{R}^n and $E \subset \partial D$ be a Borel set. Suppose u is subharmonic in D and nontangentially bounded from above on E . Then u has L^p -nontangential limit ω -a.e. on E for $1 \leq p < n/(n-2)$. If, in addition, there is an NTR mapping γ on E , then u has γ -radial limit ω -a.e. on E . Moreover, these two limits coincide ω -a.e. on E .

Proof. Let D, E and γ be as in the statement of the theorem and let $O \in D$ be as in Remark 3.3(b). First we observe that, by the corkscrew condition of γ in Definition 3.1 for each $Q \in E$ the locus $\{\gamma_Q(t) : t \in [0, \varrho]\}$ is contained in every nontangential region $\Gamma_\alpha(Q)$ if $\alpha > C_D$.

We now fix an $\alpha > C_D$, and let $\beta > C_D$ be as in Lemma 8.1(ii). Let $\varepsilon > 0$. Since u is nontangentially bounded from above, there exist constants $\alpha_1 > C_D$, $t > 0$ and $C_1 > 0$ and a set $E_1 \subset E$ with $\omega^O(E \setminus E_1) < \frac{1}{2}\varepsilon$ such that $u(X) \leq C_1$ for all $X \in S_{\alpha_1}^t(E_1)$. By Lemma 8.1(i), for the given $\beta > C_D$, there exists a closed set $F \subset E_1$ with $\omega^O(E_1 \setminus F) < \frac{1}{2}\varepsilon$ and $r > 0$ such that $S_\beta^r(F) \subset S_{\alpha_1}^t(E_1)$. Note that $S_\alpha(F) \setminus S_\alpha^r(F)$ is relatively compact in D , and thus the upper semi-continuity of u implies that there exists a constant $C > 0$ so that $u \leq C$ for all $X \in S_\beta(F)$.

Now, we apply Lemma 8.1(ii) to get an NTA domain Ω with the properties (1)–(4) listed there. The property (2) implies that $u \leq C$ on Ω , and by the properties (3) and (1) we may consider that γ is an NTR mapping on $F \subset \partial\Omega$. Therefore, by Theorem 4.1 we conclude that u has finite γ -radial limit ω_Ω -a.e. on F . The second assertion thus follows because of condition (4) and ε being arbitrary.

For the L^p -nontangential limit ($1 \leq p < n/(n-2)$), the proof is the same but we need to use Theorem 6.2 instead of Theorem 4.1. ■

In order to prove the second result (Theorem 8.5) of this section we need the following lemma. For the case of upper half-spaces, this lemma is essentially proved by Stein in [11]. Our proof is adapted from [7, Theorem 6.7].

LEMMA 8.4. Let D be an NTA domain, g a nonnegative Borel measurable function on D and $E \subset \partial D$ a Borel set. Suppose that for each $Q \in E$ there

exists $\beta > C_D$ (depending on Q) such that

$$(8.2) \quad \int_{\Gamma_\beta(Q)} g(X) dX < \infty.$$

Then, for every $\varepsilon > 0$ and $\alpha > C_D$, there exists a closed set $F \subset E$ with $\omega^O(E \setminus F) < \varepsilon$ such that

$$(8.3) \quad \int_{\Gamma_\alpha(Q)} g(X) dX < \infty$$

ω -a.e. for $Q \in F$, where $O \in D$ is some fixed point.

Proof. Let $\alpha > C_D$ be given. For any $\varepsilon > 0$, by (8.2), there is a constant $C > 0$ and a Borel set $E_1 \subset E$ with $\omega^O(E \setminus E_1) < \frac{1}{3}\varepsilon$ such that

$$\int_{\Gamma_\beta(Q)} g(X) dX < C$$

for all $Q \in E_1$. By Lemma 8.1(i), there is a closed set $E_2 \subset E_1$ satisfying $\omega^O(E_1 \setminus E_2) < \frac{1}{3}\varepsilon$ such that $S_\alpha(E_2) \subset S_{\beta/2}(E_1)$. Next, by a well-known fact on points of density, there is a constant $c > 0$ and a closed set $F \subset E_2$ such that $\omega^O(E_2 \setminus F) < \frac{1}{3}\varepsilon$ and

$$\omega^O(E_2 \cap \Delta(Q, r)) \geq c\omega^O(\Delta(Q, r))$$

for all $Q \in F$ and $r > 0$.

Now, let $X \in S_{\beta/2}(F)$. Choose $P \in F$ so that $X \in \Gamma_{\beta/2}(P)$. Let $\tilde{\Gamma}_\beta(X)$ be given by (6.4). Since $\tilde{\Gamma}_\beta(X) \supset E_2 \cap \Delta(P, \frac{1}{2}\beta\delta(X))$, we have

$$\omega^O(\tilde{\Gamma}_\beta(X)) \geq c\omega^O(\Delta(P, \frac{1}{2}\beta\delta(X))).$$

We next notice that $\Delta(\tilde{X}, \delta(X)) \subset \Delta(P, (2 + \frac{1}{2}\beta)\delta(X))$ and $\tilde{\Gamma}_\beta(X) \subset \Delta(\tilde{X}, (1 + \alpha)\delta(X))$, where \tilde{X} is a point of ∂D so that $|X - \tilde{X}| = \delta(X)$. Thus, by the doubling property of ω^O , we have

$$\begin{aligned} \omega^O(\tilde{\Gamma}_\beta(X)) &\gtrsim \omega^O(\Delta(P, \frac{1}{2}\beta\delta(X))) \\ &\gtrsim \omega^O(\Delta(P, (2 + \frac{1}{2}\beta)\delta(X))) \geq \omega^O(\Delta(\tilde{X}, \delta(X))) \\ &\gtrsim \omega^O(\Delta(\tilde{X}, (1 + \alpha)\delta(X))) \gtrsim \omega^O(\tilde{\Gamma}_\alpha(X)). \end{aligned}$$

Finally, by Fubini's theorem, we find that

$$\begin{aligned} \int_F \int_{\Gamma_\alpha(Q)} g(X) dX d\omega^O(Q) &= \int_F \int_{S_\alpha(F)} g(X) \chi_{\Gamma_\alpha(Q)}(X) dX d\omega^O(Q) \\ &= \int_{S_\alpha(F)} g(X) \omega^O(\tilde{\Gamma}_\alpha(X)) dX \end{aligned}$$

$$\begin{aligned}
 &\lesssim \int_{S_{\beta/2}(E_1)} g(X) \omega^O(\tilde{\Gamma}_\beta(X)) dX \\
 &= \int_{E_1} \int_{\Gamma_{\beta/2}(Q)} g(X) dX d\omega^O(Q) \\
 &\leq \int_{E_1} \int_{\Gamma_\beta(Q)} g(X) dX d\omega^O(Q) \leq C.
 \end{aligned}$$

From this (8.3) follows. ■

The following theorem asserts that under certain conditions on the measure μ the Green potential $G\mu$ does have nontangential limit ω -a.e. on the boundary. It has been shown in [15] that the range of the exponent p in the theorem is the best possible.

THEOREM 8.5. *Let D be an NTA domain in \mathbb{R}^n and let $G\mu$ be a Green potential in D . Suppose that $d\mu(X) = f(X)dX$ for some nonnegative measurable function f defined on D . Suppose that there is a Borel set $E \subset \partial D$ and a number $p > n/2$ such that for each $Q \in E$ there exists $\beta > C_D$ such that*

$$(8.4) \quad \int_{\Gamma_\beta(Q)} \delta(X)^{2p-n} f(X)^p dX < \infty.$$

Then $G\mu$ has nontangential limit 0 ω -a.e. on E .

Proof. Let $\alpha > C_D$ be given. We want to show that $G\mu$ has limit 0 in $\Gamma_\alpha(Q)$ ω -a.e. for $Q \in E$.

Let $O \in D$ be fixed. For any given $\varepsilon > 0$, applying the last lemma to $g(X) = \delta(X)^{2p-n} f(X)^p$ gives a constant $C > 0$ and a set $F \subset E$ with $\omega^O(E \setminus F) < \varepsilon$ such that

$$\int_{\Gamma_{2\alpha}(Q)} \delta(X)^{2p-n} f(X)^p dX < C$$

for all $Q \in F$.

As before we split $G\mu$ into two parts $G_0\mu + G_1\mu$, where $G_0 = G_{0,\varkappa}$ and $G_1 = G_{1,\varkappa}$ are given by (4.1) and (4.2), respectively, with \varkappa small enough so that $B(X, \varkappa\delta(X)) \subset \Gamma_{2\alpha}(Q)$ for all $X \in \Gamma_\alpha(Q)$ and $Q \in F$. By Lemma 4.2, $G_1\mu$ has nontangential limit 0 ω -a.e. For the singular part $G_0\mu$, we use Hölder's inequality and notice that $\delta(Y) \approx \delta(X)$ for $Y \in B(X, \varkappa\delta(X))$ and $X \in \Gamma_\alpha(Q)$. We then have

$$(G_0\mu(X))^p \lesssim \left(\int_{B(X, \varkappa\delta(X))} |X - Y|^{2-n} f(Y) dY \right)^p$$

$$\begin{aligned}
 &\leq \left(\int_{B(X, \varkappa\delta(X))} |X - Y|^{(2-n)p/(p-1)} dY \right)^{p-1} \left(\int_{B(X, \varkappa\delta(X))} f(Y)^p dY \right) \\
 &\lesssim \delta(X)^{2p-n} \int_{B(X, \varkappa\delta(X))} f(Y)^p dY \\
 &\lesssim \int_{\Gamma_{2\alpha}(Q) \cap B(X, \varkappa\delta(X))} \delta(Y)^{2p-n} f(Y)^p dY \rightarrow 0
 \end{aligned}$$

as $\delta(X) \rightarrow 0$. Therefore $\lim_{X \rightarrow Q} G\mu(X) = 0$ with $X \in \Gamma_\alpha(Q)$ for all $Q \in F$. The theorem is therefore proved. ■

References

- [1] L. Carleson, *On the existence of boundary values for harmonic functions in several variables*, Ark. Mat. 4 (1962), 393–399.
- [2] B. E. J. Dahlberg, *On estimates of harmonic measure*, Arch. Rational Mech. Anal. 65 (1977), 272–288.
- [3] —, *On the existence of radial boundary values for functions subharmonic in a Lipschitz domain*, Indiana Univ. Math. J. 27 (1978), 515–526.
- [4] J. L. Doob, *Classical Potential Theory and Its Probabilistic Counterpart*, Springer, New York, 1984.
- [5] H. Federer, *Geometric Measure Theory*, Springer, Berlin, 1969.
- [6] L. L. Helms, *Introduction of Potential Theory*, Wiley-Interscience, New York, 1969.
- [7] D. S. Jerison and C. E. Kenig, *Boundary behavior of harmonic functions in non-tangentially accessible domains*, Adv. in Math. 46 (1982), 80–147.
- [8] —, —, *Hardy spaces, A_∞ , and singular integrals on chord-arc domains*, Math. Scand. 50 (1982), 221–247.
- [9] J. E. Littlewood, *On functions subharmonic in a circle (II)*, Proc. London Math. Soc. (2) 28 (1928), 383–394.
- [10] L. Naïm, *Sur le rôle de la frontière de R. S. Martin dans la théorie du potentiel*, Ann. Inst. Fourier (Grenoble) 7 (1957), 183–281.
- [11] E. M. Stein, *On the theory of harmonic functions of several variables. II. Behavior near the boundary*, Acta Math. 106 (1961), 137–174.
- [12] —, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, N.J., 1970.
- [13] J. C. Taylor, *Fine and nontangential convergence on an NTA domain*, Proc. Amer. Math. Soc. 91 (1984), 237–244.
- [14] D. Ullrich, *Radial limits of M -subharmonic functions*, Trans. Amer. Math. Soc. 292 (1985), 501–518.
- [15] K.-O. Widman, *On the boundary behavior of solutions to a class of elliptic partial differential equations*, Ark. Mat. 6 (1967), 485–533.
- [16] R. Wittmann, *Positive harmonic functions on nontangentially accessible domains*, Math. Z. 190 (1985), 419–438.

- [17] J.-M. Wu, *L^p -densities and boundary behavior of Green potentials*, Indiana Univ. Math. J. 28 (1979), 895–911.
- [18] L. Ziomek, *On the boundary behavior in the metric L^p of subharmonic functions*, Studia Math. 29 (1967), 97–105.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
 UNIVERSITY OF MISSOURI-ST. LOUIS
 ST. LOUIS, MISSOURI 63121
 U.S.A.

Received June 30, 1992

(2967)

Operators on spaces of analytic functions

by

K. SEDDIGHI (Shiraz)

Abstract. Let M_z be the operator of multiplication by z on a Banach space of functions analytic on a plane domain G . We say that M_z is *polynomially bounded* if $\|M_p\| \leq C\|p\|_G$ for every polynomial p . We give necessary and sufficient conditions for M_z to be polynomially bounded. We also characterize the finite-codimensional invariant subspaces and derive some spectral properties of the multiplication operator in case the underlying space is Hilbert.

Introduction. Consider a Banach space \mathcal{E} of functions analytic on a plane domain G , such that for each $\lambda \in G$ the linear functional e_λ of evaluation at λ is bounded on \mathcal{E} . Assume further that \mathcal{E} contains the constant functions and that multiplication by the independent variable z defines a bounded linear operator M_z on \mathcal{E} . In case $\mathcal{E} = \mathcal{H}$ is a Hilbert space the continuity of point evaluations along with the Riesz representation theorem imply that for each $\lambda \in G$ there is a unique function $k_\lambda \in \mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda \rangle$, $f \in \mathcal{H}$. The function k_λ is the *reproducing kernel* for the point λ .

A complex-valued function φ on G for which $\varphi f \in \mathcal{E}$ for every $f \in \mathcal{E}$ is called a *multiplier* of \mathcal{E} and the collection of all multipliers is denoted by $\mathcal{M}(\mathcal{E})$. Each multiplier φ of \mathcal{E} determines a multiplication operator M_φ on \mathcal{E} by $M_\varphi f = \varphi f$, $f \in \mathcal{E}$. Each multiplier is a bounded analytic function on G . In fact $\|\varphi\|_G \leq \|M_\varphi\|$. A good source on this topic is [7].

Twenty years after the appearance of [7] it is reasonable to expect some words explaining the motivation of such a study and of any developments in the area. The description of invariant subspaces in abstract spaces has in fact appeared under some additional hypotheses and one of the first results (for simply connected domains) seems to be [6]. This kind of Beurling's theorem

1991 *Mathematics Subject Classification*: Primary 47B37; Secondary 47A25.

Key words and phrases: spaces of analytic functions, polynomially bounded, multipliers, spectral properties, cyclic subspace.

Research partially supported by a grant (no. 67-SC-520-276) from Shiraz University Research Council.