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Certain lacunary cosine series are recurrent

by

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Abstract. Let the coefficients of a lacunary cosine series be bounded and not square-summable. Then the partial sums of the series are recurrent.

In this note, we wish to consider the partial sums of a lacunary cosine series

$$s_N(x) = \sum_{j=1}^{N} a_j \cos(n_j x + \theta_j) \tag{1}$$

where we assume that a_j, θ_j are real, $|a_j| \le 1$ for every j, and the n_j satisfy $n_{j+1}/n_j \ge \lambda > 1$. We will show:

THEOREM. *Suppose that in addition to the conditions just described, also $\sum |a_j|^2 = \infty$. Then $\{s_N(x)\}$ is dense in $\mathbb{R}$ for almost every $x \in [0, 2\pi]$.*

We will restate the conclusion in probabilistic terms by saying that for almost all x the sequence $s_N(x)$ is recurrent in $\mathbb{R}$. In the case when $a_j = 1$ for all j, this question was posed by T. Murai [4] (see also Brannan and Hayman [3]), and was solved by D. Ullrich [5]. Ullrich was unable to obtain the more general case when $|a_j| \le 1$; however, the proof we give for our theorem, although shorter than that of Ullrich, does take its key idea from his method.

When the terms $\cos(n_j x + \theta_j)$ in (1) are replaced by $\exp(i n_j x)$, the most general conditions which give recurrence of the partial sums in the complex plane are far from being known. Anderson and Pitt [2] showed recurrence in this case when $a_j = 1$ and $n_j = b^j$ for some integer b. They have also shown recurrence in the complex plane when $|a_j| \to 0$ or if the a_j are bounded and $n_{j+1}/n_j \to \infty$ [1]. We stress that even though the series we consider are

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the real parts of the series considered by Anderson and Pitt, our results and techniques seem to give little insight into the complex case.

We borrow a lemma from [5]; since its proof is short, we reproduce it here for completeness.

LEMMA. *Suppose that two sequences of sets* $E_N, F_N \subset [0, 2\pi]$ *have the following property: There exists a constant* $c > 0$ *and a sequence* $\delta_N > 0$ *converging to zero such that for every* $x \in E_N$ *there is an interval* I *of length* δ_N *containing* x *with* $|F_N \cap I| \geq c|I|$. *Suppose that for almost every* $x \in [0, 2\pi]$, $x \in E_N$ *infinitely often. Then for almost every* $x \in [0, 2\pi]$, $x \in F_N$ *infinitely often.*

Proof. If we suppose the contrary, then there exists a set A with $|A| > 0$ and a K such that $A \cap (\bigcup_{N=K}^{\infty} F_N)$ is empty. Almost all points of A are points of density, so we can pick a point which is both a point of density of A and in infinitely many E_N. But then, for any interval I that contains x and $N > K$, $|I \cap A| \leq |I \cap F_N^c|$, so that as $|I| \to 0$, $|I \cap F_N^c|/|I| \to 1$, which contradicts our hypothesis.

Proof of Theorem. Let $a \in \mathbb{R}$ and $\varepsilon > 0$ be given. By Zygmund [6] (Vol. 1, p. 205) we have

$$\sup_N s_N(x) = +\infty \quad \text{and} \quad \inf_N s_N(x) = -\infty$$

for almost every x. Because of this, the set

$$E_N = \{x \in [0, 2\pi] : s_N(x) \geq a \text{ and } s_{N+1}(x) < a\}$$

covers almost every $x \in [0, 2\pi]$ infinitely often. We establish the conditions of the lemma with the sets

$$F_N = \{x \in [0, 2\pi] : |s_N(x) - a| < \varepsilon \text{ or } |s_{N+1}(x) - a| < \varepsilon\}.$$

Notice that

$$\|s'_N\|_\infty \leq \sum_{j=1}^{N} n_j \leq \frac{\lambda}{\lambda - 1} n_N.$$

Let $x \in E_N$. Let I be an interval of length $\delta_N = 2\pi/n_{N+1}$ with center x. Since $s_N(x) > a$, there are two cases.

Case I: $s_N(x_0) = a$ *for some* $x_0 \in I$. In this case, because of the estimate for the derivative of s_N, there is an interval of length $(\lambda - 1)\varepsilon/(\lambda n_N) > c|I|$ around x_0 where $|s_N - a| < \varepsilon$ and c is a constant depending only on λ. At least half of this interval is in $I \cap F_N$.

Case II: $s_N > a$ *on* I. Since $x \in E_N$, $s_N(x) > a$ and $s_{N+1}(x) < a$. Since I is long enough to contain a complete period of $\cos(n_{N+1}x + \theta_{N+1})$, there is $x_1 \in I$ such that $\cos(n_{N+1}x_1 + \theta_{N+1}) = 0$, giving

$$s_{N+1}(x_1) = s_N(x_1) + a_{N+1}\cos(n_{N+1}x_1 + \theta_{N+1}) = s_N(x_1) > a.$$

Thus $s_{N+1} - a$ has a zero in I and the argument above gives an interval of length $(\lambda - 1)\varepsilon/(\lambda n_{N+1}) > c|I|$, centered at this zero, where $|s_{N+1} - a| < \varepsilon$. So $|I \cap F_N| > \frac{c}{2}|I|$.

Applying the lemma, we see that for almost all x, $|s_N(x) - a| < \varepsilon$ infinitely often. Since $a \in \mathbb{R}$ and $\varepsilon > 0$ are arbitrary, the result is proved.

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