Finally, the ergodic theorem for measure preserving transformations tells us that this last integral is equal to

$$\int \frac{\lim_{n\to\infty} M_n(T)(f^p w)(x)}{\lim_{n\to\infty} M_n(T)w(x)} \lim_{n\to\infty} M_n(T)w(x)\,dm = \int \lim_{n\to\infty} M_n(T)(f^p w)(x)\,dm = \int f^p(x)w(x)\,dm\,.$$

3. Remarks. 1) If ϕ is not invertible (1) and (2) in Theorem 1 are still equivalent. To see this one just observes that the proof of (2)⇒(1) does not make use of the invertibility assumption, so it is enough to show that (1)⇒(2). But if (1) holds, then, using again the result of [MT] we conclude that $w^* \equiv \limsup_{n\to\infty} M_n(T)w^{1-q}$ is finite a.e. This means that X can be decomposed as the union of invariant sets X_k, where $X_k \equiv \{x : 2^{k-1} \le w^*(x) < 2^k\}$. But for each k the function w^{1-q} is in $L_1(X_k, dm)$ and then the same argument as in Theorem 1 proves (2) a.e. in X_k.

2) If ϕ is nonsingular and the operator T maps $L_p(\mu)$ into itself then the assumption that our measure μ is of the form $\mu(A) = \int_A w\,dm$ in Theorem 1 is not a restriction, because if (1) holds, i.e., if T satisfies the P.E.T. in $L_p(\mu)$, then there exists an invariant measure m equivalent to μ [AW]. If (3) holds then the measure $dm \equiv \lim M_n(T^*)\mathbf{1}\,d\mu$ is a finite invariant measure.

References

[AW] I. Assani and J. Woś, *An equivalent measure for some nonsingular transformations and applications*, Studia Math. 97 (1990), 1–12.

[MT] F. J. Martín-Reyes and A. de la Torre, *On the almost everywhere convergence of the ergodic averages*, Ergodic Theory Dynamical Systems 10 (1990), 141–149.

ANÁLISIS MATEMÁTICO
FACULTAD DE CIENCIAS
UNIVERSIDAD DE MÁLAGA
29071 MÁLAGA, SPAIN
E-mail: MARTIN_REYES@CCUMA.UMA.ES
TORRE_R@CCUMA.UMA.ES

Received November 29, 1991
Revised version June 3, 1993 (2863)

STUDIA MATHEMATICA 108 (1) (1994)

Unique continuation for elliptic equations and an abstract differential inequality

by

K. SENATOR (Warszawa)

Abstract. We consider a class of elliptic equations whose leading part is the Laplacian and for which the singularities of the coefficients of lower order terms are described by a mixed L^p-norm. We prove that the zeros of the solutions are of at most finite order in the sense of a spherical L^2-mean.

1. Introduction. Unique continuation properties of solutions of second order elliptic equations with bounded coefficients can be studied in detail with the aid of Carleman type inequalities and besides them the use of L^2-norms is sufficient (see e.g. [5]). For the case of unbounded coefficients the situation is different in general. Strong uniqueness for the Schrödinger equation with potential whose integrability exponent is minimal (i.e. equal to $n/2$ where $n \ge 3$ is the dimension of the space) can be proved by this method [8]. One also gets uniqueness theorems for elliptic equations with variable coefficients of the leading part and with coefficients of first order terms in L^{p_1}, $p_1 < \infty$ (see e.g. [6], [10]). However, it appeared that the optimal value of the exponent, $p_1 = n$, cannot be attained via Carleman type inequalities: it is possible to get $p_1 \ge (3n-2)/2$ at most [7], [2]. In fact, the estimates giving uniqueness for such values of the exponent have been found [2], [11]. T. H. Wolff has obtained uniqueness results for a case when $p_1 < (3n-2)/2$ using some modified Carleman type inequalities; namely, he proved strong uniqueness for $p_1 = \max(n, (3n-4)/2)$ [19], and unique continuation from an open set for $p_1 = n$ [20] (the assumptions on all coefficients are minimal in this case).

There are other methods giving uniqueness theorems for various classes of second order elliptic equations. Using a geometric approach N. Garofalo with F. H. Lin [3], [4] and J. L. Kazdan [9] have got strong uniqueness results for equations with coefficients of first order terms having isolated singularities of a maximal rate of growth: they may belong to L^n_{loc} without

1991 *Mathematics Subject Classification*: 35J15, 35A07.

belonging to any L^p_{loc} with $p > n$, and analogously for the zero order term. Just as in the result with mildest assumptions for the bounded coefficients case [1], the coefficients of the leading part are assumed to be Lipschitz continuous. An example in [14] exhibits that this continuity assumption is sharp. In fact, in [9] and [4] it is proved that a zero of a solution is of at most finite order in the sense of a spherical L^2-mean, which implies strong uniqueness. F. H. Lin [13] has derived this property, in the case of the Schrödinger equation with $L^{n/2}_{\mathrm{loc}}$ potential, basing on a weighted inequality of Jerison and Kenig [8] (compare Remark 4.2).

The conclusions of the uniqueness results for solutions of elliptic equations, presented in this paper, are the same as in [9] and [4]. We assume that the leading part is the Laplacian. Our assumptions on the lower order terms are less restrictive than those in [4] and [9], the singularities of the coefficients being described by means of some mixed L^p-norms. The coefficients need not be locally bounded outside a set of isolated points; instead, we postulate that they belong to L^p_{loc} for some $p < \infty$, though no improvement of the earlier uniqueness results for such coefficients is gained.

We derive the properties of zeros of solutions of elliptic equations using a lower bound for a function of a real variable with values in a Hilbert space which satisfies a differential inequality (Theorem 2.1). This function is attached to a solution of the equation as described in P. D. Lax's paper [12].

We consider the differential equation

$$\Delta u + b_1(x) \cdot Du + b_0(x) = 0 \tag{1.1}$$

where Δ is the Laplacian, $b_1(x) = (b_1^1(x), \ldots, b_1^n(x))$, $D = (D_1, \ldots, D_n)$, $D_j = \partial/\partial x_j$.

Let $\mathbb{B}_r = \mathbb{B}_r(x)$ denote the open ball with radius r centered at $x \in \mathbb{R}^n$; $\mathbb{B} = \mathbb{B}_1(0)$ is the unit ball. For a vector-valued function f on $\mathbb{B}$, and $1 \le p, q \le \infty$, define the "mixed" norm

$$\|f\|_{p,q} = \Big(\int_0^1 \Big(\int_{\partial\mathbb{B}} |f(r\omega)|^q \, dS_\omega \Big)^{p/q} r^{n-1} \, dr \Big)^{1/p} \tag{1.2}$$

where $\omega \in \partial\mathbb{B}$. It is seen that $\|f\|_{p,p} = \|f\|_{L^p(\mathbb{B})}$.

The characteristic function of a set S is denoted by χ_S.

THEOREM 1.1. *Let $p_j, q_j \ge n/(2-j)$, $n \ge 3$, $j = 0,1$ and*

$$\frac{1}{p_j} + \frac{n-1}{q_j} \le \frac{3}{2} - j\,. \tag{1.3}$$

There exists a positive number β_0 such that the condition

$$\|b_1\|_{p_1,q_1} + \|b_0\|_{p_0,q_0} \le \beta_0 \tag{1.4}$$

implies the following two assertions:

(i) *if for each $j = 0,1$, either $p_j > n/(2-j)$, or $p_j = n/(2-j)$ and*

$$\int_0^1 \|b_j \chi_{\mathbb{B}_r}\|_{p_j,q_j} \frac{dr}{r} < \infty \tag{1.5}$$

then a solution $u \in H^{1,2}(\mathbb{B})$ of equation (1.1) either vanishes identically in $\mathbb{B}$ or

$$\int_{\partial\mathbb{B}_r} u^2 \, dS \ge C r^\lambda \quad \text{for } 0 < r \le 1 \tag{1.6}$$

where C, λ are some positive constants, depending on u;

(ii) *generally, there exists a function $\eta(r) \to 0$ as $r \to 0$ such that*

$$\int_{\partial\mathbb{B}_r} u^2 \, dS \ge C \exp(-r^{-\eta(r)}) \quad \text{for } 0 < r \le 1\,. \tag{1.7}$$

The theorem is proved in Section 5. Now, we discuss some of its consequences. For any $0 < r \le 1$ put $f_r(x) = f(rx)$. If u is a solution of equation (1.1) then u_r satisfies the equation of the form (1.1) with b_1 replaced by rb_1 and b_0 by $r^2 b_0$. For any function f in $\mathbb{B}$ and $0 < r \le 1$ we have

$$\|f_r\|_{p,q} = r^{-n/p} \|f \chi_{\mathbb{B}_r}\|_{p,q}\,.$$

In particular,

$$\|f_r\|_{L^p(\mathbb{B})} = r^{-n/p} \|f\|_{L^p(\mathbb{B}_r)}\,.$$

COROLLARY 1.1. *Suppose that the assumptions of Theorem 1.1 are satisfied. If*

$$\beta \equiv \|b_1\|_{p_1,q_1} + \|b_0\|_{p_0,q_0} < \infty$$

then there exists a positive number $r_0 \le 1$, depending only on β, such that the assertions (i) *and* (ii) *of the theorem hold with "$0 < r \le 1$" replaced by "$0 < r \le r_0$".*

Proof. The number r_0 is defined by

$$\|b_{1,r_0}\|_{p_1,q_1} + \|b_{0,r_0}\|_{p_0,q_0} \le \beta_0\,.$$

It remains to show that $u_{r_0} = 0$ implies $u = 0$. This follows from the fact that then u is a solution of (1.1) with b_j replaced by $\chi_{\mathbb{B}_{r_0}} b_j$ $(j = 0,1)$.

Condition (1.3) holds when $p_j = q_j = 2n/(3-2j)$ and so, by Corollary 1.1, a zero of a solution of equation (1.1) with $b_1 \in L^{2n}_{\mathrm{loc}}$ and $b_0 \in L^{2n/3}_{\mathrm{loc}}$ is of at most finite order in the sense of the spherical L^2-mean (i.e. estimate (1.6) is valid).

As a consequence of Corollary 1.1 we get a strong uniqueness result for a class of elliptic equations; singularities of their coefficients are described by functions $c_j(x)$ in $\mathbb{B}$ such that for some p_j, q_j satisfying (1.3) and any $0<\varepsilon<1$ we have

$$\|c_j\|_{p_j,q_j} + \|c_j\|_{L^{2n/(3-2j)}(\mathbb{B}-\mathbb{B}_\varepsilon)} < \infty \tag{1.8}$$

for $j=0,1$.

THEOREM 1.2. *Let $u \in H^{1,2}(\Omega)$ be a solution of equation* (1.1) *in an open connected subset Ω of $\mathbb{R}^n$, $n \ge 3$. Suppose that for every $x \in \Omega$ there exist functions c_j in $\mathbb{B}$, $j = 0,1$, satisfying* (1.8) *and a positive number $r \le \operatorname{dist}(x, \partial\Omega)$ such that for $y \in \mathbb{B}_r(x)$,*

$$|b_j(y)| \le c_j\left(\frac{1}{r}(y-x)\right).$$

If the integral condition (1.5) *is satisfied, with c_j in place of b_j, and for some $x_0 \in \Omega$,*

$$\int\limits_{\mathbb{B}_R(x_0)} u^2\,dx = O(R^k)\,, \qquad k = 1,2,\ldots,$$

then u is identically zero in Ω.

If (1.5) *does not hold then u vanishes identically provided*

$$\int\limits_{\mathbb{B}_R(x_0)} u^2\,dx = O(\exp(-r^{-\alpha}))$$

for a positive number α.

2. An abstract differential inequality. We show a lower bound for solutions $v(t)$ of differential inequalities in a Hilbert space $\mathcal{H}$. The scalar product in $\mathcal{H}$ is denoted by $(\cdot,\cdot)$. Given a symmetric operator A in $\mathcal{H}$ we put

$$Lv = \frac{dv}{dt} - Av\,.$$

By $H^1_{\mathrm{loc}}(0,T;\mathcal{H})$ we mean the completion of $C^1(0,T;\mathcal{H})$ in the metric determined by the pseudonorms

$$\Big(\int\limits_0^\eta (\|v(t)\|^2 + \|dv/dt\|^2)\,dt\Big)^{1/2}$$

where $0<\eta<T$.

We shall use the following identities, which are easy to check and known in the logarithmic convexity method. For any function $v(t)$ such that dv/dt and Av belong to $H^1_{\mathrm{loc}}(0,T;\mathcal{H})$ we have

$$\|v\|\frac{d}{dt}\|v\| = (Av,v) + (Lv,v)\,, \tag{2.1}$$

$$\frac{d}{dt}(Av,v) = 2(Av, dv/dt)\,,$$

$$\|v\|^4\frac{d}{dt}\frac{(Av,v)}{\|v\|^2} = 2\|Av+\tfrac12 Lv\|^2\,\|v\|^2 - \tfrac12\|Lv\|^2\,\|v\|^2 - 2(Av+\tfrac12 Lv, v)^2 + \tfrac12(Lv,v)^2\,.$$

The last identity gives the estimate

$$\frac{d}{dt}\frac{(Av,v)}{\|v\|^2} \ge -\tfrac12\|Lv\|^2\,\|v\|^2\,. \tag{2.2}$$

THEOREM 2.1. *Let A be a symmetric negative definite operator in a Hilbert space $\mathcal{H}$, and $0<T\le\infty$. Suppose that dv/dt and $Av(t)$ belong to $H^1_{\mathrm{loc}}(0,T;\mathcal{H})$ and $\gamma:[0,T)\to[0,\infty)$ is a continuous function. If*

$$\|dv/dt - Av\|^2 \le \gamma(t)(|(Av,v)| + \|v\|^2)\,, \tag{2.3}$$

then

$$\|v(t)\| \ge \|v(0)\|\exp\Big(-\int\limits_0^t \alpha(\tau)\,d\tau\Big) \tag{2.4}$$

where

$$\alpha(\tau) = \gamma(\tau) + 2(|\mu|+1)\exp\Big(\frac12\int\limits_0^\tau \gamma(s)\,ds\Big) + 1\,,$$

$$\mu = (Av(0), v(0))\|v(0)\|^{-2}\,.$$

Proof. From (2.2) and (2.3) we get

$$\frac{d}{dt}\frac{(Av,v)}{\|v\|^2} \ge -\frac12\gamma(t)\Big(\frac{|(Av,v)|}{\|v\|^2}+1\Big)\,.$$

Multiplying this inequality by $\nu(t) = \exp(-\frac12\int_0^t\gamma(s)\,ds)$ and integrating, we obtain

$$\frac{|(Av,v)|}{\|v\|^2} \le \nu(t)^{-1}\Big(|\mu| + \frac12\int\limits_0^t \nu(s)\gamma(s)\,ds\Big)\,.$$

Consequently,

$$\frac{|(Av,v)|}{\|v\|^2} \le (|\mu|+1)\exp\Big(\frac12\int\limits_0^t\gamma(s)\,ds\Big)\,. \tag{2.5}$$

Using (2.1), (2.3) and (2.5) one gets

$$\begin{aligned}\frac{d}{dt}\log\|v\| &\geq \frac{(Av,v)}{\|v\|^2}-\frac{\|Lv\|}{\|v\|}\\ &\geq \frac{(Av,v)}{\|v\|^2}-\gamma(t)^{1/2}\Big(\frac{|(Av,v)|}{\|v\|^2}+1\Big)^{1/2}\\ &\geq -2\frac{|(Av,v)|}{\|v\|^2}-\gamma(t)-1\\ &\geq -2(|\mu|+1)\exp\Big(\frac{1}{2}\int_0^t\gamma(s)\,ds\Big)-\gamma(t)-1\,.\end{aligned}$$

Now, (2.4) follows by integration.

COROLLARY 2.1. (i) *If γ is an integrable function in $[0,\infty)$ then there exists a real constant λ such that*

$$\|v(t)\|\geq\|v(0)\|\exp(-\lambda(t+1))\quad \textit{for } t\geq 0\,.$$

(ii) *If $\gamma(t)\to 0$ as $t\to\infty$ then for some function $\psi(t)$ such that $\psi(t)\to 0$ as $t\to\infty$ we have*

$$\|v(t)\|\geq\|v(0)\|\exp(-\exp(t\psi(t)))\,.$$

Remark 2.1. The negativity assumption in Theorem 2.1 is not essential.

3. Estimates of operators. The functions which are in domains or ranges of the operators we consider are defined on the unit ball or on its boundary.

Let X, Y be sets with given measures, $p,q\in[1,\infty]$. The norm of an operator T from $L^p(X)$ to $L^q(Y)$ is denoted by $|||T|||_{p,q}$. If this norm is finite, one says that operator T is of *type* (p,q). If T is an integral operator with kernel $k(x,y)$ then T^* denotes the integral operator transposed to T, whose kernel is $k^*(y,x)=k(x,y)$. Let T_1 be another integral operator with kernel $k_1(x,y)$ on $X\times Y$. We say that T is *majorized* by T_1 if for a positive number C, $|k(x,y)|\leq Ck_1(x,y)$. We shall write sometimes $\|f\|_p$ instead of $\|f\|_{L^p(X)}$. For any $p\geq 1$ the number p' is given by $1/p+1/p'=1$.

Let $0<\alpha<n$. The integral operator P_α from $L^1(\partial\mathbb{B})$ to $L^1(\mathbb{B})$ is given by

$$P_\alpha\varphi(x)=\int_{\partial\mathbb{B}}|x-\omega|^{-n+\alpha}\varphi(\omega)\,dS\,.$$

The proof of the Hardy–Littlewood–Sobolev inequality in [17] may be adapted to derive L^p-estimates for P_α (we make use of the case $\alpha=1$).

LEMMA 3.1. *Let $1\leq\alpha<n$. Then*

(i) *P_α is of type (p,q) with $1<p<(n-1)/(\alpha-1)$ and*

$$\frac{1}{q}=\frac{n-1}{n}\frac{1}{p}-\frac{\alpha-1}{n}\,.$$

(ii) *P_α^* is of type (p,q) with $1<p<n/\alpha$ and*

$$\frac{1}{q}=\frac{n}{n-1}\frac{1}{p}-\frac{\alpha}{n-1}\,.$$

The norm in $H^{m,p}(\mathbb{B})$ is given by

$$\|f\|_{p,(m)}=\sum_{|\alpha|\leq m}\|D^\alpha f\|_p\,.$$

Let $1\leq p<n$. The trace operator is well-defined from $H^{1,p}(\mathbb{B})$ to $L^q(\partial\mathbb{B})$ with $1/q\geq n/(n-1)p-1/(n-1)$. The trace of a function u is denoted by $u_{|\partial\mathbb{B}}$ and $H_0^{m,p}(\mathbb{B})$ is the closure of $C_0^\infty(\mathbb{B})$ in the norm $\|\cdot\|_{p,(m)}$.

The space $H^{m,p}(\partial\mathbb{B})$ consists of functions φ on $\partial\mathbb{B}$ such that $D^\alpha\widetilde{\varphi}_{|\partial\mathbb{B}}\in L^p(\partial\mathbb{B})$ where $\widetilde{\varphi}(x)=\varphi(|x|^{-1}x)$, $|\alpha|\leq m$, and

$$\|\varphi\|_{p,(m)}=\sum_{|\alpha|\leq m}\|D^\alpha\widetilde{\varphi}\|_{L^p(\partial\mathbb{B})}\,. \tag{3.1}$$

The norm (3.1) is equivalent to

$$\|\varphi\|_p+\|(-\Delta_S)^{1/2}\varphi\|_p \tag{3.2}$$

where Δ_S is the Laplace–Beltrami operator on $\partial\mathbb{B}$ (see [15]). The set of all functions on $\partial\mathbb{B}$ of the form

$$\varphi=\sum_{0\leq k\leq l}Y_k \tag{3.3}$$

where Y_k is a surface spherical harmonic of degree k, $l=1,2,\ldots$, is a dense subset of $H^{m,p}(\partial\mathbb{B})$ for $1\leq p<\infty$.

Now, we introduce an operator Λ from $H^{1,p}(\partial\mathbb{B})$ to $L^p(\partial\mathbb{B})$. For functions of the form (3.3) we put

$$\Lambda\varphi=\sum_{1\leq k\leq l}kY_k\,.$$

As $k(k+n-2)$ is the eigenvalue of $-\Delta_S$ corresponding to the eigenfunction Y_k we infer, using the multiplier theorem for spherical harmonics [18], that the norm (3.2) is equivalent to

$$\|\varphi\|_p+\|\Lambda\varphi\|_p \tag{3.4}$$

and so the norms (3.1) and (3.4) are also equivalent.

Let $e_1,\ldots,e_n$ be the canonical basis of $\mathbb{R}^n$. The vector $\tau_j=e_j-x_jx$ is the tangent component of e_j at a point $x\in\partial\mathbb{B}$. For $j=1,\ldots,n$ we define

operators Λ_j from $H^{1,p}(\partial\mathbb{B})$ to $L^p(\partial\mathbb{B})$ by
$$\Lambda_j = \partial/\partial\tau_j + x_j\Lambda .$$

Consider the Poisson integral for the Dirichlet problem in $\mathbb{B}$,
$$u(x) = \int_{\partial\mathbb{B}} p(x,\omega)\varphi(\omega)\,dS_\omega = P\varphi(x)$$
where $p(x,\omega) = c_n'(1-|x|^2)|x-\omega|^{-n}$. (The constant c_n' is determined by $P1 = 1$.)

LEMMA 3.2. *Let $1 \le p < \infty$. For any $\varphi \in H^{1,p}(\partial\mathbb{B})$, $j = 1,\ldots,n$, we have*

(3.5) $$D_jP\varphi = P\Lambda_j\varphi \quad \textit{in } \mathbb{B} .$$

Proof. It is easy to check (3.5) for φ of the form (3.3). For any $\varphi \in H^{1,p}$ it follows by a continuity argument. For the right hand side one may use the obvious inequality $|P\varphi(x)| \le C(1-|x|)^{1-n}\|\varphi\|_1$ whereas for the left hand side of (3.5) properties of uniformly bounded sequences of harmonic functions.

The normal derivative of a function $u \in H^{1,p}(\mathbb{B})$ is given by
$$\frac{\partial u}{\partial\nu} = \sum_{1\le j\le n} \nu_j D_j u_{|\partial\mathbb{B}} ,$$
where $\nu(x) = (\nu_1(x),\ldots,\nu_n(x))$ is the outward unit normal vector at $x \in \partial\mathbb{B}$.

The operator P is majorized by P_1. By Lemmas 3.1 and 3.2 we get

COROLLARY 3.1. (i) *If $\varphi \in H^{m,p}(\partial\mathbb{B})$, $1 < p < \infty$, then $P\varphi \in H^{m,q}(\mathbb{B})$ with $q = pn/(n-1)$.*

(ii) *If $P\varphi \in H^{2,p}(\mathbb{B})$ then $D_jP\varphi_{|\partial\mathbb{B}} = \Lambda_j\varphi$ $(j = 1,\ldots,n)$ and*
$$\frac{\partial}{\partial\nu}P\varphi = \Lambda\varphi .$$

Let $g(x,y)$ be the Green function for the operator $-\Delta$ in $\mathbb{B}$. For $n \ge 3$,
$$g(x,y) = c_n(|x-y|^{2-n} - |y|^{2-n}|x-\overline{y}|^{2-n})$$
where $\overline{y} = y|y|^{-2}$ and $c_n = c_n'/(n-2)$.

We put
$$g_j(x,y) = D_{x_j}g(x,y) , \quad j = 1,\ldots,n .$$

We have

(3.6) $$|g_j(x,y)| \le C|x-y|^{1-n} .$$

For $x \in \mathbb{B}$ put

(3.7) $$Gf(x) = \int_{\mathbb{B}} g(x,y)f(y)\,dy ,$$

(3.8) $$G_jf(x) = \int_{\mathbb{B}} g_j(x,y)f(y)\,dy .$$

The integral operator G_j^0, $j = 1,\ldots,n$, is determined by the kernel $g_j(x,y)$ restricted to $\partial\mathbb{B}\times\mathbb{B}$.

LEMMA 3.3. *Let $f \in L^p(\mathbb{B})$ with $1 < p < n$, and $u = Gf$. Then*

(i) *$u \in H^{2,p}(\mathbb{B}) \cap H_0^{1,q}(\mathbb{B})$ where $1/q = 1/p - 1/n$.*
(ii) *$\Delta u = -f$.*
(iii) *$D_ju = G_jf$; $D_ju_{|\partial\mathbb{B}} = G_j^0 f$.*
(iv) *$\partial u/\partial\nu = P^*f$.*

Proof. Put $g_0(x,y) = c_n|y|^{2-n}|x-\overline{y}|^{2-n}$ and let the operator G_0 be given by (3.8) with $j = 0$. We extend f outside $\mathbb{B}$ by 0; $u = Gf$ is defined, by (3.7), on the whole $\mathbb{R}^n$. If $0 \notin \operatorname{supp} f$ then after a change of the variable y we get
$$G_0f(x) = c_n \int |x-y|^{2-n}\widetilde{f}(y)\,dy$$
where $\widetilde{f} \in L^p(\mathbb{R}^n)$ and $\widetilde{f} = 0$ in $\mathbb{B}$. Therefore $\Delta u = \widetilde{f} - f$ and so $u \in H^{2,p}$. In the remaining case, when $\operatorname{supp} f \subset \mathbb{B}$, the function G_0f is harmonic in an open set containing $\overline{\mathbb{B}}$. Also $Gf \in C^\infty(\overline{\mathbb{B}})$ when $f \in C_0^\infty(\mathbb{B})$. Thus (i) is checked. Property (iv) follows immediately from (iii) and the identity

(3.9) $$p(x,\omega) = \sum_{1\le j\le n} \omega_j g_j(\omega,x)$$

where $x \in \mathbb{B}$, $\omega \in \partial\mathbb{B}$.

We are going to discuss operators of fractional derivation on the unit ball. The *radial derivative* d_α of order $\alpha > 0$ is defined as follows: if α is an integer then
$$d_\alpha f(r\omega) = (r\partial/\partial r)^\alpha f(r\omega) , \quad \omega \in \partial\mathbb{B} ,$$
and if $\alpha = k - \beta$ with k an integer and $0 < \beta < 1$ then
$$d_\alpha f(x) = \frac{1}{\Gamma(\beta)} \int_0^1 d_k f(tx)|\log t|^{\beta-1}\frac{dt}{t} .$$

The change of variable $r = e^{-t}$ transforms the operators d_α to the usual fractional derivatives. One verifies easily the property $d_\alpha d_\beta = d_{\alpha+\beta}$. If u is a harmonic function, $u = \sum_{0\le k<\infty} r^kY_k(\omega)$, then $d_\alpha u$ is also harmonic and

(3.10) $$d_\alpha u(r\omega) = \sum_{1\le k<\infty} k^\alpha r^k Y_k(\omega) .$$

By d_0 we mean the identity operator.

If u_0 is the boundary value of a harmonic function u on $\mathbb{B}$, then $d_1u = P\Lambda u_0$. The norm (3.4) is equivalent to (3.1) with $m = 1$ and therefore there exists a constant C such that

(3.11) $$\|D_j u\|_2 \le C\|d_1 u\|_2 .$$

Denote by H the integral operator with kernel

$$h(x,y) = c_n'(1-x^2y^2)(1+x^2y^2-2x\cdot y)^{-n/2}$$

on $\mathbb{B}\times\mathbb{B}$, where $x^2=|x|^2$ and $x\cdot y$ is the usual scalar product in $\mathbb{R}^n$. We consider the commutator of the operators d_1 and G on the space of smooth functions in the closed unit ball.

LEMMA 3.4. *For each $n\ge 3$,*

(3.12) $$d_1G - Gd_1 = 2G - H .$$

Proof. Put $|x|=r$, $|y|=s$. Integration by parts in s shows that $d_1G - Gd_1$ is an integral operator with kernel

$$(r\partial/\partial r + s\partial/\partial s + n)g(x,y)$$

and (3.12) follows by direct computation.

The kernel of H may be expressed by means of the Poisson kernel. Namely,

(3.13) $$h(x,y) = p(x|y|, y/|y|) = p(y|x|, x/|x|) .$$

From this formula we get

LEMMA 3.5. *For any $f\in L^1(\mathbb{B})$, Hf is a harmonic function in $\mathbb{B}$ with boundary value P^*f and $H=PP^*$.*

LEMMA 3.6. *For any $0\le\alpha<1$ the operator $d_\alpha P$ is an integral operator majorized by $P_{1-\alpha}$.*

Proof. One verifies that for any $\omega\in\partial\mathbb{B}$ and $y\in\mathbb{B}$,

$$\left|\frac{\partial}{\partial r}p(r\omega,y)\right| \le C|r\omega-y|^{-n} .$$

It remains to show that for $\beta = 1-\alpha$,

(3.14) $$\int_0^1 |rs\omega - y|^{-n}|\log s|^{\beta-1}\,ds \le C|r\omega-y|^{-n+\beta} .$$

Suppose that $1/2\le r\le 1$ and $\omega\cdot y\ge 1/2$; for the remaining case the estimate is obvious. Put $\omega\cdot y = 1-\varepsilon^2$ and use the substitutions $r=e^{-t}$, $s=e^{-\sigma}$; then inequality (3.14) follows from

$$\int_t^\infty (\sigma-t)^{\beta-1}(\sigma^2+\varepsilon^2)^{-n/2}\,d\sigma \le C(t^2+\varepsilon^2)^{(-n+\beta)/2} .$$

This inequality may be reduced to the case $\varepsilon=1$, and the latter is derived easily from the case $\varepsilon=0$ for which we have equality.

For each $0<\alpha<n$ denote by H_α the integral operator whose kernel is $(x^2y^2-2x\cdot y+1)^{(-n+\alpha)/2}$, $x,y\in\mathbb{B}$. From Lemma 3.6 and (3.13) we obtain

COROLLARY 3.2. *For each $0\le\alpha<1$ the operator $d_\alpha H$ is an integral operator on $\mathbb{B}$ majorized by $H_{1-\alpha}$.*

We derive bounds for the operators H_α from bounds of their "traces" on $\partial\mathbb{B}$ by means of Minkowski's inequality.

LEMMA 3.7. *Let $K_{r,s}$, $0<r,s<1$, be linear operators on $\partial\mathbb{B}$ and*

$$|||K_{r,s}|||_{\sigma,\varrho} \le \mathcal{K}(r,s) .$$

Then for the operator T on $\partial\mathbb{B}\times(0,1)$ given by

$$Tf(\cdot,t) = \int K_{t,s}f(\cdot,s)\,ds$$

and the integral operator M on $\mathbb{R}^+$ defined by

$$Mg(t) = \int \mathcal{K}(t,s)g(s)\,ds$$

we have

$$|||T|||_{\sigma,\varrho} \le |||M|||_{\sigma,\varrho} .$$

THEOREM 3.1. *Let $0<\alpha\le 1$ and*

$$\frac12 \le \frac1\sigma + \frac1p < 1, \quad \frac12\le\frac1\sigma+\frac1q\le 1 .$$

If

(3.15) $$\frac1p + \frac{n-1}{q} \le \alpha + n\left(\frac12 - \frac1\sigma\right)$$

then

(3.16) $$\|H_\alpha(af)\|_2 \le C\|a\|_{p,q}\|f\|_\sigma .$$

Proof. Suppose that both sides of (3.15) are equal. It can be seen from the identity $x^2y^2-2x\cdot y+1 = |x-y|^2+(1-x^2)(1-y^2)$ that (3.16) follows from the corresponding estimate for the integral operator $\widetilde{H}_\alpha$ on $\partial\mathbb{B}\times(0,1/2)$, with kernel

(3.17) $$k(\omega,t,\xi,s) = (|\omega-\xi|+t+s)^{-n+\alpha}\,\widetilde{a}(\xi,s)$$

where $\omega,\xi\in\partial\mathbb{B}$, $t=1-|x|$, $\widetilde{a}(\xi,s) = a((1-s)\xi)$.

By Young's inequality, the integral operators on $\partial\mathbb{B}$ with kernels $k_1(\omega,\xi) = (|\omega-\xi|+t+s)^{-n+\alpha}$ map L^{q_1} into L^2 for $1/q_1 = 1/\sigma+1/q$ and their norms are bounded by

$$\|(|\omega|+t+s)^{-n+\alpha}\|_{L^\tau(\mathbb{R}^{n-1}_\omega)} = C(t+s)^{-1/w}$$

where $1/q_1 = 1/2+1/\tau$ and $1/w = 1/p'+1/2-1/\sigma$.

Let $K_{t,s}$ be the integral operator on $\partial\mathbb{B}$ given by (3.17). By Hölder's inequality,

$$|||K_{t,s}|||_{\sigma,2} \le C\|\widetilde{a}(\xi,s)\|_{L^q(\partial\mathbb{B}_\xi)}(t+s)^{-1/w}\,.$$

We apply Young's inequality for multiplicative convolution operators on $\mathbb{R}^+$ to the integral operator M_1 on $\mathbb{R}^+$ given by

$$M_1 g(t) = \int (t+s)^{-1/w} g(s)\,ds\,.$$

If $1 \le w < 2$ then it is of type $(p_1, 2)$ with $1/p_1 = 1/\sigma + 1/p = 1/2 + 1/w'$. Therefore the operator $M = M_1 \circ \|\widetilde{a}(\cdot, s)\|_p$ is of type $(\sigma, 2)$ and so, according to Lemma 3.7, estimate (3.16) is proved.

4. A formula for the Dirichlet problem. The Dirichlet problem for equation (1.1) in the unit ball is uniquely solvable in $H^{1,2}$ if the norms $\|b_0\|_{n/2}$, $\|b_1\|_n$ are sufficiently small (see e.g. [16]). Set

$$B = b_1(x)D + b_0(x)\,, \qquad F = \sum_{0\le k<\infty} (BG)^k$$

where $(BG)^0 = I$, the identity operator.

We shall use the inequalities of Hölder, Young and Hardy–Littlewood–Sobolev without mention.

THEOREM 4.1. *Let $b_j \in L^{n/(2-j)}$, $j = 0, 1$, $n \ge 3$. If the norms $\|b_j\|_{n/(2-j)}$ are sufficiently small then the operator F is of type (t,t) with $1 < t < n/2$ (with $1 < t < n$ if $b_0 = 0$). If $u_0 \in H^{1,p}(\partial\mathbb{B})$ with $1 < p < n-1$ (with $1 < p < \infty$ when $b_0 = 0$) then the function*

$$u = Pu_0 + GFBPu_0 \tag{4.1}$$

is of class $H^{1,s}$ with $s = pn/(n-1)$ and u is a solution of the equation $\Delta u + Bu = 0$ with trace u_0 on $\partial\mathbb{B}$.

Proof. We have $BG = \sum_{1\le j\le n} b_1^j G_j + b_0 G$. The estimate for the operator F follows from

$$|||b_1^j G_j|||_{t,t} \le C\|b_1^j\|_n \qquad \text{for } 1 < t < n\,,$$
$$|||b_0 G|||_{t,t} \le C\|b_0\|_{n/2} \qquad \text{for } 1 < t < n/2\,.$$

By the Sobolev imbedding theorem and Lemma 3.1,

$$\|Pu_0\|_q \le C\|u_0\|_{p,(1)}$$

where $1 < p < n-1$ and $1/q = (n-1)/pn - 1/n$.

Now, from $BP = \sum b_1^j P\Lambda_j + b_0 P$ and Lemma 3.1 we infer that the operator FBP maps $H^{1,p}(\partial\mathbb{B})$ into $L^t(\mathbb{B})$ with $1/t = 1/s + 1/n$ and so $u \in H^{1,s}$. By Lemma 3.3, $u_{|\partial\mathbb{B}} = u_0$ and $\Delta u = -FBPu_0$. The identity $F = I + BGF$ gives $Bu = FBPu_0$ and thus $\Delta u + Bu = 0$.

COROLLARY 4.1. *Let u be a solution of equation (1.1). If the norms $\|b_j\|_{n/(2-j)}$ $(j = 0, 1)$ are small enough and v is a harmonic function on $\mathbb{B}$ such that $u_{|\partial\mathbb{B}} = v_{|\partial\mathbb{B}}$ then*

$$\|u\|_{2,(1)} \le 2\|v\|_{2,(1)}\,. \tag{4.2}$$

If $b_1 = 0$ then

$$\|u\|_2 \le 2\|v\|_2\,. \tag{4.3}$$

Proof. From (4.1) we get

$$D_k u = D_k v + G_k F b_1 D v + G_k F b_0 v\,.$$

Thus (4.2) is a consequence of the estimates

$$|||G_k F b_1|||_{2,2} \le C\|b_1\|_n\,, \qquad |||G_k F b_0|||_{\sigma,2} \le C\|b_0\|_{n/2}\,,$$

where $\sigma = 2n/(n-2)$, and (4.3) follows from

$$|||GFb_0|||_{2,2} \le C\|b_0\|_{n/2}\,.$$

Remark 4.1. A function $u \in H^{1,2}(\mathbb{B})$ is a solution of the Dirichlet problem for equation (1.1) with boundary value u_0 if and only if

$$u - GBu = Pu_0\,, \tag{4.4}$$

and the solution of (4.4) is represented by the series

$$u = \sum_k (GB)^k Pu_0\,.$$

Formula (4.1) is obtained from this formally.

Remark 4.2. F. H. Lin [13] showed that a zero of a solution of the Schrödinger equation $-\Delta u + V(x)u = 0$ with $V \in L^{n/2}_{\mathrm{loc}}$ is of at most finite order in the sense that

$$\int_{\mathbb{B}_r} u^2\,dx \ge Cr^\lambda\,. \tag{4.5}$$

This condition follows from (1.6). The converse does not hold in general but one obtains easily (1.6) from (4.5) using (4.3) and the following estimate:

$$\int_{\partial\mathbb{B}} v^2\,dS \ge \int_{\mathbb{B}} v^2\,dx\,.$$

5. Proof of Theorem 1.1. We apply Theorem 2.1 taking $A = -d_1$, and with $\mathcal{H}$ being the space of harmonic functions in $\mathbb{B}$ with norm given by

$$\|v\|^2_{\mathcal{H}} = \int_{\partial\mathbb{B}} v^2\,dS\,.$$

Denote by φ_r the trace of u_r on $\partial\mathbb{B}$ and define for $t \ge 0$,

$$v(t) = P\varphi_r\,, \qquad r = e^{-t}\,.$$

The assumptions concerning the function $v(t)$ in Theorem 2.1 are satisfied because $u \in H^{1,2}$.

It is easy to verify that

$$P\varphi_r = (I + G\Delta)u_r\,.$$

Let u^ε, $0 < \varepsilon < 1$, be smooth functions in $\mathbb{B}$ such that $u^\varepsilon \to u$ in $H^{2,2}$ as $\varepsilon \to 0$. We have

$$\frac{dv}{dt} = -P\frac{\partial u_r}{\partial \nu} = -\lim_{\varepsilon\to 0}(I + G\Delta)d_1 u_r^\varepsilon\,.$$

Using the identity $\Delta d_1 - d_1\Delta = 2\Delta$, Lemma 3.4 and equation (1.1) we get

(5.1) $$dv/dt + d_1 v = -HB_r u_r$$

where

$$B_r = rb_{1,r}(x)D + r^2 b_{0,r}(x)\,.$$

It is easy to show, using (3.10), that the norm in $\mathcal{H}$ is equivalent to

$$\|d_{1/2}v\|_2 + \|v\|_2\,.$$

By Theorem 3.1 we have

$$\|H_{1/2}(b_{1,r}Du_r)\|_2 \le C\|b_{1,r}\|_{p_1,q_1}\|Du_r\|_2\,.$$

Applying Theorem 3.1 with $\sigma = 2n/(n-2)$ and the Sobolev Theorem we get

$$\|H_{1/2}(b_{0,r}u_r)\|_2 \le C\|b_{0,r}\|_{p_0,q_0}\|u_r\|_{2,(1)}\,.$$

The operators d_α on $\mathcal{H}$ are symmetric and positive definite. The pseudo-norm $(d_{1/2}v, v)_{\mathcal{H}}^{1/2}$ is equivalent to $\|d_{1/2}v\|_2$ and therefore, by (3.11), to $\|Dv\|_2$.

From Corollary 4.1 we get

(5.2) $$\|dv/dt + d_1 v\|_{\mathcal{H}}^2 \le C\beta(r)((d_1 v, v)_{\mathcal{H}} + \|v\|_2^2)$$

where $C = C(n)$ and

$$\beta(r) = r^2\|b_{1,r}\|_{p_1,q_1}^2 + r^4\|b_{0,r}\|_{p_0,q_0}^2\,.$$

If condition (1.5) is satisfied then the function $\gamma(t) = C\beta(e^{-t})$ is integrable and by Corollary 2.1,

$$\|v(t)\|^2 \ge \|v(0)\|^2\exp(-\lambda(1+t))\,.$$

Hence

$$\int_{\partial\mathbb{B}} u_r^2\,dS \ge Cr^\lambda \int_{\partial\mathbb{B}} u^2\,dS\,.$$

This estimate gives (1.6). The remaining case is treated similarly.

Remark 5.1. It follows from (4.1) and Lemma 3.3(iv) that

$$\partial u_r/\partial\nu = \Lambda\varphi_r + P^*FB_r v\,,$$

and consequently we have the formula

$$dv/dt + d_1 v = -PP^*FB_r v\,,$$

analogous to (5.1) and from which we can also deduce inequality (5.2).

References

[1] N. Aronszajn, A. Krzywicki and J. Szarski, *A unique continuation theorem for exterior differential forms on Riemannian manifolds*, Ark. Mat. 4 (1962), 417–453.

[2] B. Barcelo, C. E. Kenig, A. Ruiz and D. Sogge, *Weighted Sobolev inequalities and unique continuation for the Laplacian plus lower order terms*, Illinois J. Math. 32 (1988), 230–245.

[3] N. Garofalo and F. H. Lin, *Monotonicity properties of variational integrals, A_p weights and unique continuation*, Indiana Univ. Math. J. 35 (1986), 245–268.

[4] —, —, *Unique continuation for elliptic operators: a geometric-variational approach*, Comm. Pure Appl. Math. 90 (1987), 347–366.

[5] L. Hörmander, *The Analysis of Linear Partial Differential Operators*, Vol. III, Springer, New York, 1985.

[6] —, *Uniqueness theorems for second order elliptic differential equations*, Comm. Partial Differential Equations 8 (1983), 21–64.

[7] D. Jerison, *Carleman inequalities for the Dirac and Laplace operators and unique continuation*, Adv. in Math. 62 (1986), 118–134.

[8] D. Jerison and C. E. Kenig, *Unique continuation and absence of positive eigenvalues for Schrödinger operators*, Ann. of Math. 121 (1985), 463–488.

[9] J. L. Kazdan, *Unique continuation in geometry*, Comm. Pure Appl. Math. 91 (1988), 667–681.

[10] C. E. Kenig, *Restriction theorems, Carleman estimates, uniform Sobolev inequalities and unique continuation*, in: Lecture Notes in Math. 1384, Springer, 1989, 69–90.

[11] Y. M. Kim, MIT thesis, 1989.

[12] P. D. Lax, *A stability theorem for solutions of abstract differential equations, and its application to the study of the local behaviour of solutions of elliptic equations*, Comm. Pure Appl. Math. 9 (1956), 747–766.

[13] F. H. Lin, *A uniqueness theorem for parabolic equations*, Comm. Pure Appl. Math. 93 (1990), 127–136.

[14] A. Pliś, *On non-uniqueness in Cauchy problem for an elliptic second order differential equation*, Bull. Acad. Polon. Sci. 11 (1963), 95–100.

[15] R. T. Seeley, *Singular integrals on compact manifolds*, Amer. J. Math. 81 (1959), 658–690.

[16] G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, Ann. Inst. Fourier (Grenoble) 15 (1) (1965), 189–258.

[17] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, 1970.

[18] R. Strichartz, *Multipliers for spherical harmonic expansions*, Trans. Amer. Math. Soc. 167 (1972), 115–124.
[19] T. H. Wolff, *Unique continuation for* $|\Delta u| \le V|\nabla u|$ *and related problems*, Rev. Mat. Iberoamericana 6 (1990), 155–200.
[20] —, *A property of measures in* $\mathbb{R}^n$ *and an application to unique continuation*, preprint.

INSTITUTE OF MATHEMATICS
UNIVERSITY OF WARSAW
BANACHA 2
02-097 WARSZAWA, POLAND

Received November 29, 1991 (2867)
Revised version November 23, 1992

STUDIA MATHEMATICA 108 (1) (1994)

Certain lacunary cosine series are recurrent

by

D. J. GRUBB (DeKalb, Ill.) and
CHARLES N. MOORE (Manhattan, Kan.)

Abstract. Let the coefficients of a lacunary cosine series be bounded and not square-summable. Then the partial sums of the series are recurrent.

In this note, we wish to consider the partial sums of a lacunary cosine series

$$s_N(x) = \sum_{j=1}^{N} a_j \cos(n_j x + \theta_j) \tag{1}$$

where we assume that a_j, θ_j are real, $|a_j| \le 1$ for every j, and the n_j satisfy $n_{j+1}/n_j \ge \lambda > 1$. We will show:

THEOREM. *Suppose that in addition to the conditions just described, also* $\sum |a_j|^2 = \infty$. *Then* $\{s_N(x)\}$ *is dense in* $\mathbb{R}$ *for almost every* $x \in [0, 2\pi]$.

We will restate the conclusion in probabilistic terms by saying that for almost all x the sequence $s_N(x)$ is recurrent in $\mathbb{R}$. In the case when $a_j = 1$ for all j, this question was posed by T. Murai [4] (see also Brannan and Hayman [3]), and was solved by D. Ullrich [5]. Ullrich was unable to obtain the more general case when $|a_j| \le 1$; however, the proof we give for our theorem, although shorter than that of Ullrich, does take its key idea from his method.

When the terms $\cos(n_j x + \theta_j)$ in (1) are replaced by $\exp(in_j x)$, the most general conditions which give recurrence of the partial sums in the complex plane are far from being known. Anderson and Pitt [2] showed recurrence in this case when $a_j = 1$ and $n_j = b^j$ for some integer b. They have also shown recurrence in the complex plane when $|a_j| \to 0$ or if the a_j are bounded and $n_{j+1}/n_j \to \infty$ [1]. We stress that even though the series we consider are

1991 *Mathematics Subject Classification*: Primary 42A55.
Research of the second author partially supported by NSF grant DMS-9204529.