But this implies (24) since

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} w_k = 0. \blacksquare$$

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Montel and reflexive preduals of spaces of holomorphic functions on Fréchet spaces

by

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Abstract. For U open in a locally convex space E it is shown in [31] that there is a complete locally convex space $G(U)$ such that $G(U)'_i = (\mathcal{H}(U), \tau_\delta)$. Here, we assume U is balanced open in a Fréchet space and give necessary and sufficient conditions for $G(U)$ to be Montel and reflexive. These results give an insight into the relationship between the τ_0 and τ_ω topologies on $\mathcal{H}(U)$.

1. Introduction. Let U be an open subset of a locally convex space E. We denote by $\mathcal{H}(U)$ the space of holomorphic functions from U to $\mathbb{C}$. We shall say that a seminorm p on $\mathcal{H}(U)$ is τ_δ-*continuous* if for each countable increasing open cover $\{U_n\}_n$ of U there is a positive integer n_0 and $C > 0$ such that $p(f) \le C\|f\|_{U_{n_0}}$ for every $f \in \mathcal{H}(U)$. In [31], $G(U)$ is defined to be the space of linear forms on $\mathcal{H}(U)$ which are τ_0-continuous when restricted to the locally bounded sets. We give $G(U)$ the topology of uniform convergence on locally bounded subsets of $\mathcal{H}(U)$. Mujica and Nachbin prove that $G(U)'_i = (\mathcal{H}(U), \tau_\delta)$ and then proceed to show that the topological properties of $G(U)$ are useful in characterizing the topological properties of $\mathcal{H}(U)$. This result is a topological generalization of a result of Mazet [27] who had previously shown that $G(U)' = \mathcal{H}(U)$. In [14], the author further investigated the space $G(U)$ and obtained necessary and sufficient conditions for the inductive dual of $G(U)$ to be equal to its strong dual and thus for $(\mathcal{H}(U), \tau_\delta)$ to be equal to $G(U)'_b$. One of the conditions for this to happen is that $G(U)$ be distinguished. We investigate necessary and sufficient conditions for $G(U)$ to be Montel and reflexive. Among the conditions for $G(U)$ to be Montel is that the τ_0 and τ_ω topologies coincide on $\mathcal{H}(U)$ while among the conditions for reflexivity is that the τ_0 and τ_ω topologies are compatible on $\mathcal{H}(U)$. This implies that for U balanced open in Tsirelson's space we have $(\mathcal{H}(U), \tau_0)' = (\mathcal{H}(U), \tau_\omega)'$ while $\tau_0 \ne \tau_\omega$. In the final section we give further examples of Fréchet spaces with $\tau_0 \ne \tau_\omega$ but with both of these topologies

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being compatible. We finish by studying $\mathcal{H}(K)$ for K balanced compact in a Fréchet space with the density condition.

In [14] the author shows that for each locally convex space E and each integer n there is a complete locally convex space, $Q(^nE)$, such that $Q(^nE)'_i = (P(^nE), \tau_\omega)$ and for each compact subset K of E there is a complete locally convex space, $G(K)$, such that $G(K)'_i = (\mathcal{H}(K), \tau_\omega)$. The space $(P(^nE), \tau_\omega)$ is the space of n-homogeneous polynomials on E with the topology induced by τ_δ. If U (resp. K) is balanced open (resp. compact) then $\{Q(^nE)\}_n$ is an $\mathcal{S}$-absolute decomposition for $G(U)$ (resp. $G(K)$) (see Propositions 4 and 5 of [14]).

We refer the reader to [19] for background material on infinite-dimensional holomorphy and to [24], [25] and [34] for background material on locally convex spaces.

2. Montel preduals of locally convex spaces. In this section we give necessary and sufficient conditions on $\mathcal{H}(U)$ for $G(U)$ to be Montel. We first state a technical lemma which is part of Lemma 6 of [7] and which we will find useful in subsequent sections.

LEMMA 1. *Let E be a (complete) infrabarrelled locally convex space. Then E is topologically isomorphic to a (closed) subspace of $(E'_i)'_b$.*

The following theorem characterizes the Fréchet spaces for which $G(U)$ is Montel. Part of the following theorem is proved in [21] and [4].

THEOREM 2. *Let E be a Fréchet space. Then the following are equivalent:*

(a) *$\tau_0 = \tau_\omega$ on $P(^nE)$ for every integer n.*

(b) *$\tau_0 = \tau_\omega$ on $\mathcal{H}(K)$ for one (and hence every) balanced compact subset K of E.*

(c) *$\tau_0 = \tau_\omega$ on $\mathcal{H}(U)$ for one (and hence every) balanced open subset U of E.*

(d) *$(\mathcal{H}(K), \tau_\omega)$ is Montel for one (and hence every) balanced compact subset K of E.*

(e) *$(\mathcal{H}(U), \tau_\omega)$ is semi-Montel for one (and hence every) balanced open subset U of E.*

(f) *$(\mathcal{H}(U), \tau_\delta)$ is Montel for one (and hence every) balanced open subset U of E.*

(g) *$(P(^nE), \tau_\omega)$ is Montel for every integer n.*

(h) *$Q(^nE)$ is Montel for every integer n.*

(i) *$G(U)$ is Montel for one (and hence every) balanced open subset U of E.*

(j) *$G(K)$ is Montel for one (and hence every) balanced compact subset K of E.*

Proof. The equivalence of (a) to (g) follows from Lemma 3.5 of [21], whereas the equivalence of (h), (i) and (j) follows from the remarks preceding Lemma 3.4 of [21]. If $G(U)$ is Montel, then $G(U)$ is distinguished and therefore $G(U)'_b = G(U)'_i = (\mathcal{H}(U), \tau_\delta)$ by Theorem 9 of [14]. Since the strong dual of a Montel space is Montel it follows that (i) implies (f). By Lemma 1, $G(U)$ is a closed subspace of $(\mathcal{H}(U), \tau_\delta)'_b$. Therefore if (f) holds, $(\mathcal{H}(U), \tau_\delta)'_b$ and hence $G(U)$ will be Montel. ∎

From Theorem 2 we see that $G(U)$ is Montel if and only if the τ_0 and τ_ω topologies coincide on $\mathcal{H}(U)$. The problem of the coincidence of τ_0 and τ_ω on $\mathcal{H}(U)$ has been considered by various authors. Ansemil and Taskinen [5] gave the first example of a Fréchet Montel space for which $\tau_0 \neq \tau_\omega$ on any balanced open subset. For classes of Fréchet spaces where $\tau_0 = \tau_\omega$ on $\mathcal{H}(U)$ for every open subset see [9], [30] and [36]. For classes of Fréchet spaces where $\tau_0 = \tau_\omega$ on $\mathcal{H}(U)$ for every balanced open subset see [8], [12], [28], [4], [20], [22], [13], [29], [16].

3. Reflexive preduals of the spaces of holomorphic functions. By construction, when U is a balanced open subset of a Fréchet space E, $G(U)$ is a subspace of $(\mathcal{H}(U), \tau_\delta)'_b$. The following theorem gives necessary and sufficient conditions for $G(U)$ to be equal to $(\mathcal{H}(U), \tau_\delta)'_b$. One of these conditions is that $G(U)$ is reflexive, and therefore Theorem 3 may be seen as a reflexive version of Theorem 2.

THEOREM 3. *Let E be a Fréchet space. Then the following are equivalent:*

(a) *$(P(^nE), \tau_0)' = (P(^nE), \tau_\omega)'$ for every integer n.*

(b) *$(\mathcal{H}(U), \tau_0)' = (\mathcal{H}(U), \tau_\omega)'$ for one (and hence every) balanced open subset U of E.*

(c) *$(\mathcal{H}(K), \tau_0)' = (\mathcal{H}(K), \tau_\omega)'$ for one (and hence every) balanced compact subset K of E.*

(d) *$(P(^nE), \tau_\omega)$ is reflexive for every integer n.*

(e) *$(\mathcal{H}(K), \tau_\omega)$ is reflexive for one (and hence every) balanced compact subset K of E.*

(f) *$(\mathcal{H}(U), \tau_\omega)$ is semi-reflexive for one (and hence every) balanced open subset U of E.*

(g) *$(\mathcal{H}(U), \tau_\delta)$ is reflexive for one (and hence every) balanced open subset U of E.*

(h) *$Q(^nE)$ is reflexive for every integer n.*

(i) *$G(K)$ is reflexive for one (and hence every) balanced compact subset K of E.*

(j) *$G(U)$ is reflexive for one (and hence every) balanced open subset U of E.*

(k) $(\mathcal{H}(U),\tau_\omega)' \subseteq G(U)$ *for one (and hence every) balanced open subset* U *of* E.

(l) $G(U) = (\mathcal{H}(U),\tau_\delta)'_b$ *for one (and hence every) balanced open subset* U *of* E.

Proof. The equivalence of (d) to (g) follows from the fact that $\{(P(^nE),\tau_\omega)\}_n$ is a Schauder decomposition for $(\mathcal{H}(U),\tau_\omega)$, $(\mathcal{H}(U),\tau_\delta)$ and $(\mathcal{H}(K),\tau_\omega)$, and Theorem 3.2 of [26]. The equivalence of (h) to (j) also follows from Propositions 4 and 5 of [14] and Theorem 3.2 of [26].

(h)⇒(d). Follows from the fact that each $Q(^nE) = \widehat{\bigotimes}_{s,n,\pi} E$ is a distinguished Fréchet space, and the fact that the strong dual of a reflexive Fréchet space is reflexive.

(e)⇒(i). As $(\mathcal{H}(K),\tau_\omega)$ is reflexive, $(\mathcal{H}(K),\tau_\omega)'_b$ is reflexive and therefore $G(K)$ is reflexive by Lemma 1.

(i)⇒(c). By [29], $G(K) = (\mathcal{H}(K),\tau_0)'_b$. Therefore, if $G(K)$ is reflexive, it is distinguished and

$$(\mathcal{H}(K),\tau_0)'_b = G(K) = (G(K)'_b)'_b = (\mathcal{H}(K),\tau_\omega)'_b\,.$$

(c)⇒(b). For U a balanced open subset of E we have

$$(\mathcal{H}(U),\tau_0) = \varprojlim_{K\subset U} (\mathcal{H}(K),\tau_0)$$

and

$$(\mathcal{H}(U),\tau_\omega) = \varprojlim_{K\subset U} (\mathcal{H}(K),\tau_\omega)\,,$$

where both limits are taken over all compact balanced subsets of U. By the argument of Proposition 7 of [14] we can show that both these projective limits are reduced. Thus, by IV.4.4 of [34] we have the algebraic equivalences

$$(\mathcal{H}(U),\tau_0)' = \varinjlim_{K\subset U} (\mathcal{H}(K),\tau_0)'$$

and

$$(\mathcal{H}(U),\tau_\omega)' = \varinjlim_{K\subset U} (\mathcal{H}(K),\tau_\omega)'\,.$$

Hence if $(\mathcal{H}(K),\tau_0)' = (\mathcal{H}(K),\tau_\omega)'$ for every K we have (b).

(b)⇒(a). As τ_ω is finer than τ_0, we have $(P(^nE),\tau_0)' \subseteq (P(^nE),\tau_\omega)'$ for every integer n. We know that $\{(P(^nE),\tau_0)'_b\}_n$ is an $\mathcal{S}$-absolute decomposition of $(\mathcal{H}(U),\tau_0)'_b$, while $\{(P(^nE),\tau_\omega)'_b\}_n$ is an $\mathcal{S}$-absolute decomposition of $(\mathcal{H}(U),\tau_\omega)'_b$, thus if $(P(^nE),\tau_0)'$ is strictly contained in $(P(^nE),\tau_\omega)'$ for some n, we cannot have $(\mathcal{H}(U),\tau_0)'$ equal to $(\mathcal{H}(U),\tau_\omega)'$, and so (a) holds.

(a)⇒(d). Since $(P(^nE),\tau_0)' \subseteq Q(^nE) \subseteq (P(^nE),\tau_\omega)'$, (a) implies $Q(^nE) = (P(^nE),\tau_\omega)'_b$, whence $((P(^nE),\tau_\omega)'_b)' = Q(^nE)' = P(^nE)$.

(j)⇒(l). If $G(U)$ is reflexive it is distinguished and so $G(U)'_b = (\mathcal{H}(U),\tau_\delta)$. Taking strong duals we get $G(U) = (G(U)'_b)'_b = (\mathcal{H}(U),\tau_\delta)'_b$.

(l)⇒(g). If $G(U) = (\mathcal{H}(U),\tau_\delta)'_b$, then $\mathcal{H}(U) = G(U)' = ((\mathcal{H}(U),\tau_\delta)'_b)'$ and so $(\mathcal{H}(U),\tau_\delta)$ is semireflexive. Since it is infrabarrelled it is reflexive.

(b)⇒(k). Follows from the fact that $(\mathcal{H}(U),\tau_0)'$ is always contained in $G(U)$.

(k)⇒(f). It follows from Grothendieck's Completeness Theorem, Theorem 3.11.1 of [24], that $G(U)$ is the completion of $(\mathcal{H}(U),\tau_0)'_b$. Therefore (k) implies that $G(U)$ is also the completion of $(\mathcal{H}(U),\tau_\omega)'_b$. It now follows that $((\mathcal{H}(U),\tau_\omega)'_b)' = G(U)' = \mathcal{H}(U)$, which is (f). ∎

If E is a Banach space, then Proposition 5.4 of [33] implies that all the above conditions are equivalent to the condition that any entire holomorphic function on E with values in any Banach space F is weakly compact, i.e. for each $f: E \to F$ and each $x \in E$ there is a neighbourhood V_x of x such that $f(V_x)$ is weakly compact in F. Theorem 2 tells us that $G(U)$ is Montel if and only if the τ_0 and τ_ω topologies coincide on $\mathcal{H}(U)$, while Theorem 3 tells us that $G(U)$ is reflexive if and only if the two topologies are compatible. We give an example showing Theorem 3 is not included in Theorem 2. In 1973 Tsirelson [38] constructed an infinite-dimensional reflexive Banach space with an unconditional basis that does not contain a copy of c_0 or ℓ_p, $1 < p < \infty$, as a subspace. We will denote this space by T^* and refer to it as Tsirelson's space [1]. In [1] Alencar, Aron and Dineen proved that if U is a balanced open subset of T^*, then $(\mathcal{H}(U),\tau_\delta)$ is reflexive. Corollary 2.8 of [6] shows that $(\mathcal{H}(U),\tau_\delta)$ is reflexive for U balanced open in any quotient of T^* [2]. As infinite-dimensional Banach spaces cannot be Montel, T^* is an example of a Fréchet space with the property that $G(U)$ is reflexive but not Montel for any balanced open subset U, thus τ_0 and τ_ω topologies are compatible on $\mathcal{H}(U)$, without being equal.

If U (resp. K) is a balanced open (resp. compact) subset of a Fréchet space E it follows from Corollary 3 of [2] and Proposition 5 of [3] together with Proposition I.6.2 of [35] that τ_δ (resp. τ_ω) is the infrabarrelled topology associated with τ_0 on $\mathcal{H}(U)$ (resp. $\mathcal{H}(K)$). It will therefore follow that if $\tau_0 \neq \tau_\omega$, $(\mathcal{H}(U),\tau_0)$ and $(\mathcal{H}(K),\tau_0)$ cannot be infrabarrelled. It is, however, still possible that they may have the weaker property of being Mackey spaces. From Theorem 3, we see that the spaces $(\mathcal{H}(U),\tau_0)$, $(\mathcal{H}(K),\tau_0)$ and $(P(^nT^*),\tau_0)$ are not Mackey for U balanced open in T^*, K balanced compact in T^* and n any positive integer.

In the theory of infinite-dimensional Banach spaces, the fact that T^* does not contain a copy of c_0 of ℓ_p is useful in finding counterexamples. In order to prove that $G(U)$ is reflexive the fact that E does not contain a

[1] This is Tsirelson's original space. Some authors refer to Tsirelson's space as the space T which is the dual of T^*.

[2] By [15], there are quotients of T^* which are not isomorphic to T^*.

copy of ℓ_p is in fact necessary. This is because a result of Aron, quoted in [33], says: *For any integer p the space* $(P(^n\ell_p),\tau_\omega)$ *contains a copy of* ℓ_∞ *for any integer n with* $n>p$. Consequently, for any balanced open subset U of ℓ_p, $1\le p<\infty$, we have $G(U)\subsetneq(\mathcal{H}(U),\tau_\delta)'_{\rm b}$. In [14] we proved that $G(U)'_{\rm b}=(\mathcal{H}(U),\tau_\delta)$ for U a balanced open subset of a Banach space with unconditional basis. Therefore we have examples where $G(U)'_{\rm b}=(\mathcal{H}(U),\tau_\delta)$ and $(\mathcal{H}(U),\tau_\delta)'_{\rm b}\neq G(U)$.

By definition $G(U)$ is the space of all linear forms on $\mathcal{H}(U)$ which when restricted to locally bounded sets are τ_0-continuous. Therefore if $G(U)$ is reflexive, as happens with Tsirelson's space, we see, from (1), that every τ_δ-continuous linear form on $\mathcal{H}(U)$ is τ_0-continuous on locally bounded sets.

It is interesting to note the relationship between conditions (b), (j) and (k) of Theorem 3. We see, from (b) and (j), that for any Fréchet space for which $G(U)$ is not reflexive (this includes all nonreflexive Fréchet spaces and the ℓ_p spaces), there is a τ_ω-continuous linear form on $\mathcal{H}(U)$ which is not τ_0-continuous. Since $G(U)$ is the completion of $(\mathcal{H}(U),\tau_0)'_{\rm b}$, we see by (b) and (k) that if there is a τ_ω-continuous linear form on $\mathcal{H}(U)$ which is not τ_0-continuous, then there is also a τ_ω-continuous linear form which is not the limit in $(\mathcal{H}(U),\tau_\omega)'_{\rm b}$ of a net in $(\mathcal{H}(U),\tau_0)'$, i.e., if $(\mathcal{H}(U),\tau_0)'_{\rm b}$ is not equal to $(\mathcal{H}(U),\tau_\omega)'_{\rm b}$ it is not even dense in $(\mathcal{H}(U),\tau_\omega)'_{\rm b}$.

4. Further examples of Montel and reflexive preduals. In the previous two sections we gave necessary and sufficient conditions for $G(U)$ to be Montel and reflexive. In this section we show how to construct new Fréchet spaces with the property that $G(U)$ is Montel or reflexive out of spaces where we know that this property holds.

A necessary and sufficient condition, by Theorem 2, in order to have $G(U)$ Montel for U balanced open in a Fréchet space E is that $Q(^nE)=\widehat{\bigotimes}_{s,n,\pi}E$ is Montel for every integer n. However, all the situations where we know that $G(U)$ is Montel are deduced from the (possibly stronger) fact that $\widehat{\bigotimes}_{n,\pi}E$ is Montel for every integer n. This observation motivates the following definition.

Definition 4. Let $\mathcal{M}$ (resp. $\mathcal{R}$, $\mathcal{DC}$) be the collection of Fréchet spaces E with the property that $\widehat{\bigotimes}_{n,\pi}E$ is Montel (resp. reflexive, has the density condition) for every integer n.

The collection $\mathcal{M}$ is precisely the collection of all Fréchet spaces E such that E has property (BB) n-times as defined in [16]. It contains

(a) all Fréchet Schwartz spaces,

(b) all hilbertizable Fréchet Montel spaces,

(c) all ε-spaces,

(d) all Montel (Fo)-spaces as defined by Peris [32].

It follows from Corollary 2.8 of [6] that every quotient of T^* is in $\mathcal{R}$ and therefore we have $\mathcal{M}\subsetneq\mathcal{R}$. Since, by Proposition 2.4.4 of [32], the tensor product of two (Fo)-spaces is an (Fo)-space and the pair has the (BB) property it follows by Corollary 7 of [10] that every (Fo)-space with the density condition is in $\mathcal{DC}$. In [37] it is shown that there is a Fréchet Montel space F such that $F\widehat{\otimes}_\pi F$ is not distinguished, and therefore not in $\mathcal{DC}$.

In [16] it is shown that if E is in $\mathcal{M}$, then $E^{\mathbb{N}}$ is in $\mathcal{M}$. The corresponding result also holds for $\mathcal{R}$ and $\mathcal{D}$ with similar proofs. In particular, $(T^*)^{\mathbb{N}}$ is an example of a non-Montel Fréchet space with the property that $G(U)$ is reflexive for every balanced open subset U.

The following proposition is perhaps the most useful method of obtaining new spaces in $\mathcal{M}$, $\mathcal{R}$ and $\mathcal{DC}$.

Proposition 5. *Let E be any of the following:*

(a) *a Montel decomposable* (FG)*-space*,

(b) *a Fréchet Schwartz space with the bounded approximation property*,

(c) *a Fréchet nuclear space*,

and let F be in $\mathcal{M}$ *(resp.* $\mathcal{R}$, $\mathcal{DC}$*). Then* $E\widehat{\otimes}_\pi F$ *and* $E\times F$ *are in* $\mathcal{M}$ *(resp.* $\mathcal{R}$, $\mathcal{DC}$*).*

Proof. By repeated use of Corollary 6 of [18] in the case (a), Theorem 12 of [17] in the case (b) and Proposition 2.3.2.13 of [23] and Corollary 7 of [10] in the case (c) we find that $E\widehat{\otimes}_\pi F$ is in $\mathcal{M}$ (resp. $\mathcal{R}$, $\mathcal{DC}$) for $F\in\mathcal{M}$ (resp. $F\in\mathcal{R}$, $\mathcal{DC}$). Using induction, Theorem 15.4.1 of [25] and the fact that $E\widehat{\otimes}_\pi F\cong F\widehat{\otimes}_\pi E$, we see that

$$\Big(\widehat{\bigotimes_{n,\pi}}E\Big)\,\widehat{\otimes}_\pi\Big(\widehat{\bigotimes_{m,\pi}}F\Big)$$

is Montel (resp. reflexive, has the density condition) for every $(n,m)\in\mathbb{N}\times\mathbb{N}$. For each positive integer n,

$$\widehat{\bigotimes_{n,\pi}}(E\times F)\cong\Big(\widehat{\bigotimes_{n,\pi}}E\Big)\times\Big(\widehat{\bigotimes_{n-1,\pi}}E\,\widehat{\otimes}_\pi F\Big)^n \times\Big(\widehat{\bigotimes_{n-2,\pi}}E\Big)\widehat{\otimes}_\pi\Big(\widehat{\bigotimes_{2,\pi}}F\Big)^{\binom{n}{2}}\times\dots \times\Big(E\,\widehat{\otimes}_\pi\widehat{\bigotimes_{n-1,\pi}}F\Big)^n\times\Big(\widehat{\bigotimes_{n,\pi}}F\Big).$$

Since the product of Montel spaces (resp. reflexive spaces, spaces having the density condition) is Montel (resp. reflexive, has the density condition) it follows that $\widehat{\bigotimes}_{n,\pi}(E\times F)$ is Montel (resp. reflexive, has the density condition) for every n. ∎

If we take F in $\mathcal{M}$ in the above proposition, we obtain new examples where $\tau_0=\tau_\omega$. We have $\tau_0=\tau_\omega$ for U balanced open in $E\times F$, where E satisfies (a), (b) or (c) above and F is one of the following:

(i) a Fréchet Schwartz space,
(ii) a hilbertizable Fréchet Montel space,
(iii) an ε-space,
(iv) a Montel (Fo)-space as defined by Peris [32].

Until now, the known spaces E with $G(U)$ reflexive for any balanced open subset U have either had $\tau_0=\tau_\omega$ on $\mathcal{H}(U)$, which is the case when E is in $\mathcal{M}$, or satisfied $\tau_\omega=\tau_\delta$ on $\mathcal{H}(U)$, which is the case with T^*. We give an example of a Fréchet space F in $\mathcal{R}$ with the property that $\tau_0<\tau_\omega<\tau_\delta$ on $\mathcal{H}(U)$ for each balanced open subset U of F.

Let E be a Fréchet nuclear space or a decomposable (FG)-space which does not admit a continuous norm; $\mathbb{C}^{\mathbb{N}}$ is an example of such a space. By Proposition 5, $E\times T^*$ is in $\mathcal{R}$ and therefore $G(U)$ is reflexive for U balanced open in $E\times T^*$. We claim that $\tau_0<\tau_\omega<\tau_\delta$ on $\mathcal{H}(U)$. Since $E\times T^*$ is not Montel, τ_0 is not equal to τ_ω. As E does not admit a continuous norm, $E\times T^*$ does not admit a continuous norm. By Example 2.52 of [19], $\tau_\omega<\tau_\delta$ on $\mathcal{H}(U)$. In particular, from Theorem 2, we see that $E\times T^*$ is an example of a Fréchet space with $\tau_0<\tau_\omega<\tau_\delta$ on each balanced open subset U of $E\times T^*$ but $(\mathcal{H}(U),\tau_0)'=(\mathcal{H}(U),\tau_\omega)'$.

Following Díaz and Miñarro [17] we say that a Schauder decomposition $\{E_n\}_n$ of the locally convex space E has property (M) if

$$\lim_{n\to\infty}\sup_{x\in B}p\Big(\sum_{j=n}^{\infty}x_j\Big)=0$$

for every bounded set B in E and $p\in$ c.s.(E), where $x_j=x|_{E_j}$. We show that the Schauder decomposition $\{Q(^nE)\}_n$ of $G(K)$ satisfies condition (M) for K compact balanced in a locally convex space.

LEMMA 6. *Let K be a balanced compact subset of a locally convex space E. Then the Schauder decomposition $\{Q(^nE)\}_n$ of $G(K)$ has property* (M).

Proof. By the definition of the topology on $G(K)$ we may assume that every bounded set B in $G(K)$ is a set of linear maps $\phi:\mathcal{H}(K)\to\mathbb{C}$ which is uniformly bounded on each B_W for $W\supset K$, where $B_W=\{f\in\mathcal{H}^\infty(W):\|f\|_W\le 1\}$, and each seminorm p on $G(K)$ is defined by

$$p(\phi)=\sup_{f\in B_V}|\phi(f)|$$

for $\phi\in G(K)$, where V is an open neighbourhood of K. As in Proposition 3.13 of [19] it follows that

$$\widetilde{B}_V=\Big\{\sum_{n=m}^{\infty}\frac{\widehat{d}^nf(0)}{n!}:m\in\mathbb{N},\ f\in B_V\Big\}$$

is defined and uniformly bounded on some neighbourhood of K. For every ϕ in B we have

$$p\Big(\sum_{n=m}^{\infty}\phi_n\Big)=\sup_{f\in B_V}\Big|\phi\Big(\sum_{n=m}^{\infty}\frac{\widehat{d}^nf(0)}{n!}\Big)\Big|\le\frac{1}{m^2}\|\phi\|_{\widetilde{B}_V}.$$

As $\sup_{\phi\in B}\|\phi\|_{\widetilde{B}_V}$ is bounded we have

$$\lim_{m\to\infty}\sup_{\phi\in B}p\Big(\sum_{n=m}^{\infty}\phi_n\Big)\le\lim_{m\to\infty}\frac{1}{m^2}\sup_{\phi\in B}\|\phi\|_{\widetilde{B}_V}=0.$$

Thus $G(K)$ satisfies condition (M). ∎

PROPOSITION 7. *Let K be a balanced compact subset of a Fréchet space E. Then $E\in\mathcal{DC}$ if and only if $G(K)$ satisfies the density condition.*

Proof. The result follows by applying Proposition 4 of [17] to Lemma 6. ∎

COROLLARY 8. *Let K be a balanced compact subset of a Fréchet space E. Then $E\in\mathcal{DC}$ if and only if the bounded subsets of $(\mathcal{H}(K),\tau_\omega)$ are metrizable.*

This result is known for E a quasinormable metrizable locally convex space, since by Proposition 6.18 of [19], the inductive limit $\mathcal{H}(K)=\varinjlim_{V\supset K}(\mathcal{H}^\infty(V),\|\cdot\|_V)$ is boundedly retractive.

COROLLARY 9. *Let K be as above. Then $(\mathcal{H}(K),\tau_\omega)$ admits a continuous norm.*

Proof. As $G(K)$ satisfies the density condition it is distinguished. Therefore $\mathcal{H}(K)=G(K)'_b$ and by Lemma 3 of [17], $\mathcal{H}(K)$ admits a continuous norm. ∎

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