into $Q(D,D)$ satisfies (5.3), (5.7) and (5.8). Now the theorem follows upon applying Proposition 5.2.

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INSTITUTE OF APPLIED MATHEMATICS AND MECHANICS
WARSAW UNIVERSITY
BANACHA 2
02-097 WARSZAWA, POLAND
E-mail: RUSINEK@MIMUW.EDU.PL

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On the representation of uncountable symmetric basic sets and its applications

by

FRANCISCO L. HERNANDEZ (Madrid) and
STANIMIR L. TROYANSKI (Sofia)

Abstract. It is shown that every uncountable symmetric basic set in an F-space with a symmetric basis is equivalent to a basic set generated by one vector. We apply this result to investigate the structure of uncountable symmetric basic sets in Orlicz and Lorentz spaces.

I. Introduction. There are three results about Banach spaces with an uncountable symmetric basis having some relevance to the subject of this paper. Firstly, it was shown by renorming arguments in [T$_1$] that if X is a Banach space with a symmetric basis $\{e_\alpha\}_{\alpha\in A}$ which contains a subspace isomorphic to $c_0(\Gamma)$ (resp. to $\ell^1(\Gamma)$) for an uncountable set Γ then X itself is isomorphic to $c_0(A)$ (resp. to $\ell^1(A)$). Later, using this result and combinatorial considerations, Drewnowski [D$_1$] proved that for nonseparable Banach spaces with a symmetric basis, all uncountable symmetric bases are equivalent. Recently, in the special context of Orlicz spaces, Rodriguez-Salinas [R] has given necessary and sufficient conditions for isomorphic embeddings of Orlicz spaces $h_N(\Gamma)$ into a space $h_M(A)$ for uncountable sets $\Gamma\subseteq A$.

Our aim in the present paper is to analyze the above results in a general framework, that is, to generalize them to F-spaces, i.e. complete metric linear spaces, to determine the context of validity of possible connected extensions, and finally, to give some new applications.

For example, we prove that for the class of F-spaces the above mentioned result of [T$_1$] on spaces containing an isomorphic copy of $c_0(\Gamma)$ can be

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extended, while the corresponding version for spaces containing $\ell^1(\Gamma)$ is not true for F-spaces. We also show that, for every $0 < p < \infty$, there exist Orlicz spaces $h_M(A)$ with a symmetric basis, different from $\ell^p(A)$, containing an isomorphic copy of $\ell^p(\Gamma)$ for uncountable sets $\Gamma \subseteq A$ (Proposition 7). However, unlike the case of Orlicz spaces, it turns out that the Lorentz spaces $d(w,p,A)$ do not contain any isomorphic copy of $\ell^p(\Gamma)$ for uncountable sets $\Gamma \subseteq A$ and $0 < p < \infty$ (Proposition 10).

Our main tool is a structure theorem for isomorphic embeddings of spaces with an uncountable symmetric basis into F-spaces with a symmetric basis.

Let X be an F-space with a symmetric basis $\{e_\alpha\}_{\alpha\in A}$. A basic set $\{v_\gamma\}_{\gamma\in\Gamma}$ in X is called a *block basis generated by a vector* $x = \sum_{n=1}^\infty a_n e_{\alpha_n}$ if there exist disjoint infinite subsets $\{\gamma(i)\}_{i=1}^\infty$ of A, for $\gamma \in \Gamma$, where $\gamma \neq \delta$ or $i \neq j$ implies $\gamma(i) \neq \delta(j)$, such that

$$v_\gamma = \sum_{i=1}^\infty a_i e_{\gamma(i)} \quad \text{for each } \gamma \in \Gamma .$$

MAIN THEOREM. *Let X be an F-space with a symmetric basis $\{e_\alpha\}_{\alpha\in A}$. If $\{u_\gamma\}_{\gamma\in\Gamma}$ is an uncountable symmetric basic set in X then $\{u_\gamma\}_{\gamma\in\Gamma}$ is equivalent to a block basis $\{v_\gamma\}_{\gamma\in\Gamma}$ generated by one vector.*

In the case of Banach spaces this result was reported at the Vth Spring Conference of Bulgarian Mathematicians [T$_2$] and is presented in the Doctoral Thesis of the second author.

II. Proof of the main result.

Let us start with recalling some definitions. By X we will denote an F-space, i.e. a complete metric linear space, with an F-norm $\|\cdot\|$.

DEFINITION 1. A family $\{e_\alpha\}_{\alpha\in A}$ of vectors in X is said to be a *symmetric basis* of X if:

(i) it is an unconditional basis of X, i.e. for every $x \in X$ there is a unique family $\{a_\alpha\}_{\alpha\in A}$ of scalars such that

$$x = \sum_{\alpha\in A} a_\alpha e_\alpha \quad \text{(in the sense of sumability), and}$$

(ii) for any two sequences $\{\alpha_n\}$ and $\{\alpha'_n\}$ in A, the basic sequences $\{e_{\alpha_n}\}$ and $\{e_{\alpha'_n}\}$ are equivalent.

Denote by Π_A the family of all one-to-one correspondences π from A onto A and by Σ_A the family of all functions σ defined on A taking values ± 1.

Suppose that the symmetric basis $\{e_\alpha\}_{\alpha\in A}$ is *regular* (cf. [KPR]), i.e. for every $\lambda > 0$,

$$(*) \qquad \varphi(\lambda) > 0, \quad \text{where} \quad \varphi(\lambda) = \inf_{\alpha\in A} \|\lambda e_\alpha\|.$$

Then, as in the case of Banach spaces, it turns out that each $x \in X$ can be represented in a unique way in the form $x = \sum_\alpha a_\alpha e_\alpha$, where the convergence is unconditional and the families $\{\widehat{\pi}\}_{\pi\in\Pi_A}$, $\{\widehat{\sigma}\}_{\sigma\in\Sigma_A}$ of operators defined by

$$\widehat{\pi}(x) = \sum_\alpha a_\alpha e_{\pi(\alpha)} \quad \text{and} \quad \widehat{\sigma}(x) = \sum_\alpha \sigma(\alpha) a_\alpha e_\alpha$$

are equicontinuous. The proof is the same as in the case of Banach spaces, taking into account that for every $x = \sum_\alpha a_\alpha e_\alpha$ the set $\{a_\alpha e_\beta\}_{(\alpha,\beta)\in A^2}$ is bounded (see [LT]).

Remark. The regularity condition $(*)$ is essential: let ω be the space of all scalar sequences and $\{e_n\}_{n=1}^\infty$ be the natural basis of ω. Obviously, $\{e_n\}$ is a symmetric basis, but for every $\lambda > 0$, $\inf_n \|\lambda e_n\| = 0$ and $\{\widehat{\pi}\}$ is not an equicontinuous family of operators.

From now on we only consider uncountable symmetric bases. For such bases the regularity condition $(*)$ is obviously satisfied.

In an F-space X with an uncountable symmetric basis one can introduce an F-norm by the formula

$$(1) \qquad |x| = \sup_{\sigma\in\Sigma_A} \sup_{\pi\in\Pi_A} \|\widehat{\sigma}\widehat{\pi}x\| .$$

Since $\{\widehat{\sigma}\}$ and $\{\widehat{\pi}\}$ are equicontinuous, the F-norm $|\cdot|$ is equivalent to $\|\ \|$, i.e. they generate the same topology.

For our purposes we shall assume, without loss of generality, that the initial F-norm in X satisfies

$$(2) \qquad \|\lambda x\| \le \|x\| \quad \text{if } |\lambda| \le 1 ,$$

$$(3) \qquad \|\widehat{\sigma}\widehat{\pi}x\| = \|x\| \quad \text{for } \sigma \in \Sigma_A,\ \pi \in \Pi_A .$$

The existence of an equivalent F-norm with (3) follows from (1). (For (2) see [Ro] or [KPR].)

Let σ take values $0, \pm 1$. Then it follows from (2) and (3) that

$$(4) \qquad \|\widehat{\sigma}x\| \le 2\|x/2\| \le 2\|x\| \quad \text{and} \quad \lim_{\lambda\to 0} \varphi(\lambda) = 0 .$$

LEMMA 1. *Let X be an F-space and $\{e_m\}_{m=1}^\infty$ be an unconditional basis in X with biorthogonal system $\{e_m^*\}_{m=1}^\infty$. Let $\{x_n\}_{n=1}^\infty$ be an unconditional basic sequence in X such that $\lim_{n\to\infty} e_m^*(x_n) = e_m^*(x)$ for some $x \in X$ and for every $m \in \mathbb{N}$. Then the sequence $\{y_n\}_{n=1}^\infty$ with $y_n = x_n - x$ for $n \in \mathbb{N}$ is equivalent to $\{x_n\}_{n=1}^\infty$.*

Proof. We proceed in a standard way (cf. [M]). If $x \notin [x_n]_{n=1}^\infty$ then we consider the sequence $x_0 = 2x, x_1, x_2, \ldots$ and put $Y = [x_n]_{n=0}^\infty$. Let $\{x_n^*\}_{n=0}^\infty$ be the biorthogonal system to $\{x_n\}_{n=0}^\infty$. Since $x \neq 0$, there exists a natural number m such that $e_m(x) \neq 0$. We may assume without loss of generality that $e_m^*(x_n) \geq a > 0$ for some $a > 0$. According to (4), for every $y \in Y$, $|e_m^*(y)| \leq 2\|y\|$. Then, if $x_n^*(y) \geq 0$ for every $n \in \mathbb{N}$ we have

$$(5) \qquad \sum_n x_n^*(y) \leq a^{-1} \sum_n x_n^*(y) e_m^*(x_n) = a^{-1} e_m^*(y) \leq 2a^{-1}\|y\|.$$

Since $\{x_n\}_{n=1}^\infty$ is an unconditional basis of Y, it follows from (5) that for every $y \in Y$ we have

$$\sum_n |x_n^*(y)| < \infty$$

and the linear functional f defined by $f(y) = \sum_{n=0}^\infty x_n^*(y)$ is continuous. Let n_0 be such that

$$\Big|\sum_{n>n_0} x_n^*(x)\Big| < \frac{1}{2}$$

and take $Y_0 = [x_0, x_{n_0+1}, x_{n_0+2}, \ldots]$. Let $g(y) = x_0^*(y) + \sum_{n>n_0} x_n^*(y)$. Since $x_0^*(x) = 1/2$, we have

$$(6) \qquad |g(x)| < 1.$$

The operator $T : Y_0 \to Y_0$ defined by $Ty = g(y)x$ is linear and continuous. By (6), the operator $U : Y_0 \to Y_0$ given by the von Neumann formula $Uy = \sum_{n=0}^\infty T^n y$ is linear and continuous and it is the inverse of $I - T$. Thus, the sequences $\{x_n\}_{n>n_0}$ and $\{y_n\}_{n>n_0}$ are equivalent. ∎

LEMMA 2. *Let $\{e_m\}_{m=1}^\infty$ be a symmetric basis of an F-space X with biorthogonal system $\{e_m^*\}_{m=1}^\infty$. Let $\{x_n\}_{n=1}^\infty$ be a symmetric basic sequence in X, $x \in X$ a nonnull vector, and $\{\pi_n\}_{n=1}^\infty$ a sequence of elements of $\Pi_{\mathbb{N}}$ such that*

$$(7) \qquad \lim_{n\to\infty} \|\widehat{\pi}_n x_n - x\| = 0.$$

Then $\{x_n\}_{n=1}^\infty$ is equivalent to the symmetric basic sequence $\{y_n\}_{n=1}^\infty$, where $y_n = \sum_{m=1}^\infty e_m^*(x) e_{\tau(m,n)}$ for $n \in \mathbb{N}$ and $\tau(m,n)$ are different natural numbers.

(In other words, the lemma says that the sequence $\{x_n\}_{n=1}^\infty$ is equivalent to a block basis generated by the vector x.)

Proof. By passing to a subsequence if necessary, we can assume that $\lim_{n\to\infty} e_m^*(x_n) = a_m$ exists for every $m \in \mathbb{N}$. Let $\mathbb{M} = \{m \in \mathbb{N} : a_m \neq 0\}$. We shall show that there exists a one-to-one correspondence θ from $\mathbb{M}$ into $\mathbb{N}$ such that $e_{\theta(m)}^*(x) = a_m$.

Assume the opposite. Then there exists $a \neq 0$ such that if $\operatorname{Card} A$ denotes the cardinal number of a set A, then $\operatorname{Card} A > \operatorname{Card} B$, where $A = \{m : a_m = a\}$ and $B = \{m : e_m^*(x) = a\}$. Put $\varepsilon = \inf\{|e_m^*(x) - a| : m \notin B\} > 0$,

and let n_0 be great enough so that for $n > n_0$ and every $m \in A$,

$$|e_m^*(x_n) - a| < \varepsilon/2.$$

Then for $n > n_0$, $m \in A$ and $k \notin B$,

$$(8) \qquad |e_m^*(x_n) - e_k^*(x)| > \varepsilon/2.$$

Since $\operatorname{Card} A > \operatorname{Card} B$, for each n there exists s_n such that $s_n \in \pi_n(A)\setminus B$. Write $\mu_n = \pi_n^{-1}$. Then for $n > n_0$ according to (2), (4) and (8) we have

$$2\|\widehat{\pi}_n x_n - x\| \geq \|e_{s_n}^*(\widehat{\pi} x_n - x)e_{s_n}\| = \|(e_{\mu_n(s_n)}^*(x_n) - e_{s_n}^*(x))e_{s_n}\| \geq \varphi(\varepsilon/2) > 0,$$

which contradicts (7). Hence, there exists a correspondence $\theta : \mathbb{M} \to \mathbb{N}$ such that $e_{\theta(m)}^*(x) = a_m$ and θ is invertible. As the basis $\{e_m\}_{m=1}^\infty$ is symmetric, there exists $x_0 \in X$ such that $e_m^*(x_0) = a_m$ for $m \in \mathbb{N}$. Now, by Lemma 1, we can assume, without loss of generality, that

$$(9) \qquad \lim_{n\to\infty} e_m^*(x_n) = 0 \quad \text{for every } m \in \mathbb{N}.$$

We can find an increasing sequence $\{m_i\}_{i=1}^\infty$ of natural numbers such that

$$(10) \qquad \Big\|\sum_{m=m_i+1}^\infty e_m^*(x)e_m\Big\| < \frac{1}{2^i} \quad \text{for each } i \in \mathbb{N}.$$

Now, let us define recursively an increasing sequence (n_i) of natural numbers in the following way: Put $n_0 = 0$. Having chosen $n_1 < \ldots < n_{i-1}$, by (7) and (9), we can find n_i so that

$$(11) \qquad \varphi(|e_{\mu_{n_j}(m)}^*(x_{n_i})|) \leq \frac{1}{2^i m_i}$$

for $m = 1, \ldots, m_j$ and $j = 1, \ldots, i-1$, and

$$(12) \qquad \|\widehat{\pi}_{n_i} x_{n_i} - x\| < \frac{1}{2^i}.$$

Define $A_0 = \emptyset$, $A_i = \{\mu_{n_i}(m) : m \leq m_i\}$ and

$$B_i = \Big\{m \in \mathbb{N} : \mu_{n_i}(m) \in A_i \cap \bigcup_{j=1}^{i-1} A_j\Big\} \quad \text{for } i \in \mathbb{N}.$$

We shall show that

$$(13) \qquad \varphi(|e_m^*(\widehat{\pi}_{m_i} x_{n_i})|) \leq \frac{1}{2^i m_i} \quad \text{for each } m \in B_i.$$

Indeed, since $m \in B_i$, there exist $j < i$ and $s \leq m_j$ such that $\mu_{n_i}(m) = \mu_{n_j}(s)$. But

$$e_m^*(\widehat{\pi}_{n_i} x_{n_i}) = e_{\mu_{n_i}(m)}^*(x_{n_i}) = e_{\mu_{n_j}(s)}^*(x_{n_i}).$$

Therefore (11) implies (13).

Denote by $\mathbb{N}_0$ the set

$$\{k \in \mathbb{N} : \text{there exists } m \text{ such that } k = \mu_n(m),\ e_k^*(x_n) = 0 \text{ for all } n \in \mathbb{N}\}.$$

Without affecting the generality we may suppose that $\mathbb{N}_0$ is infinite. (If $\mathbb{N}_0$ is finite or empty we may consider the space $X\times X$ which has a symmetric basis equivalent to $\{e_n\}$, and the elements $(x_n, 0), (x, 0)$ and $\widehat{\pi}_n(u,v) = (\widehat{\pi}_n u, v)$ for $(u, v) \in X \times X$, instead of X, x_n, x, and $\widehat{\pi}_n$.)

Let $\{k_m\}_{m\in B_i}$ be disjoint subsets of $\mathbb{N}_0$. Consider

$$\tau_{m,i} = \begin{cases} k_m & \text{for } m \in B_i, \\ \mu_{n_i}(m) & \text{for } m \notin B_i, \end{cases}$$

and the vectors

$$u_i = \sum_{m=1}^{m_i} e_m^*(x) e_{\tau_{m,i}}, \qquad v_i = \sum_{m=m_i+1}^{\infty} e_m^*(x) e_{\tau_{m,i}},$$

and

$$w_i = \sum_{m=1}^{\infty} e_m^*(\widehat{\pi}_{n_i} x_{n_i}) e_{\tau_{m,i}}.$$

Since

$$x_{n_i} = \sum_{m=1}^{\infty} e^*_{\mu_{n_i}(m)}(x_{n_i}) e_{\mu_{n_i}},$$

we have

$$x_{n_i} - w_i = \sum_{m\in B_i} e^*_{\mu_{n_i}(m)}(x_{n_i})(e_{\mu_{n_i}(m)} - e_{k_m}).$$

As Card $B_{n_i} \le m_i$, it follows from (3) and (13) that

$$\|x_{n_i} - w_i\| < \frac{1}{2^{i-1}} \tag{14}$$

and using (3), (10) and (12) we get

$$\|w_i - (u_i + v_i)\| = \|\widehat{\pi}_{n_i} x_{n_i} - x\| < \frac{1}{2^i}, \quad \|v_i\| < \frac{1}{2^i} \quad \text{for each } i \in \mathbb{N}. \tag{15}$$

Therefore it follows from (14) and (15) that $\|x_{n_i} - u_i\| < 2^{-i+2}$ for $i \in \mathbb{N}$. Thus, the sequences $\{x_{n_i}\}_{i=1}^\infty$ and $\{u_i\}_{i=1}^\infty$ are equivalent. As $\tau_{m,i}$, for $m = 1, \ldots, m_i$ and $i \in \mathbb{N}$, are different natural numbers and $\{e_n\}_{n=1}^\infty$ is a symmetric basis, we find that $\{u_i\}_{i=1}^\infty$ is equivalent to $\{z_i\}_{i=1}^\infty$, where $z_i = \sum_{m=1}^{m_i} e_m^*(x) e_{\tau(m,i)}$ for $i \in \mathbb{N}$. But from (3) and (10) we deduce that $\|z_i - y_i\| < 2^{-i}$ for $i \in \mathbb{N}$. Therefore, the sequences $\{y_i\}_{i=1}^\infty$ and $\{z_i\}_{i=1}^\infty$ are equivalent, whence it follows that $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$ are equivalent. ■

Proof of Main Theorem. By Lemma 2 it is sufficient to find sequences $\{\pi_i\}_{i=1}^\infty \subset \Pi_\Gamma$, $\{\gamma_i\}_{i=1}^\infty \subset \Gamma$ and a vector $x \in X$ such that

$$\lim_{i\to\infty} \|\widehat{\pi}_i u_{\gamma_i} - x\| = 0, \quad \text{where } \gamma_i \ne \gamma_j \text{ for } i \ne j. \tag{16}$$

Let us introduce in X an equivalence relation: we say that $x_1, x_2 \in X$ are equivalent if there exists a bijection $\pi \in \Pi_A$ such that $\widehat{\pi} x_1 = x_2$. Denote by $[X]$ the resulting quotient space, and introduce a metric ϱ in $[X]$ by

$$\varrho([x],[y]) = \inf\{\|x - y\| : x \in [x],\ y \in [y]\}.$$

It is easily seen, by (3), that ϱ satisfies the triangle inequality and that $[X]$ is separable. Let ν be the natural quotient map from X onto $[X]$, and let $U = \{u_\gamma\}_{\gamma\in\Gamma}$. If $\nu(U)$ is a countable set, then there exists an uncountable subset $\Gamma_0 \subset \Gamma$ and $\gamma_0 \in \Gamma_0$ such that $\nu(u_\gamma) = \nu(u_{\gamma_0})$ for every $\gamma \in \Gamma_0$, whence there exists $\pi_\gamma \in \Pi_\Gamma$ such that $\widehat{\pi}_\gamma u_\gamma = u_{\gamma_0}$ for each $\gamma \in \Gamma_0$. This implies directly the assertion in this case by taking $x = u_{\gamma_0}$.

Assume now that $\nu(U)$ is uncountable. Since the quotient space $[X]$ is Lindelöf, it follows that there exists an $x \in X$ such that the intersection of an arbitrary neighborhood of $\nu(x)$ with $\nu(U)$ is uncountable. Therefore we can take, for each natural number i, maps π_i' and π_i in Π_A and a vector $u_{\gamma_i} \in U$ such that

$$\|\widehat{\pi}_i x - \widehat{\pi}_i' u_{\gamma_i}\| - \frac{1}{i} \le \varrho(\nu(x), \nu(u_{\gamma_i})) < \frac{1}{i}.$$

Hence

$$\|x - \widehat{\pi}_i \widehat{\pi}_i' u_{\gamma_i}\| \to 0 \quad \text{as } i \to \infty,$$

which concludes the proof. ■

COROLLARY 3. *In a nonseparable F-space X with a symmetric basis all symmetric bases are equivalent.*

In the case of Banach spaces the above result was obtained by Drewnowski in [D₁] with a different method using combinatorial arguments.

III. Applications.

In this section we shall present some applications of our main result and some examples.

COROLLARY 4. *Let X be an F-space with a symmetric basis $\{e_\alpha\}_{\alpha\in A}$ which contains a subspace isomorphic to $c_0(\Gamma)$ for uncountable Γ. Then X is isomorphic to $c_0(A)$.*

Proof. By the Main Theorem there exists a symmetric basic set $\{u_\gamma\}_{\gamma\in\Gamma}$ in X, where

$$u_\gamma = \sum_{i=1}^{\infty} a_i e_{\gamma(i)}, \quad \gamma(i) \in A, \quad \gamma(i) \ne \delta(j) \quad \text{if } i \ne j \text{ or } \gamma \ne \delta,$$

which is equivalent to the natural basis in $c_0(\Gamma)$. Let $a_1 \neq 0$; then, by the regularity condition (*) and (4), we obtain

$$\frac{1}{2}\sup_{\gamma}\|b_\gamma e_{\gamma(1)}\| \le \Big\|\sum_{\gamma} b_\gamma e_{\gamma(1)}\Big\| \le 2\Big\|a_1^{-1}\sum_{\gamma} b_\gamma u_\gamma\Big\|.$$

This implies that the basis $\{e_\alpha\}_{\alpha\in A}$ is equivalent to the natural basis of $c_0(A)$. ∎

Remark. The validity of the above corollary for Banach spaces X with an uncountable unconditional basis has been analyzed in [D$_2$].

Remark. As mentioned in the introduction, an analogous result to Corollary 4 is valid in the context of Banach spaces when the space $c_0(\Gamma)$ is replaced by $\ell^1(\Gamma)$ ([T$_1$]). In the general framework of F-spaces such a result is not true: we shall show below that there exist (nonconvex) Orlicz spaces $h_M(A)$, with a symmetric basis, containing an isomorphic copy of $\ell^1(\Gamma)$.

Recall that an *Orlicz function* M is a nondecreasing function from $[0,\infty)$ into $[0,\infty)$, left continuous for $t>0$, continuous at 0, with $M(0)=0$, $M(1) = 1$, and $M(t)>0$ for $t>0$. For each Orlicz function M we define the *Orlicz space* $l_M(A)$ consisting of all real-valued functions defined on A for which

$$\overline{M}\Big(\frac{x}{\lambda}\Big) = \sum_{\alpha\in A} M\Big(\frac{|x(\alpha)|}{\lambda}\Big) < \infty$$

for some $\lambda>0$. The space $l_M(A)$ is an F-space with F-norm defined by the formula

$$\|x\| = \inf\{\lambda>0 : \overline{M}(x/\lambda)\le\lambda\}.$$

In the case of being M convex, one can introduce a norm in $l_M(A)$ by

$$\|x\| = \inf\{\lambda>0 : \overline{M}(x/\lambda)\le 1\},$$

which is equivalent to the above F-norm $\|\cdot\|$.

Let e_α be the function defined by $e_\alpha(\beta)=\delta_{\alpha,\beta}$ for $\alpha,\beta\in A$. We shall denote by $h_M(A)$ the subspace of $l_M(A)$ generated by the family $\{e_\alpha\}_{\alpha\in A}$. It is easy to see that $h_M(A)$ consists of those vectors $x\in l_M(A)$ for which $\overline{M}(x/\lambda)<\infty$ for every $\lambda>0$.

We will say that an Orlicz function M satisfies the Δ_2^0-*condition* if there exists a constant $C>0$ and a $t_0>0$ such that $M(2t)\le CM(t)$ for $0\le t\le t_0$. If M satisfies the Δ_2^0-condition then $l_M(A)=h_M(A)$ and the unit vectors $\{e_\alpha\}_{\alpha\in A}$ are a symmetric basis in $l_M(A)$. If A is a countable set we write h_M and l_M as usual. We refer to [Ro] and [LT] for other basic properties of Orlicz spaces.

The study of the symmetric structure of convex Orlicz sequence spaces h_M has been mainly carried out by Lindenstrauss and Tzafriri (cf. [LT]), while the nonconvex case has been done by Kalton ([K]). In particular, the

Orlicz sequence spaces h_N which can be embedded isomorphically into an Orlicz space h_M are entirely characterized in terms of the associated set $C_{M,1}$ (cf. [LT], Thm. 4.a.8). Recall that the compact set $C_{M,1}$ is defined as $\overline{\mathrm{conv}}\, E_{M,1}$ in the space $C[0,1/2]$, where

$$E_{M,1} = \overline{\{M(st)/M(s) : 0<s<1\}}.$$

Recently, the question of whether this result can be extended to Orlicz spaces $h_M(A)$, for an uncountable set A, has been answered in the negative by Rodriguez-Salinas [R]. In his proof the compactness of $C_{M,1}$ plays an important role. Here, we will get the main result of [R] as a consequence of our representation theorem (also we extend the result of [R] to the nonconvex case):

PROPOSITION 5. *Let M and N be Orlicz functions (not necessarily convex) and let A and Γ be sets with* $\operatorname{Card}A\ge\operatorname{Card}\Gamma>\aleph_0$. *The following statements are equivalent*:

(i) *$h_N(\Gamma)$ is isomorphic to a subspace of $h_M(A)$.*

(ii) *There exists $x=\{x_i\}_{i=1}^\infty\in h_M$ such that the functions N and Q are equivalent at* 0, *where*

$$Q(t)=\sum_{i=1}^\infty M(x_i t).$$

These imply

(iii) *There exists a positive function $W\in L^1[0,1]$ such that the functions N and R are equivalent at* 0, *where*

$$R(t)=\int_0^1 \frac{M(st)}{M(s)}W(s)\,ds.$$

In addition, if M satisfies the Δ_2^0-condition and is p-convex at 0 *for some $p>0$ then* (iii)⇔(ii).

Proof. The implication (i)⇒(ii) follows from the Main Theorem. One can easily verify directly that (ii)⇒(i).

(ii)⇒(iii). First assume that $x_i=0$ for $i>n$. Then we put $W(s)=M(s)$. We have $R(t)=\int_0^1 M(st)\,ds$, and

$$\frac{1}{2}M\Big(\frac{t}{2}\Big)\le\int_{1/2}^1 M(st)\,ds\le R(t)\le M(t).$$

Assume now that $x=\{x_i\}_{i=1}^\infty$ contains infinitely many nonzero elements. Since $\overline{M}(x/\lambda)<\infty$ for every $\lambda>0$, we get $x_i\to 0$. Without loss of generality we may assume that $1=x_1\ge x_2\ge\ldots\ge 0$. We can find an increasing sequence of integers (m_k) so that $m_1=1$, $x_{m_1}>x_{m_2}>\ldots$ and $x_i=x_{m_k}$

if $m_k \leq i < m_{k+1}$. Now we take a decreasing sequence $\{y_j\}_{j=1}^{\infty}$ in $[0,1]$ containing $\{x_{m_k}\}_{k=1}^{\infty}$ as a subsequence, so that

$$y_1 = 1 \quad \text{and} \quad y_{j+1}/y_j \geq 1/2 .$$

Let

$$n_j = \begin{cases} m_{k+1} - m_k & \text{if } y_j = x_{m_k}, \\ 0 & \text{if } y_j \notin \{x_{m_k}\}_{k=1}^{\infty}. \end{cases}$$

Evidently for $t > 0$,

$$\sum_{i=1}^{\infty} M(x_i t) = \sum_{j=1}^{\infty} n_j M(y_j t) .$$

We define a positive function W on (0,1] as follows:

$$W(s) = n_j \Big(\int_{y_{j+1}}^{y_j} \frac{dv}{M(v)} \Big)^{-1} \quad \text{if } s \in (y_{j+1}, y_j] .$$

We shall show that W is the required function. Since

$$\sum_{j=1}^{\infty} n_j (y_j - y_{j+1}) \Big(\int_{y_{j+1}}^{y_j} \frac{dv}{M(v)} \Big)^{-1} \leq \sum_{j=1}^{\infty} n_j M(y_j) < \infty ,$$

we get $W \in L^1[0,1]$. Obviously

$$R(t) = \int_0^1 \frac{M(st)}{M(s)} W(s)\, ds = \sum_{j=1}^{\infty} n_j \Big(\int_{y_{j+1}}^{y_j} \frac{dv}{M(v)} \Big)^{-1} \int_{y_{j+1}}^{y_j} \frac{M(st)}{M(s)}\, ds .$$

This implies

$$\sum_{j=1}^{\infty} n_j M(y_j t/2) \leq \sum_{j=1}^{\infty} n_j M(y_{j+1} t) \leq R(t) \leq \sum_{j=1}^{\infty} n_j M(y_j t) .$$

Hence R is equivalent to $Q(t) = \sum_{i=1}^{\infty} M(x_i t)$, so to N at 0.

(iii)⇒(ii). Take

$$\lambda_k = \int_{2^{-(k+1)}}^{2^{-k}} \frac{W(s)}{M(s)}\, ds .$$

We may assume that $\lambda_0 = 1$. For $t < 1$ we get

$$\sum_{k=0}^{\infty} \lambda_k M(2^{-(k+1)} t) \leq R(t) \leq \sum_{k=0}^{\infty} \lambda_k M(2^{-k} t) .$$

Now, by the p-convexity of M, there exists $C > 0$ such that $M(\lambda t) \leq C\lambda^p M(t)$ for $\lambda, t \in [0,1]$. Thus

$$\begin{aligned} \sum_{k=0}^{\infty} \lambda_k M(2^{-k-1} t) &\leq \sum_{k=0}^{\infty} ([\lambda_k] + 1) M(2^{-k-1} t) \leq \sum_{k=0}^{\infty} (\lambda_k + 1) M(2^{-k} t) \\ &\leq \sum_{k=0}^{\infty} \lambda_k M(2^{-k} t) + \sum_{k=0}^{\infty} C 2^{-kp} M(t) \\ &\leq \Big(1 + \frac{C}{1 - 2^{-p}}\Big) \sum_{k=0}^{\infty} \lambda_k M(2^{-k} t) , \end{aligned}$$

where $[x]$ is the integer part of x. Hence we conclude that

$$Q(t) = \sum_{k=0}^{\infty} ([\lambda_k] + 1) M(2^{-k-1} t)$$

is equivalent to $R(t)$ at 0. ∎

Remark. It follows immediately from Proposition 5 that if $h_M(A)$ and $h_N(A)$ are isomorphic and A is uncountable then the functions M and N are equivalent at 0. In the countable case this is not true: there exist isomorphic Orlicz sequence spaces h_M and h_N defined by two Orlicz functions M and N which are not equivalent at 0 ([LT]).

Remark. Given an Orlicz function M, denote by $\Sigma_{M,1}$ the set of all Orlicz functions Q satisfying the statement (ii) in the above result. It is clear that $\Sigma_{M,1}$ is contained in $C_{M,1}$. In general, $\Sigma_{M,1}$ is not compact: If $M(t) = t^p/|\log t|$ at 0, $0 < p < \infty$, then reasoning as in ([LT], p. 158) we find that the family of functions $\{N_q\}_{0<q\leq 1}$ for $N_q(t) = t^p/|\log t|^q$ is contained in $\Sigma_{M,1}$. But it is not difficult to see that the function t^p is not equivalent to any function in $\Sigma_{M,1}$ (see [R]).

COROLLARY 6. *Let M be an Orlicz function with the Δ_2^0-condition and p-convex at 0 for some $p > 0$, and A be a set with* Card $A > \aleph_0$. *Then M is a submultiplicative function at 0 iff every subspace of $h_M(A)$ with an uncountable symmetric basis $\{y_i\}_{i\in\Gamma}$ with $\Gamma \subset A$ is isomorphic to $h_M(\Gamma)$.*

Proof. If $\{y_i\}_{i\in\Gamma}$ is an uncountable symmetric basic set in $h_M(A)$ then it follows from the Main Theorem that $\{y_i\}_{i\in\Gamma}$ is equivalent to the canonical basis of some Orlicz space $h_N(\Gamma)$, where $N \in \Sigma_{M,1}$. Hence $N > M$ at 0. Now, by the submultiplicativity of M at 0, we have $N(t) = \sum M(x_i t) \leq C(\sum M(x_i)) M(t)$. Therefore $h_N(\Gamma) = h_M(\Gamma)$.

Conversely, assume $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ with $x_n \searrow 0, y_n \searrow 0$, and $M(x_n y_n) > 4^n M(x_n) M(y_n)$ for each $n \in \mathbb{N}$. Consider the Orlicz function

N defined by

$$N(t)=\sum_{i=1}^{\infty}\frac{1}{2^i}\,\frac{M(x_it)}{M(x_i)}\quad\text{for } 0<t<1\,.$$

Then $N\in\Sigma_{M,1}$, so, by Proposition 5, $h_N(A)\subsetneq h_M(A)$. Moreover, since $N(y_n)\ge 2^nM(y_n)$, N is not equivalent to M at 0, which is a contradiction. ∎

The following proposition will show in particular that the results of [T$_1$] on Banach spaces containing copies of $\ell^1(\Gamma)$, or $c_0(\Gamma)$, do not carry over to $\ell^p(\Gamma)$, $p>1$.

PROPOSITION 7. *For every $p>0$ there exists an Orlicz function M_p, which does not satisfy the Δ_2^0-condition, such that the Orlicz space $h_{M_p}(A)$ contains an isomorphic copy of $\ell^p(A)$. Moreover, if $p>1$ then M_p is convex.*

In the proof we need the following two lemmas:

LEMMA 8. *There exist strictly increasing sequences $\{k_i\}_{i=0}^{\infty}$, $\{m_i\}_{i=0}^{\infty}$ of integers such that $m_i>k_i=\sum_{j=0}^{i-1}m_j$, for $i=1,2,\ldots$,*

$$\lim_{i\to\infty}(m_{i+1}-m_i)=\infty \tag{17}$$

and

$$\sum_{i=0}^{\infty}f(x+k_i)=1\quad\textit{for } x\ge 1\,, \tag{18}$$

where

$$f(x)=\sum_{i=0}^{\infty}\chi_{[m_i,m_i+1)}(x)\,. \tag{19}$$

P r o o f. We proceed by induction. Define $f_0:\mathbb{N}\to\{0,1\}$ by $f_0(1)=1$, $f_0(n)=0$ for $n>1$, and put $k_0=0$, $m_0=1$, $l_0=1$ and $F_0(n)=f_0(n)$. Assume that natural numbers $k_0<k_1<\ldots<k_j$, $m_0<m_1<\ldots<m_j$ and $l_0<l_1<\ldots<l_j$, and maps $f_i:\mathbb{N}\to\{0,1\}$, $i=1,\ldots,j$, have been chosen such that

$$f_i(n)=f_{i-1}(n)\quad\text{for } n<m_i;\qquad f_i(m_i)=1;\qquad f_i(n)=0\quad\text{for } n>m_i\,,$$

and the maps $F_i:\mathbb{N}\to\{0,1\}$ defined by

$$F_i(n)=\sum_{s=0}^{i}f_i(n+k_s)$$

satisfy $F_i(n)=1$ for $n\le l_i$, $l_i\ge i+1$, and $F_i(l_i+1)=0$ for $i=0,1,\ldots,j$.

Now take $m_{j+1}=m_j+k_j+l_j+1$, $k_{j+1}=m_j+k_j$, and $f_{j+1}(n)=f_j(n)$ for $n<m_{j+1}$, $f_{j+1}(m_{j+1})=1$, $f_{j+1}(n)=0$ for $n>m_{j+1}$. Let

$$F_{j+1}(n)=\sum_{s=0}^{j+1}f_{j+1}(n+k_s)\,.$$

Since $f_{j+1}(n+k_{j+1})=1$ only for $n=l_j+1$, it is clear that

$$F_{j+1}(l_j+1)=1\quad\text{and}\quad F_{j+1}(n)=\sum_{s=0}^{j}f_{j+1}(n+k_s)\quad\text{for } n\ne l_j+1\,.$$

For $n\le m_j$, $n\ne l_j+1$, we have $f_{j+1}(n+k_s)=f_j(n+k_s)$, therefore $F_{j+1}(n)=F_j(n)$. Let $n>m_j$, then $f_{j+1}(n)\ne 0$ only for $n=m_{j+1}$. Since for $n>m_j$ we have $n+k_s>m_j$, it follows that $n+k_s=m_{j+1}$ at most for one s, and thus, only one of the numbers $f_{j+1}(n+k_s)$ may be 1. So the induction step is made.

Finally, we define $f(x)=f_j([x])$ if $[x]\in[1,m_j]$. Obviously, f is of the form (19) and satisfies (18). ∎

LEMMA 9. *For each $p>0$ there exists an Orlicz function M_p, which does not satisfy the Δ_2^0-condition, such that*

$$\sum_{k=1}^{\infty}a_kM_p(b_kt)=t^p\quad\textit{for } 0\le t\le 1/2\,,$$

where $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$ are two suitable sequences of positive numbers with $a_k>1$. Moreover, if $p>1$ then the function M_p is convex.

P r o o f. Let f be the function of Lemma 8. Define a function g on $(0,1/2]$ by $g(t)=f(-\log_2 t)$. Then

$$g(t)=\sum_{i=0}^{\infty}\chi_{(c_i,d_i]}(t)\quad\text{and}\quad\sum_{i=0}^{\infty}g(b_it)=1\quad\text{for } t\in(0,1/2]\,,$$

where

$$c_i=\frac{1}{2^{m_i+1}}\,,\quad d_i=2c_i\,,\quad b_i=\frac{1}{2^{k_i}}\,.$$

Let $q>-1$ and consider the increasing function G_q defined by

$$G_q(x)=\int_0^x t^qg(t)\,dt\,.$$

Using the Beppo Levi Theorem, we get

$$\sum_{i=0}^{\infty}\int_0^x t^qg(b_it)\,dt=\int_0^x\sum_{i=0}^{\infty}t^qg(b_it)\,dt=\frac{x^{q+1}}{q+1}\,,$$

which implies that

$$(20)\qquad \sum_{i=0}^{\infty}\frac{1}{b_i^{q+1}}G_q(b_ix)=\frac{x^{q+1}}{q+1}.$$

Now, consider the convex function $H_q(x)=\int_0^x G_q(t)\,dt$. Using the above equality and Beppo Levi Theorem again, we get

$$(21)\qquad \sum_{i=0}^{\infty}\frac{1}{b_i^{q+2}}H_q(b_ix)=\sum_{i=0}^{\infty}\int_0^x\frac{G_q(b_it)}{b_i^{q+1}}\,dt=\frac{x^{q+2}}{(q+1)(q+2)}.$$

Finally, let us show that G_q and H_q do not satisfy the Δ_2^0-condition. Indeed,

$$\frac{G_q(2c_i)}{G_q(c_i)}=\frac{G_q(d_i)}{G_q(d_{i+1})}\ge\frac{\int_{c_i}^{d_i}t^q\,dt}{\int_0^{d_{i+1}}t^q\,dt}\ge 2^{(q+1)(m_{i+1}-m_i-1)},$$

so, by (17), we get the result for G_q. Now, as

$$H_q(3x)\ge\int_{2x}^{3x}G_q(t)\,dt\ge G_q(2x)x$$

and $H_q(x)\le G_q(x)x$, we have

$$\frac{H_q(3c_i)}{H_q(c_i)}\ge\frac{G_q(2c_i)}{G_q(c_i)},$$

which shows that H_q also fails the Δ_2^0-condition. ∎

Proof of Proposition 7. Given $q>-1$, we consider the function

$$G_q(x)=\int_0^x t^q g(t)\,dt$$

as defined in Lemma 9. Hence

$$\sum_{i=0}^{\infty}\frac{1}{b_i^{q+1}}G_q(b_ix)=\frac{x^{q+1}}{q+1}\quad\text{for } 0<x\le\frac12 \text{ and } b_i=\frac{1}{2^{k_i}}.$$

Denote by $\{b_i'\}_{i=0}^{\infty}$ the sequence

$$(\overbrace{b_0,\dots,b_0}^{[2^{k_0(q+1)}]},\ \overbrace{b_1,\dots,b_1}^{[2^{k_1(q+1)}]},\dots).$$

Then

$$\sum_{i=0}^{\infty}G_q(b_i'x)\le\sum_{i=0}^{\infty}\frac{1}{b_i^{q+1}}G_q(b_ix)\le 2\sum_{i=0}^{\infty}G_q(b_i'x).$$

Hence, by Proposition 5, it remains to prove that $\{b_i'\}_{i=0}^{\infty}\in h_{G_q}$.

First, let us prove

$$(22)\qquad \sum_{j=0}^{\infty}g(b_j2^su)\le(s+2)^2<\infty\quad\text{for } 2^{-(s+1)}\le u\le 1.$$

Indeed, for $-(s+1)\le\theta\le 0$,

$$\sum_{j=0}^{\infty}g(b_j2^s2^\theta)=\sum_{j=0}^{\infty}f(k_j-s-\theta)=\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\chi_{[m_i,m_i+1)}(k_j-s-\theta).$$

If $k_j-m_i\ge s+\theta\ge s-(s+1)=-1$, we have $k_j>m_i-1$, so, by the definitions of m_j and k_j (Lemma 8), we get $j>i$. Now from $k_j-(m_i+1)<s+\theta<s$, we deduce

$$s>k_j-(m_i+1)=\sum_{l=0,l\ne i}^{j-1}m_l-1>j-2.$$

Hence $i<j<s+2$, and we have

$$\sum_{j=0}^{\infty}g(b_j2^{s+\theta})=\sum_{j=0}^{s+1}\sum_{i<j}\chi_{[m_i,m_i+1)}(k_j-s-\theta)\le\sum_{j=0}^{s+1}(s+2)$$
$$=(s+2)^2\quad\text{for } 0>\theta>-(s+1).$$

Now let us show

$$\sum_{i=0}^{\infty}G_q(b_i'2^s)<\infty\quad\text{for every } s\in\mathbb{N}.$$

We have

$$\sum_{i=0}^{\infty}\frac{1}{b_i^{q+1}}G(b_i2^s)=\sum_{i=0}^{\infty}\int_0^1 g(b_i2^su)u^q2^{s(q+1)}\,du$$
$$=\Big(\int_0^{2^{-(s+1)}}+\int_{2^{-(s+1)}}^1\Big)\sum_{i=0}^{\infty}g(b_i2^su)u^q2^{s(q+1)}\,du$$
$$\le\int_0^{2^{-(s+1)}}u^q2^{s(q+1)}\,du+\int_{2^{-(s+1)}}^1(s+2)^2u^q2^{s(q+1)}\,du<\infty$$

since $q>-1$, (22) holds and $\sum_{i=0}^{\infty}g(b_it)=1$ for $t\in(0,1/2]$.

Finally, for the convex function H_q we take the sequence $\{b_i^*\}_{i=0}^{\infty}$ defined by

$$(\overbrace{b_0,\dots,b_0}^{[2^{-k_0(q+2)}]},\ \overbrace{b_1,\dots,b_1}^{[2^{-k_1(q+2)}]},\dots).$$

Then for $x > 0$,

$$\sum_{i=0}^{\infty} H_q(b_i^* x) \le \sum_{i=0}^{\infty} \frac{1}{b_i^{q+2}} H_q(b_i x) \le 2 \sum_{i=0}^{\infty} H_q(b_i^* x),$$

and

$$\sum_{i=0}^{\infty} \frac{1}{b_i^{q+2}} H_q(b_i x) = \sum_{i=0}^{\infty} \int_0^x \frac{1}{b_i^{q+1}} G_q(b_i t)\, dt \le x \sum_{i=0}^{\infty} \frac{1}{b_i^{q+1}} G_q(b_i x) < \infty.$$

Hence, by (21) and Proposition 5, we conclude that, for $p = q+2 > 1$, $\ell^p(A)$ is isomorphic to a subspace of $h_{H_q}(A)$. ■

Remark. Notice that Proposition 7 in the special case of $p = 1$ shows that, unlike for Banach spaces, there exist nonconvex F-spaces X with an uncountable symmetric basis $\{e_\alpha\}_{\alpha\in A}$ containing an isomorphic copy of $\ell^1(\Gamma)$ for $\operatorname{Card}\Gamma > \aleph_0$ and $X \ne \ell^1(A)$.

PROBLEM. A natural question arises of whether there exists an F-space Y, with a symmetric basis, different from $c_0(\Gamma)$, with the same property as $c_0(\Gamma)$ in Corollary 4.

Let $0 < p < \infty$ and $w = \{w_i\}_{i=1}^\infty$ be a nonincreasing sequence of positive scalars such that $\lim_i w_i = 0$ and

$$\sum_{i=1}^{\infty} w_i = \infty.$$

We denote by $d(w,p,A)$ the *Lorentz space* of all real functions $x = x(\alpha)$ defined on the set A for which

$$\|x\| = \sup\Big\{\sum_{j=1}^{\infty} w_j |x(\alpha_j)|^p\Big\}^{1/p'} < \infty \tag{23}$$

where the supremum is taken over all sequences $\{\alpha_j\}_{j=1}^\infty$ of different elements of A, and $p' = \max(1,p)$. From (23) we deduce that there exists a sequence $\{\alpha_i\}_{i=1}^\infty$ such that $|x(\alpha_1)| \ge |x(\alpha_2)| \ge \ldots$, $\lim_{i\to\infty} x(\alpha_i) = 0$, $x(\alpha) = 0$ if $\alpha \ne \alpha_i$, $i = 1,2,\ldots$, and

$$\|x\| = \Big\{\sum_{i=1}^{\infty} w_i |x(\alpha_i)|^p\Big\}^{1/p'}.$$

As in the sequence case (cf. [LT]), the space $d(w,p,A)$ is a Banach space (in the case $0 < p < 1$, a p-Banach space) and the canonical basis $\{e_\alpha\}_{\alpha\in A}$ is a symmetric basis in $d(w,p,A)$. If A is countable we just write $d(w,p)$.

Unlike in the countable case, it turns out that the Lorentz spaces $d(w,p,A)$ cannot contain isomorphic copies of $\ell^p(\Gamma)$-spaces for uncountable sets Γ:

PROPOSITION 10. *Let $0 < p < \infty$. The Lorentz space $d(w,p,A)$ contains an isomorphic copy of $\ell^p(\Gamma)$ iff $\operatorname{Card}\Gamma \le \aleph_0$.*

Proof. The sufficiency part is well known ([ACL], and [P] for the nonconvex case).

Conversely, assume that $\operatorname{Card}\Gamma > \aleph_0$. Let $\{u_\gamma\}_{\gamma\in\Gamma}$ be a symmetric basic set in $d(w,p,A)$ equivalent to the natural basis of $\ell^p(\Gamma)$. It follows from the Main Theorem that there exists a sequence $\{a_i\}_{i=1}^\infty$ of reals and a sequence $\{\gamma(i)\}_{i=1}^\infty$ of different elements of A such that $\{u_\gamma\}_{\gamma\in\Gamma}$ and $\{v_\gamma\}_{\gamma\in\Gamma}$ are equivalent, where

$$v_\gamma = \sum_{i=1}^{\infty} a_i^{1/p} e_{\gamma(i)}.$$

In particular, the block basic sequence $\{v_k\}_{k=1}^\infty$ generated by the vector

$$x = \sum_{i=1}^{\infty} a_i^{1/p} e_i$$

in $d(w,p)$ is equivalent to the natural basis of ℓ^p. But this is not possible because

$$\lim_{n\to\infty} \frac{\|\sum_{k=1}^n v_k\|}{n^{1/p'}} = 0. \tag{24}$$

Indeed, we can assume w.l.o.g. that $\{a_i\}_{i=1}^\infty$ is a nonincreasing positive sequence. It is easy to see that

$$\Big\|\sum_{k=1}^{n} v_k\Big\|^{p'} = \sum_{k=1}^{n}\sum_{i=1}^{\infty} w_{k+(i-1)n} a_i.$$

Now, given $\varepsilon > 0$ take a natural number N_ε such that

$$\sum_{i\ge N_\varepsilon} w_i a_i < \frac{\varepsilon}{3}.$$

Then, for each $1 \le k \le n$,

$$\sum_{i=1}^{\infty} w_{k+(i-1)n} a_i \le \Big(w_k a_1 + \sum_{i=2}^{N_\varepsilon} w_{(i-1)n} a_i + \sum_{i>N_\varepsilon} w_{i-1} a_{i-1}\Big).$$

Since $w_k \to 0$, there exists a natural number $n_0(\varepsilon)$ such that for $n > n_0(\varepsilon)$ we have $\sum_{i=2}^{N_\varepsilon} w_n a_i < \varepsilon/2$. Hence for $n > n_0(\varepsilon)$,

$$\Big\|\sum_{k=1}^{n} v_k\Big\|^{p'} \le \sum_{k=1}^{n} (w_k a_1 + \varepsilon) = \Big(\sum_{k=1}^{n} w_k\Big) a_1 + n\varepsilon.$$

But this implies (24) since

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} w_k = 0. \blacksquare$$

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DEPT. MATHEMATICAL ANALYSIS
FACULTY OF MATHEMATICS
COMPLUTENSE UNIVERSITY
28040 MADRID, SPAIN

DEPT. MATHEMATICS & INFORMATICS
SOFIA UNIVERSITY
5, JAMES BOURCHIER BLVD.
1126 SOFIA, BULGARIA

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Montel and reflexive preduals of spaces of holomorphic functions on Fréchet spaces

by

CHRISTOPHER BOYD (Dublin)

Abstract. For U open in a locally convex space E it is shown in [31] that there is a complete locally convex space $G(U)$ such that $G(U)'_i = (\mathcal{H}(U), \tau_\delta)$. Here, we assume U is balanced open in a Fréchet space and give necessary and sufficient conditions for $G(U)$ to be Montel and reflexive. These results give an insight into the relationship between the τ_0 and τ_ω topologies on $\mathcal{H}(U)$.

1. Introduction. Let U be an open subset of a locally convex space E. We denote by $\mathcal{H}(U)$ the space of holomorphic functions from U to $\mathbb{C}$. We shall say that a seminorm p on $\mathcal{H}(U)$ is τ_δ-*continuous* if for each countable increasing open cover $\{U_n\}_n$ of U there is a positive integer n_0 and $C > 0$ such that $p(f) \le C\|f\|_{U_{n_0}}$ for every $f \in \mathcal{H}(U)$. In [31], $G(U)$ is defined to be the space of linear forms on $\mathcal{H}(U)$ which are τ_0-continuous when restricted to the locally bounded sets. We give $G(U)$ the topology of uniform convergence on locally bounded subsets of $\mathcal{H}(U)$. Mujica and Nachbin prove that $G(U)'_i = (\mathcal{H}(U), \tau_\delta)$ and then proceed to show that the topological properties of $G(U)$ are useful in characterizing the topological properties of $\mathcal{H}(U)$. This result is a topological generalization of a result of Mazet [27] who had previously shown that $G(U)' = \mathcal{H}(U)$. In [14], the author further investigated the space $G(U)$ and obtained necessary and sufficient conditions for the inductive dual of $G(U)$ to be equal to its strong dual and thus for $(\mathcal{H}(U), \tau_\delta)$ to be equal to $G(U)'_b$. One of the conditions for this to happen is that $G(U)$ be distinguished. We investigate necessary and sufficient conditions for $G(U)$ to be Montel and reflexive. Among the conditions for $G(U)$ to be Montel is that the τ_0 and τ_ω topologies coincide on $\mathcal{H}(U)$ while among the conditions for reflexivity is that the τ_0 and τ_ω topologies are compatible on $\mathcal{H}(U)$. This implies that for U balanced open in Tsirelson's space we have $(\mathcal{H}(U), \tau_0)' = (\mathcal{H}(U), \tau_\omega)'$ while $\tau_0 \ne \tau_\omega$. In the final section we give further examples of Fréchet spaces with $\tau_0 \ne \tau_\omega$ but with both of these topologies

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