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On the Pointwise Ergodic Theorem in L_p

by

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Abstract. Let (X, F, μ) be a finite measure space, and ϕ an invertible, nonsingular transformation on (X, F, μ). We prove that the Pointwise Ergodic Theorem (P.E.T.) in $L_p(d\mu)$ holds for the operator $Tf = f\circ\phi$ if, and only if, it holds for the formal adjoint of T in $L_q(d\mu)$ $(1/p+1/q = 1)$. We also characterize the P.E.T. in terms of the Radon–Nikodym derivative of the measure μ with respect to an invariant measure.

1. Introduction. Let (X, F, m) be a finite measure space, and ϕ an invertible, measure preserving transformation. Let w be an integrable, positive function defined on X, and consider the measure $\mu(A) = \int_A w\,dm$, and the associated space $L_p(\mu)$.

We say that a linear operator T, defined on $L_p(\mu)$, satisfies the Pointwise Ergodic Theorem (P.E.T.) in $L_p(\mu)$ if, for every f in $L_p(\mu)$, the averages $M_n(T)f \equiv (f + Tf + \ldots + T^{n-1}f)/n$ converge a.e. to a function Pf which belongs to $L_p(\mu)$.

In [AW], Assani and Woś characterized those ϕ for which the P.E.T. holds in $L_p(\mu)$, $1 < p < \infty$, for the operator $Tf = f\circ\phi$, in terms of two conditions, one on the adjoint of T, plus another on the averages of the measures of the pre-images of measurable sets. In that paper they raised the question of finding a characterization in terms of w. In this note we give an answer to this question. We show that T satisfies the P.E.T. in $L_p(\mu)$ if, and only if, w satisfies the following condition:

There exists C such that the following holds a.e.:

$$\lim_{n\to\infty} (M_n(T)w)^{1/p}(M_n(T)w^{1-q})^{1/q}(x) \le C, \quad p+q = pq.$$

We also show that T satisfies the P.E.T. in $L_p(\mu)$ if, and only if, T^*, the formal adjoint of T, satisfies the P.E.T. in $L_q(\mu)$.

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Throughout this paper the letter C will denote a positive, finite constant, not necessarily the same at each occurrence, p will denote a number bigger than 1, and q will be its conjugate exponent, i.e. the number $p/(p-1)$.

2. Main result. Let (X,F,m) and ϕ be as above, and let T be the linear operator defined on measurable functions by $Tf = f\circ\phi$. We observe that from

$$\int Tfg\,d\mu = \int f(\phi(x))g(x)w(x)\,dm = \int f(x)g(\phi^{-1}(x))w(\phi^{-1}(x))\,dm,$$

it follows that the formal adjoint of T is the operator defined as $T^*(g) = w^{-1}T^{-1}(gw)$.

THEOREM 1. *The following are equivalent*:

(1) *T satisfies the P.E.T. in $L_p(\mu)$.*

(2) *There exists C such that the following holds a.e.*:

$$\lim_{n\to\infty}(M_n(T)w)^{1/p}(M_n(T)w^{1-q})^{1/q} \le C.$$

(3) *T^* satisfies the P.E.T. in $L_q(\mu)$.*

Proof. (1)⇒(3). It follows from the definition of T^* that $(T^*)^i(g) = w^{-1}T^{-i}(gw)$. Therefore $M_n(T^*)g = w^{-1}M_n(T^{-1})(gw)$, and the limit of the right hand side exists for g in $L_q(\mu)$ because then $gw\in L_1(m)$.

The operator $f\to Pf = \lim_{n\to\infty}M_n(T)f$ from $L_p(\mu)$ into itself, being linear and positive, is bounded. Now let g be a positive function in $L_q(\mu)$, and h a positive bounded function. Fatou's lemma, plus the fact that T is bounded in L_∞, gives

$$\int h\lim_{n\to\infty}M_n(T^*)g\,d\mu \le \lim_{n\to\infty}\int hM_n(T^*)g\,d\mu$$
$$= \lim_{n\to\infty}\int gM_n(T)h\,d\mu = \int g\lim_{n\to\infty}M_n(T)h\,d\mu = \int gPh\,d\mu,$$

which implies (3) because P is bounded in $L_p(\mu)$.

(3)⇒(1). The function $\mathbf{1}$ that maps every x into 1 is clearly in $L_q(\mu)$, and thus $\lim M_n(T^*)\mathbf{1}$ exists and is in $L_q(\mu)$. But $\lim M_n(T^*)\mathbf{1} = w^{-1}\lim M_n(T^{-1})w \equiv w^{-1}h$. Therefore the function h^qw^{1-q} is integrable with respect to the invariant measure dm. Observe also that the function h is invariant and positive.

Our formula for $M_n(T^*)$ gives that (3) can be written as

$$\int|\lim_{n\to\infty}M_n(T^{-1})g|^qw^{1-q}\,dm \le C\int|g|^qw^{1-q}\,dm,$$

and, since h is invariant and positive, this is equivalent to

$$\int|\lim_{n\to\infty}M_n(T^{-1})g|^qh^qw^{1-q}\,dm \le C\int|g|^qh^qw^{1-q}\,dm,$$

for all g in $L_q(h^qw^{1-q}\,dm)$. This is nothing but statement (1) with T^{-1}, q, and h^qw^{1-q} instead of T, p, and w respectively. Using what we have already proved, we obtain

$$\int|\lim_{n\to\infty}M_n(T)g|^ph^{-p}w\,dm \le C\int|g|^ph^{-p}w\,dm,$$

which implies (1), because h is invariant and positive.

(1)⇒(2). We know from the proof of (3)⇒(1) that there exists a function h, invariant and different from zero a.e. such that $h^qw^{1-q}\in L_1(dm)$. This means that $h^{q/p}w^{1-q}\in L_p(d\mu)$. It follows from (1) that for every invariant set A,

$$\int_A(\lim_{n\to\infty}M_n(T)h^{q/p}w^{1-q})^pw\,dm \le C\int_A h^qw^{p-pq}w\,dm$$
$$= C\int_A h^qw^{1-q}\,dm,$$

and since the function $\lim_{n\to\infty}M_n(T)(h^{q/p}w^{1-q})(x)$ is invariant, we obtain

$$\int_A(\lim_{n\to\infty}M_n(T)h^{q/p}w^{1-q})^p\lim_{n\to\infty}M_n(T)w\,dm$$
$$\le C\int_A\lim_{n\to\infty}M_n(T)(h^qw^{1-q})\,dm.$$

Since A is any invariant set, and the integrands are also invariant, we get

$$(\lim_{n\to\infty}M_n(T)h^{q/p}w^{1-q})^p\lim_{n\to\infty}M_n(T)w \le C\lim_{n\to\infty}M_n(T)(h^qw^{1-q}),$$

and (2) follows from the fact that h is invariant and positive.

(2)⇒(1). Observe that $\lim_{n\to\infty}(M_n(T)w)(x)$ exists and is different from zero a.e. because w is a positive, integrable function with respect to the invariant measure dm. Therefore (2) implies that $\lim_{n\to\infty}(M_n(T)w^{1-q})(x)$ exists and is finite a.e., which gives that $\sup_n M_n(T)w^{1-q}(x)$ is finite a.e., and according to [MT], this implies that $Pf(x) = \lim_{n\to\infty}M_n(T)f(x)$ exists and is finite a.e., for all $f\in L_p(wdm)$.

The only thing left is to prove that $Pf\in L_p(wdm)$. Let f be a positive function. Hölder's inequality gives that

$$(M_n(T)f)^p(x) \le (M_n(T)(f^pw))(x)(M_n(T)w^{1-q})^{p/q}(x).$$

It follows from (2) that

$$(Pf)^p(x) \le C\frac{\lim_{n\to\infty}M_n(T)(f^pw)(x)}{\lim_{n\to\infty}M_n(T)w(x)}$$

and

$$\int(Pf)^p(x)w(x)\,dm \le C\int\frac{\lim_{n\to\infty}M_n(T)(f^pw)(x)}{\lim_{n\to\infty}M_n(T)w(x)}w(x)\,dm.$$

Finally, the ergodic theorem for measure preserving transformations tells us that this last integral is equal to

$$\int \frac{\lim_{n\to\infty} M_n(T)(f^p w)(x)}{\lim_{n\to\infty} M_n(T)w(x)} \lim_{n\to\infty} M_n(T)w(x)\,dm = \int \lim_{n\to\infty} M_n(T)(f^p w)(x)\,dm = \int f^p(x)w(x)\,dm\,.$$

3. Remarks. 1) If ϕ is not invertible (1) and (2) in Theorem 1 are still equivalent. To see this one just observes that the proof of (2)⇒(1) does not make use of the invertibility assumption, so it is enough to show that (1)⇒(2). But if (1) holds, then, using again the result of [MT] we conclude that $w^* \equiv \limsup_{n\to\infty} M_n(T)w^{1-q}$ is finite a.e. This means that X can be decomposed as the union of invariant sets X_k, where $X_k \equiv \{x : 2^{k-1} \le w^*(x) < 2^k\}$. But for each k the function w^{1-q} is in $L_1(X_k, dm)$ and then the same argument as in Theorem 1 proves (2) a.e. in X_k.

2) If ϕ is nonsingular and the operator T maps $L_p(\mu)$ into itself then the assumption that our measure μ is of the form $\mu(A) = \int_A w\,dm$ in Theorem 1 is not a restriction, because if (1) holds, i.e., if T satisfies the P.E.T. in $L_p(\mu)$, then there exists an invariant measure m equivalent to μ [AW]. If (3) holds then the measure $dm \equiv \lim M_n(T^*)\mathbf{1}\,d\mu$ is a finite invariant measure.

References

[AW] I. Assani and J. Woś, *An equivalent measure for some nonsingular transformations and applications*, Studia Math. 97 (1990), 1–12.

[MT] F. J. Martín-Reyes and A. de la Torre, *On the almost everywhere convergence of the ergodic averages*, Ergodic Theory Dynamical Systems 10 (1990), 141–149.

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Unique continuation for elliptic equations and an abstract differential inequality

by

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Abstract. We consider a class of elliptic equations whose leading part is the Laplacian and for which the singularities of the coefficients of lower order terms are described by a mixed L^p-norm. We prove that the zeros of the solutions are of at most finite order in the sense of a spherical L^2-mean.

1. Introduction. Unique continuation properties of solutions of second order elliptic equations with bounded coefficients can be studied in detail with the aid of Carleman type inequalities and besides them the use of L^2-norms is sufficient (see e.g. [5]). For the case of unbounded coefficients the situation is different in general. Strong uniqueness for the Schrödinger equation with potential whose integrability exponent is minimal (i.e. equal to $n/2$ where $n \ge 3$ is the dimension of the space) can be proved by this method [8]. One also gets uniqueness theorems for elliptic equations with variable coefficients of the leading part and with coefficients of first order terms in L^{p_1}, $p_1 < \infty$ (see e.g. [6], [10]). However, it appeared that the optimal value of the exponent, $p_1 = n$, cannot be attained via Carleman type inequalities: it is possible to get $p_1 \ge (3n-2)/2$ at most [7], [2]. In fact, the estimates giving uniqueness for such values of the exponent have been found [2], [11]. T. H. Wolff has obtained uniqueness results for a case when $p_1 < (3n-2)/2$ using some modified Carleman type inequalities; namely, he proved strong uniqueness for $p_1 = \max(n, (3n-4)/2)$ [19], and unique continuation from an open set for $p_1 = n$ [20] (the assumptions on all coefficients are minimal in this case).

There are other methods giving uniqueness theorems for various classes of second order elliptic equations. Using a geometric approach N. Garofalo with F. H. Lin [3], [4] and J. L. Kazdan [9] have got strong uniqueness results for equations with coefficients of first order terms having isolated singularities of a maximal rate of growth: they may belong to L^n_{loc} without

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