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Some integral and maximal operators related to starlike sets

by

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Abstract. We prove two-weight norm estimates for fractional integrals and fractional maximal functions associated with starlike sets in Euclidean space. This is seen to include general positive homogeneous fractional integrals and fractional integrals on product spaces. We consider both weak type and strong type results, and we show that the conditions imposed on the weight functions are fairly sharp.

0. Introduction. This paper is concerned with studying weighted norm inequalities for certain generalizations of the Riesz fractional integral operators and associated maximal operators. One such operator is the following: on $\mathbb{R}^n$, $n > 1$, define

$$I_{\alpha,\beta}f(x) = f * k_{\alpha,\beta}(x)\,,$$

where

$$k_{\alpha,\beta}(x) = \frac{1}{|x|^{n-1-\alpha}|x_n|^{1-\beta}}\,, \tag{0.1}$$

for $x = (x_1, \ldots, x_n)$. Here, $-\beta < \alpha < n-1$ and $0 < \beta < 1$. We may think of these operators as interpolating between an n-dimensional Riesz fractional integral operator and a 1-dimensional Riesz fractional integral operator in the last coordinate.

We shall derive results for our operators from corresponding results for more standard operators. For example, we derive weak and strong type estimates for $I_{\alpha,\beta}$ from corresponding results for the ordinary fractional integral $I_{\alpha+\beta}$. The necessary requirement for this derivation is that we have precise control over the operator norms of the standard operators in terms of the constants appearing in the conditions on the weights.

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The operators $I_{\alpha,\beta}$ are a special case of the more general operators

$$I_{\Omega,\mu}f(x) = f * k_{\Omega,\mu}(x),$$

where

$$k_{\Omega,\mu}(x) = \frac{\Omega(x)}{|x|^{n-\mu}},$$

for $0 < \mu < n$, and for Ω a nonnegative function which is homogeneous of degree 0. With some modifications, the technique we develop for $I_{\alpha,\beta}$ and its associated operators may be applied to $I_{\Omega,\mu}$ and its associated operators. Product space fractional integral operators are included in this more general case.

The kernel $k_{\Omega,\mu}$ has associated with it a starlike set which plays a significant role in our analysis of $I_{\Omega,\mu}$: let $\varrho(\theta) = \Omega(\theta)^{1/(n-\mu)}$, and let $S = S_\varrho = \{r\theta : \theta \in \mathbf{S}^{n-1},\ 0 \le r < \varrho(\theta)\} = \{x : k_{\Omega,\mu}(x) > 1\}$. Then S is starlike with respect to the origin. Further, $|S| < \infty$ if and only if $\varrho \in L^n(\mathbf{S}^{n-1})$, or equivalently iff $\Omega \in L^{n/(n-\mu)}(\mathbf{S}^{n-1})$. By homogeneity, we see that if $|S| < \infty$, then $k_{\Omega,\mu}$ satisfies the distributional estimate $|\{x : k_{\Omega,\mu}(x) > s\}| \le Cs^{-n/(n-\mu)}$, and therefore by [St], p. 121, Comment 1.4, we may apply the proof of the unweighted Sobolev fractional integral theorem to see that $I_{\Omega,\mu}$ satisfies the same unweighted L^p-L^q mapping properties as the Riesz fractional integral of order μ: that is, if $1 < p, q < \infty$ and $1/q = 1/p - \mu/n$, then

$$\|I_{\Omega,\mu}f\|_q \le C\|f\|_p, \tag{0.2}$$

and at $p = 1$,

$$|\{x : |I_{\Omega,\mu}f(x)| > s\}| \le \left(\frac{C\|f\|_1}{s}\right)^{n/(n-\mu)}. \tag{0.3}$$

For $I_{\alpha,\beta}$ we have $\mu = \alpha + \beta$, $\Omega(x) = (|x|/|x_n|)^{1-\beta}$, and if we let $\gamma = (n-1-\alpha)/(1-\beta)$, then $\varrho(\theta) = |\theta_n|^{-1/(\gamma+1)}$, so that $S_\varrho = \{r\theta : \theta \in \mathbf{S}^{n-1},\ 0 \le r < |\theta_n|^{-1/(\gamma+1)}\}$ depends only on γ. In this case $|S| < \infty$ if and only if $\gamma + 1 > n$, which in turn is equivalent to $\beta > \alpha/(n-1)$.

The starlike set $S = S_\varrho$, for $\varrho(\theta) = \Omega(\theta)^{1/(n-\mu)}$, also appears in a useful representation for $I_{\Omega,\mu}$, and this in turn will lead us to several related operators. When E is a set of positive measure in $\mathbb{R}^n$, then whenever $x \in \mathbb{R}^n$, and $t > 0$, let $x + E$ denote the set $\{x + y : y \in E\}$ and let tE denote the set $\{ty : y \in E\}$. We use this definition rather than the standard convention in weighted norm theory (in which tE denotes the set E dilated by a factor of t around the center of E) for reasons which will become clear immediately. Let

$$A_{E,t}f(x) = t^{-n}\int_{tE} f(x-y)\,dy = \int_E f(x-ty)\,dy.$$

Then we claim that

$$I_{\Omega,\mu}f(x) = (n-\mu)\int_0^\infty t^\mu A_{S,t}f(x)\,\frac{dt}{t}. \tag{0.4}$$

To see this, use polar coordinates and a linear change of variables in the t integral, together with several changes in the order of integration, to get

$$\begin{aligned}\int_0^\infty t^\mu A_{S,t}f(x)\,\frac{dt}{t} &= \int_{\mathbf{S}^{n-1}}\int_0^{\varrho(\theta)}\int_0^\infty t^\mu f(x-tr\theta)\,\frac{dt}{t}\,r^{n-1}\,dr\,d\theta\\ &= \int_{\mathbf{S}^{n-1}}\int_0^\infty t^\mu f(x-t\theta)\int_0^{\varrho(\theta)} r^{n-\mu-1}\,dr\,\frac{dt}{t}\,d\theta\\ &= \frac{1}{n-\mu}\int_{\mathbf{S}^{n-1}}\int_0^\infty f(x-t\theta)t^{\mu-1}\Omega(\theta)\,dt\,d\theta\\ &= \frac{1}{n-\mu}I_{\Omega,\mu}f(x).\end{aligned}$$

In the case of the ordinary fractional integral $I_\mu f(x) = f * |x|^{n-\mu}$, we see this formula holds with $\varrho \equiv 1$ and $S =$ the unit ball in $\mathbb{R}^n$.

We now consider several maximal operators associated with $I_{\Omega,\mu}f$. For E a set of positive measure, define the *centered fractional maximal operator* of order μ for E to be

$$M^{\sim}_{E,\mu}f(x) = \sup_{t>0}|tE|^{\mu/n-1}\Big|\int_{x-tE} f(y)\,dy\Big| = |E|^{\mu/n-1}\sup_{t>0} t^\mu|A_{E,t}f(x)|.$$

For us, it will be more useful to deal with an unnormalized version of this operator. For E a set of possibly infinite measure, define

$$M_{E,\mu}f(x) = \sup_{t>0} t^{\mu-n}\Big|\int_{x-tE} f(y)\,dy\Big| = \sup_{t>0} t^\mu|A_{E,t}f(x)|.$$

When $|E| < \infty$, this differs from $M^{\sim}_{E,\mu}f(x)$ by a factor of $|E|^{\mu/n-1}$. Finally, whenever F is a nonempty set, we define the *uncentered* (by F) *fractional maximal operator* for the set E to be

$$M_{E,F,\mu}f(x) = \sup_{\substack{t>0,\ z\in\mathbb{R}^n\\ x\in z-tF}} t^{\mu-n}\Big|\int_{z-tE} f(y)\,dy\Big|.$$

The operators that we are interested in are $M_{S,\mu}$ and $M_{S,Q_1,\mu}$, where S is a starlike set and Q_1 is the cube of edgelength 1 centered at the origin. Unweighted estimates for $M_{S,0}$ for S starlike with finite measure have been considered by Calixto P. Calderón in [Ca], M. Christ in [Ch], and M. Christ and J. L. Rubio de Francia in [ChR]. Weighted estimates for $M_{S,0}$ in the context of A_p weights have been obtained in [W1] and [W2]. The normalized version of $M_{E,F,\mu}$ for the case $\mu = 0$ is a special case of a maximal operator considered by Córdoba in [Cor]. When both E, F are either the unit ball or the unit cube, $M_{E,F,\mu}$ and $M_{E,\mu}$ are comparable and play a role in the study of the ordinary fractional integral.

The operators $M_{S,\mu}$ and $M_{S,Q_1,\mu}$ are not generally comparable, and as we shall see, exhibit distinctly different behavior. The connections between $M_{S,\mu}$ and $I_{\Omega,\mu}$ are fairly clear: Let $f \geq 0$, and suppose S and Ω are such that $S = S_\varrho$ for $\varrho(\theta) = \Omega(\theta)^{1/(n-\mu)}$. Then $M_{S,\mu} f(x)$ is the supremum of the integrand in (0.4). On the other hand, if $r < t$, then $A_{S,r} f(x) \leq (t/r)^n A_{S,t} f(x)$ since S is starlike, and therefore

$$r^\mu A_{S,r} f(x) \leq C \int_r^{2r} t^\mu A_{S,t} f(x) \frac{dt}{t} \leq C I_{\Omega,\mu} f(x),$$

for all $r > 0$, giving $M_{S,\mu} f(x) \leq C I_{\Omega,\mu} f(x)$.

We would like to thank Prof. Carlos Kenig for his encouragement and for pointing out to us that the operators $I_{\alpha,\beta}$ might be of general interest.

1. The theorems in the simplest case. Since all the operators under consideration are positive operators, let us henceforth always assume that $f \geq 0$. We use the fairly standard notation in which $\|f\|_{p,w}$ denotes the L^p-norm of f on $\mathbb{R}^n$ with respect to the measure $w(x)\,dx$. We will use $\|f\|_{L^{(p,\infty)}(w)}$ to denote the Lorentz space $L^{(p,\infty)}$-norm of f with respect to the measure $w(x)\,dx$. We recall that $L^{(p,\infty)}(w)$ is a Banach space for $p > 1$, and that $\|f\|_{L^{(p,\infty)}(w)}$ is comparable to $\sup_{t>0}\{tw\{x : |f(x)| > t\}^{1/p}\}$. We begin by considering our model operators, namely the fractional integral $I_{\alpha,\beta}$ and the related maximal operators $M_{S,\mu}$ and $M_{S,Q_1,\mu}$, where $S = S_\varrho$ for $\varrho(\theta) = |\theta_n|^{-1/(\gamma+1)}$. With these operators we associate a group of linear transformations $\{\delta_a\}_{a>0}$, defined by $\delta_a x = (ax_1, \ldots, ax_{n-1}, a^{-\gamma} x_n)$. We also associate with these operators rectangles $R_a = \delta_a Q_1$. Finally, whenever E is a set of positive measure, define $\mathfrak{B}(E) = \{z + tE : z \in \mathbb{R}^n,\ t > 0\}$ to be the collection of all translates and dilates of E.

We begin with the result having the most straightforward proof.

THEOREM 1 (Weighted norm estimates for $M_{S_\varrho,\mu}$ for $\varrho(\theta) = |\theta_n|^{-1/(\gamma+1)}$). *For $\gamma > 0$, let $S = S_\varrho$ for $\varrho(\theta) = |\theta_n|^{-1/(\gamma+1)}$. Suppose that $1 \leq p \leq q < \infty$, $0 \leq \mu < n$, and let v, w be weights.*

(A) *Suppose that $M_{S,\mu}$ satisfies the weak type estimate*

$$w\{x : M_{S,\mu} f(x) > s\} \leq \left(\frac{B\|f\|_{p,v}}{s}\right)^q. \tag{1.1}$$

Then there is a constant C such that whenever $a \geq 1$ and $R \in \mathfrak{B}(R_a)$, v, w satisfy

$$\begin{aligned} &|R|^{\mu/n-1}\Big(\int_R w\Big)^{1/q}\Big(\int_R v^{-p'/p}\Big)^{1/p'} \leq CBa^{(\gamma-n+1)(1-\mu/n)}, \quad p > 1,\\ &|R|^{\mu/n-1}\Big(\int_R w\Big)^{1/q} \operatorname*{ess\,sup}_{x\in R} v(x)^{-1} \leq CBa^{(\gamma-n+1)(1-\mu/n)}, \quad p = 1. \end{aligned} \tag{1.2}$$

Here, C is a constant independent of v, w, and a.

(B) *Conversely, suppose that there is a monotone function $C(a)$ such that if $a \geq 1$ and $R \in \mathfrak{B}(R_a)$, then*

$$\begin{aligned} &|R|^{\mu/n-1}\Big(\int_R w\Big)^{1/q}\Big(\int_R v^{-p'/p}\Big)^{1/p'} \leq C(a)a^{(\gamma-n+1)(1-\mu/n)}, \quad p > 1,\\ &|R|^{\mu/n-1}\Big(\int_R w\Big)^{1/q} \operatorname*{ess\,sup}_{x\in R} v(x)^{-1} \leq C(a)a^{(\gamma-n+1)(1-\mu/n)}, \quad p = 1. \end{aligned} \tag{1.3}$$

If $C(a)$ also satisfies

$$\begin{aligned} &\int_1^\infty C(a)\frac{da}{a} = b, \quad q > 1,\\ &\int_1^\infty C(a)(1 + \log^+ C(a))\frac{da}{a} = b, \quad p = q = 1, \end{aligned} \tag{1.4}$$

then $M_{S,\mu}$ satisfies the weak type estimate (1.1), with $B \leq cb$ for some constant c independent of v, w and f.

(C) *Suppose that $1 < p \leq q < \infty$, and v, w satisfy the following strengthening of (1.3): for some $r > 1$,*

$$|R|^{\mu/n-1/p}\Big(\int_R w\Big)^{1/q}\Big(\frac{1}{|R|}\int_R v^{-rp'/p}\Big)^{1/(rp')} \leq C(a)a^{(\gamma-n+1)(1-\mu/n)}$$

for all $R \in \mathfrak{B}(R_a)$ and $a \geq 1$, where $C(a)$ satisfies (1.4). Then $M_{S,\mu}$ satisfies the strong type inequality

$$\|M_{S,\mu} f\|_{q,w} \leq B\|f\|_{p,v}. \tag{1.5}$$

Remark. In special cases, it is possible to give sufficient conditions for weak or strong type inequalities for $M_{S,\mu}$ which only involve integrating w and $v^{-p'/p}$ over cubes. While results of this kind are not very sharp, and consequently not of much interest, we would like to point out one situation where there is a relatively simple statement of this type. It involves the case when $w = v$, $\mu = 0$, and $p = q$. We will show at the end of §3 that if $\gamma+1 > n$

and $1 < p < \infty$, then

$$\|M_{S,0}f\|_{p,w} \le c\|f\|_{p,w}$$

provided either

$$w \in A_{p(1-n/(\gamma+1))} \quad \text{and} \quad p\left(1-\frac{n}{\gamma+1}\right) > 1$$

or

$$w \in A_{p'(1-n/(\gamma+1))} \quad \text{and} \quad p'\left(1-\frac{n}{\gamma+1}\right) > 1\,,$$

where for $1 < p < \infty$, A_p denotes the class of weights w which satisfy

$$\left(\frac{1}{|Q|}\int_Q w\right)^{1/p}\left(\frac{1}{|Q|}\int_Q w^{-p'/p}\right)^{1/p'} \le C$$

for all cubes Q (see [M]).

In Theorem 1, the sufficient conditions (1.3), (1.4) for weak type norm inequalities are slightly stronger than the necessary conditions (1.2). We may see from the following that this separation between the necessary and sufficient conditions is very slight.

COROLLARY 1. *For $0 < \overline{\gamma} < \gamma$, let $S' = S_{\varrho'}$ for $\varrho'(\theta) = |\theta_n|^{-1/(\overline{\gamma}+1)}$. The necessary condition corresponding to (1.2) for $M_{S',\mu}$ (that is, replacing γ by $\overline{\gamma}$ in (1.2), such replacement occurring also in the definition of the dilations δ_a and the rectangles R_a) is a sufficient condition for $M_{S,\mu}$ to satisfy the weak type estimate (1.1).*

For any $\gamma > 0$, the collections $\mathfrak{B}(R_a)$ for $a \ge 1$ range over all rectangles of the form $I \times J$, with I a cube in $\mathbb{R}^{n-1}$, and J an interval in $\mathbb{R}$ whose length is at most the side length of I, so (1.2) and (1.3) are actually conditions over arbitrary rectangles of this kind, with constants that depend only upon the eccentricity and γ.

Let $M_{a,\mu} = M_{R_a,\mu}$. Theorem 1 will be proved by showing that

$$M_{S,\mu}f(x) \ge cM_{a,\mu}f(x)\,, \quad a \ge 1\,, \tag{1.6}$$

and that

$$M_{S,\mu}f(x) \le C\sum_{k=0}^{\infty} M_{2^k,\mu}f(x)\,. \tag{1.7}$$

These two comparisons are proven by showing that the rectangles R_a for $a \ge 1$ are contained in a large dilation of S and that $\{R_a\}_{a\ge1}$ covers a small dilation of S. (This is similar to an argument used by Calixto P. Calderón in [Ca].) The operators $M_{a,\mu}$ in turn are compared to $M_{Q_1,\mu}$ by a scaling argument using the transformations δ_a. This together with some well-known

results for $M_{Q_1,\mu}$ give the first two parts of the theorem. A strong type result of C. Perez [P] for $M_{Q_1,\mu}$ likewise gives the strong type result (C) of the theorem.

The scaling argument comparing $M_{a,\mu}$ to $M_{Q_1,\mu}$ is the following: when Λ is an invertible linear transformation on $\mathbb{R}^n$, let us also identify Λ with the operator

$$\Lambda f(x) = |\det\Lambda| f(\Lambda x)\,, \tag{1.8}$$

and observe that

$$\Lambda^{-1}A_{E,t}\Lambda f = |\det\Lambda|^{-1}A_{\Lambda E,t}f\,, \tag{1.9}$$

$$\Lambda^{-1}M_{E,\mu}\Lambda f = |\det\Lambda|^{-1}M_{\Lambda E,\mu}f\,. \tag{1.10}$$

Therefore, if we identify δ_a with an operator as in (1.8), then we see that

$$M_{a,\mu}f(x) = a^{n-1-\gamma}\delta_a^{-1}M_{Q_1,\mu}\delta_a f(x)\,. \tag{1.11}$$

A similar comparison and rescaling argument will be used to derive weak and strong type estimates for $I_{\alpha,\beta}$.

Rescaling the operators also leads naturally to rescaling the weights. A simple change of variables shows that (1.2) is equivalent to

$$\tag{1.2$'$}\begin{aligned} |Q|^{\mu/n-1}\Big(\int_Q \delta_a w\Big)^{1/q}\Big(\int_Q (\delta_a v)^{-p'/p}\Big)^{1/p'} &\le CBa^{\gamma-n+1}\,, \quad p>1\,,\\ |Q|^{\mu/n-1}\Big(\int_Q \delta_a w\Big)^{1/q} \operatorname*{ess\,sup}_Q(\delta_a v)^{-1} &\le CBa^{\gamma-n+1}\,, \quad p=1\,,\end{aligned}$$

for cubes Q. We note for future reference that $|R_a| = a^{n-1-\gamma}$, so that the right-hand side of (1.2$'$) may be written as $CB/|R_a|$. Similarly, (1.3) is equivalent to

$$\tag{1.3$'$}\begin{aligned} |Q|^{\mu/n-1}\Big(\int_Q \delta_a w\Big)^{1/q}\Big(\int_Q (\delta_a v)^{-p'/p}\Big)^{1/p'} &\le C(a)a^{\gamma-n+1}\,, \quad p>1\,,\\ |Q|^{\mu/n-1}\Big(\int_Q \delta_a w\Big)^{1/q} \operatorname*{ess\,sup}_Q(\delta_a v)^{-1} &\le C(a)a^{\gamma-n+1}\,, \quad p=1\,.\end{aligned}$$

Writing conditions in terms of these operations on weights has the effect of simplifying the proofs of our theorems, as well as simplifying some of the terms occurring in our next two theorems, which we may now state.

THEOREM 2 (Weak type estimates for $I_{\alpha,\beta}$). *Let $1 < p \le q < \infty$, $-\beta < \alpha < n-1$, $0 < \beta < 1$. Let $\gamma = (n-1-\alpha)/(1-\beta)$, and let δ_a be as given prior to Theorem 1. Identify δ_a with an operator as in (1.8).*

If $I_{\alpha,\beta}$ *satisfies the weak type inequality*

$$w\{x : |I_{\alpha,\beta} f(x)| > s\} \le \left(\frac{B\|f\|_{p,v}}{s}\right)^q, \tag{1.12}$$

then w *and* v *satisfy*

$$\Big(\int_Q \delta_a w\Big)^{1/q} \Big(\int_{Q^c} \frac{(\delta_a v(x))^{-p'/p}}{|x - x_Q|^{(n-\alpha-\beta)p'}}\,dx\Big)^{1/p'} \le CBa^{\gamma-n+1}, \tag{1.13}$$

for every cube Q *and every* $a \ge 1$, *where* x_Q *is the center of* Q, *and* C *is a constant independent of* Q *and* a. *Conversely, if* $1 < p < q < \infty$ *and there is a monotone function* $C(a)$ *such that* v, w *satisfy*

$$\Big(\int_Q \delta_a w\Big)^{1/q} \Big(\int_{Q^c} \frac{(\delta_a v(x))^{-p'/p}}{|x - x_Q|^{(n-\alpha-\beta)p'}}\,dx\Big)^{1/p'} \le C(a)a^{\gamma-n+1},$$

for every cube Q *and every* $a \ge 1$, *and if* $C(a)$ *satisfies the integral condition* (1.4), *then* (1.12) *holds, with the constant* B *less than a fixed multiple of the integral in* (1.4).

We could have phrased our conditions on the weights in terms of integrals over rectangles $R \in \mathfrak{B}(R_a)$, using the change of variables $\widetilde{x} = \delta_a x$, except that this would lead to some awkward-looking terms. The separation between the necessary and sufficient conditions is again quite slight, in a sense similar to Corollary 1: see Corollary 2 following Theorem 3.

Our next result is also more naturally stated in terms of averages over cubes.

THEOREM 3 (Strong type estimates for $I_{\alpha,\beta}$). *Let* α, β, γ *be as in Theorem* 2, *and suppose* $1 < p \le q < \infty$.

(A) *If* $I_{\alpha,\beta}$ *satisfies the strong type inequality*

$$\|I_{\alpha,\beta} f\|_{q,w} \le B\|f\|_{p,v} \tag{1.14}$$

then for every cube Q *and every* $a \ge 1$, w, v *satisfy the condition*

$$|Q|^{1-(\alpha+\beta)/n}\Big(\int s_Q(x)^q \delta_a w(x)\,dx\Big)^{1/q} \times \Big(\int s_Q(x)^{p'} (\delta_a v(x))^{-p'/p}\,dx\Big)^{1/p'} \le cBa^{\gamma-n+1} \tag{1.15}$$

where $s_Q(x) = (|Q|^{1/n} + |x - x_Q|)^{\alpha+\beta-n}$, *and* x_Q *is the center of* Q. *Hence* w, v *also satisfy the weaker condition*

$$|Q|^{(\alpha+\beta)/n-1}\Big(\int_Q \delta_a w\Big)^{1/q}\Big(\int_Q (\delta_a v)^{-p'/p}\Big)^{1/p'} \le cBa^{\gamma-n+1}.$$

In each case, c *is a constant independent of* a. *Conversely, if there exists an* $r > 1$ *such that for every cube* Q *and for all* $a \ge 1$, *the weights* v, w *satisfy*

$$|Q|^{(\alpha+\beta)/n+1/q-1/p}\Big(\frac{1}{|Q|}\int_Q (\delta_a w)^r\Big)^{1/(qr)} \times \Big(\frac{1}{|Q|}\int_Q (\delta_a v)^{-rp'/p}\Big)^{1/(rp')} \le C(a)a^{\gamma-n+1}, \tag{1.16}$$

where $C(a)$ *satisfies* (1.4), *then* $I_{\alpha,\beta}$ *satisfies the strong type inequality* (1.14), *with* B *bounded by a fixed multiple of the integral in* (1.4).

(B) *If* $p < q$, *and*

$$|Q|^{1-(\alpha+\beta)/n}\Big(\int s_Q(x)^q \delta_a w(x)\,dx\Big)^{1/q} \times \Big(\int s_Q(x)^{p'} (\delta_a v(x))^{-p'/p}\,dx\Big)^{1/p'} \le C(a)a^{\gamma-n+1},$$

where $C(a)$ *is a monotone function satisfying* (1.4), *then* (1.14) *holds.*

COROLLARY 2. *Let* α, β, γ *be as in Theorems* 2 *and* 3, *and suppose that* $\overline{\alpha}, \overline{\beta}$ *satisfy the same requirements, respectively, as* α, β. *Suppose also that* $\overline{\alpha} + \overline{\beta} = \alpha + \beta$, *and that* $\overline{\gamma} = (n - 1 - \overline{\alpha})/(1 - \overline{\beta})$ *satisfies* $0 < \overline{\gamma} < \gamma$ (*or equivalently,* $\overline{\beta} < \beta$). *Then*

(i) *The condition corresponding to* (1.13) *for the operator* $I_{\overline{\alpha},\overline{\beta}}$ *is sufficient for* $I_{\alpha,\beta}$ *to satisfy the weak type estimate* (1.12). *That is, if* (1.13) *holds with* α, β, γ *replaced by* $\overline{\alpha}, \overline{\beta}, \overline{\gamma}$ (*in particular,* γ *is replaced with* $\overline{\gamma}$ *in the definition of* δ_a *as well*), *then* $I_{\alpha,\beta}$ *satisfies the weak type estimate* (1.12). *In other words, for weak type estimates the necessary condition for* $I_{\overline{\alpha},\overline{\beta}}$ *is a sufficient condition for* $I_{\alpha,\beta}$.

(ii) *When* $p < q$, *the condition corresponding to* (1.15) *for the operator* $I_{\overline{\alpha},\overline{\beta}}$ *is sufficient for* $I_{\alpha,\beta}$ *to satisfy the strong type inequality* (1.14). *In addition, if* $p \le q$ *and the condition corresponding to* (1.16) *for* $I_{\overline{\alpha},\overline{\beta}}$ *holds with* $C(a)$ *bounded, then* $I_{\alpha,\beta}$ *satisfies the strong type inequality* (1.14).

The necessity statements in Theorems 2 and 3 can be strengthened. For Theorem 3, we will show in §2 that the following condition is necessary for (1.14):

$$\Big(\frac{|R|}{|R_a|}\Big)^{1-(\alpha+\beta)/n}\Big(\int \widetilde{s}_R(x)^q w(x)\,dx\Big)^{1/q} \times \Big(\int \widetilde{s}_R(x)^{p'} v(x)^{-p'/p}\,dx\Big)^{1/p'} \le C \tag{1.15'}$$

for all $R \in \mathfrak{B}(R_a)$, for all $a \ge 1$, where for a rectangle R of the form $I \times J$

with I a cube in $\mathbb{R}^{n-1}$ and J an interval in $\mathbb{R}$, with center x_R, we define

$$\widetilde{s}_R(y) = (|I|^{1/(n-1)} + |y - x_R|)^{-(n-1-\alpha)}(|J| + |y_n - x_{R,n}|)^{-(1-\beta)}.$$

To see that (1.15′) implies (1.15), first note that the change of variables $y = \delta_a x$ transforms (1.15) into

$$|Q|^{1-(\alpha+\beta)/n}\Big(\int s_Q(\delta_a^{-1}y)^q w(y)\,dy\Big)^{1/q} \times \Big(\int s_Q(\delta_a^{-1}y)^{p'} v(y)^{-p'/p}\,dy\Big)^{1/p'} \le C.$$

If $R = \delta_a Q$, then $R \in \mathfrak{B}(R_a)$ and $|Q| = |R|/|R_a|$. Therefore (1.15) will follow from (1.15′) if we show that

$$s_Q(\delta_a^{-1}y) \le c\widetilde{s}_R(y) \tag{1.17}$$

if $a \ge 1$ and $R = \delta_a Q$, with c independent of a and Q. This inequality will be proved in §2.

Similarly, the necessity statement in Theorem 2 can be strengthened. In fact, we will show in §2 that the following condition is necessary for (1.12):

$$\Big(\int_R w\Big)^{1/q}\Big(\int \widetilde{s}_R(y)^{p'} v(y)^{-p'/p}\,dy\Big)^{1/p'} \le C \tag{1.13′}$$

for all $R \in \mathfrak{B}(R_a)$, for all $a \ge 1$. To see that (1.13′) implies (1.13), it suffices to observe that the change of variables $y = \delta_a x$ transforms (1.13) into

$$\Big(\int_R w(y)\,dy\Big)^{1/q}\Big(\int_{R^c} \frac{v(y)^{-p'/p}}{|\delta_a^{-1}y - x_Q|^{(n-\alpha-\beta)p'}}\,dy\Big)^{1/p'} \le CB, \qquad R = \delta_a Q,$$

and then to note that it follows from (1.17) that

$$|\delta_a^{-1}y - x_Q|^{-(n-\alpha-\beta)p'} \le c\widetilde{s}_R(y)$$

if $a \ge 1$ and $y \notin R = \delta_a Q$, with c independent of a and Q. We note again in passing that *any* rectangle R of the form $I \times J$ with I a cube in $\mathbb{R}^{n-1}$, J an interval in $\mathbb{R}$, and $|I|^{1/(n-1)} \ge |J|$ belongs to $\mathfrak{B}(R_a)$ for some $a \ge 1$. Consequently, (1.13′) and (1.15′) hold for all such rectangles, and the ratio $|R|/|R_a|$ in (1.15′) may be written as $|R|e_R^{(n-1-\gamma)/(\gamma+1)}$, where $e_R = |J||I|^{-1/(n-1)}$ is the eccentricity of R.

Our result for the uncentered maximal operator has a slightly different proof from the previous theorems. For E, F measurable sets, let $\mathfrak{B}(E,F) = \{(z + tE, z + tF) : z \in \mathbb{R}^n,\ t > 0\}$ be the collection of all pairs of joint translates and dilates of E and F.

THEOREM 4 (Weak type estimates for $M_{S,Q_1,\mu}$). *Let S, γ, μ, p, q be as in Theorem 1, and let v and w be weights.*

(A) *Suppose that $M_{S,Q_1,\mu}$ satisfies the weak type estimate*

$$w\{x : M_{S,Q_1,\mu}f(x) > s\} \le \Big(\frac{B\|f\|_{p,v}}{s}\Big)^q. \tag{1.18}$$

Let $R_a^* = \delta_a^* Q_1$ for $\delta_a^* x = (ax_1, \ldots, ax_{n-1}, x_n)$. Then for every $a \ge 1$ and every pair of rectangles $(R, R^*) \in \mathfrak{B}(R_a, R_a^*)$, v, w satisfy

$$\begin{aligned} &|R|^{\mu/n-1}\Big(\int_{R^*} w\Big)^{1/q}\Big(\int_R v^{-p'/p}\Big)^{1/p'} \le Ca^{(\gamma-n+1)(1-\mu/n)}, \quad p > 1,\\ &|R|^{\mu/n-1}\Big(\int_{R^*} w\Big)^{1/q} \operatorname*{ess\,sup}_R v^{-1} \le Ca^{(\gamma-n+1)(1-\mu/n)}, \quad p = 1. \end{aligned} \tag{1.19}$$

We may take C to be less than a fixed multiple of B independent of v, w, and a.

(B) *Conversely, suppose that there is a monotone function $C(a)$ such that if $a \ge 1$, and $(R, R^*) \in \mathfrak{B}(R_a, R_a^*)$, then*

$$\begin{aligned} &|R|^{\mu/n-1}\Big(\int_{R^*} w\Big)^{1/q}\Big(\int_R v^{-p'/p}\Big)^{1/p'} \le C(a)a^{(\gamma-n+1)(1-\mu/n)}, \quad p > 1,\\ &|R|^{\mu/n-1}\Big(\int_{R^*} w\Big)^{1/q} \operatorname*{ess\,sup}_R v^{-1} \le C(a)a^{(\gamma-n+1)(1-\mu/n)}, \quad p = 1. \end{aligned} \tag{1.20}$$

If $C(a)$ also satisfies (1.4), then $M_{S,Q_1,\mu}$ satisfies the weak type estimate (1.18), with B bounded by some fixed multiple of the integral in (1.4). In particular, if $S' = S_{\gamma'}$ where $0 < \gamma' < \gamma$, then the necessary condition (1.19) for $M_{S',Q_1,\mu}$ is a sufficient condition for $M_{S,Q_1,\mu}$ to satisfy the weak type estimate (1.18).

We could also phrase (1.19) in an equivalent formulation involving cubes as we did in (1.2′). This may easily be seen to be the same as (1.2′), but with $\delta_a w$ replaced with $\delta_a^* w$, and $\delta_a v$ left as is.

2. Results for general S or Ω. We now consider results for our operators which involve starlike sets S more general than those considered in §1. We will first consider the operators $M_{S,\mu}$ and $M_{S,Q_1,\mu}$ for S an arbitrary starlike set, and $I_{\Omega,\mu}$ for appropriate Ω. We only derive necessary conditions when S is open, and in this case our results show strong parallels with the results of §1, giving sufficient conditions which differ from the necessary conditions by a convergence factor.

All the conditions on S are stated in terms of the boundary function ϱ of S. When we say a set is *starlike*, we will always mean that the set is starlike with respect to the origin. We see that except for at most a set of measure 0 on the boundary, starlike sets S are always of the form $S = S_\varrho = \{r\theta : \theta \in \mathbf{S}^{n-1},\ 0 \le r < \varrho(\theta)\}$, and that S_ϱ is open iff ϱ is lower semicontinuous and $\varrho \ge c > 0$.

DEFINITION. Given a starlike set S centered at the origin, we will say that a collection of open rectangles $\{R_j\}$ with arbitrary orientation is a *starlike cover* of S if: (i) For each j, R_j contains the origin on its major axis, (ii) $S \subseteq \bigcup_j R_j$, and (iii) $\sum |R_j| \le C|S|$ for some $C > 0$. If in addition we can choose $\{R_j\}$ so that for some $c < 1$, $cR_j \subseteq S$ for all j, then we will say that $\{R_j\}$ is a *proper starlike cover* of S. Finally, if each of the rectangles R_j is centered at the origin, then we say that $\{R_j\}$ is a *centered* starlike cover.

We will prove the following lemma in §3.

LEMMA 1. (A) *Every starlike set S has a starlike cover $\{R_j\}$, and we may in fact choose $\{R_j\}$ to be a centered starlike cover.*

(B) *If S is starlike and open, then it admits a proper starlike cover $\{R_j\}$, and we may in fact choose $\{R_j\}$ so that $cR_j \subseteq S$ for each j, with $c < 1$ depending only on n, and $|S| \approx \sum |R_j|$, with constants of comparability depending only on n. Finally, if S is also symmetric with respect to the origin, then we may in addition choose the proper starlike cover so that it is also a centered starlike cover.*

With every rectangle R containing the origin we may associate a linear transformation δ_R of positive determinant such that $R = \delta_R Q_R$, where Q_R is a cube of unit edgelength containing the origin. This only specifies δ_R, Q_R up to a rotation, and any choice will work, but for the sake of definiteness let us specify that δ_R is diagonal with respect to the orthonormal basis of unit vectors parallel to the edges of R, and that the diagonal entry corresponding to an edge of R is the length of R in that direction. Then $\det \delta_R = |R|$. In what follows, Theorems 5–7 are analogous to Theorems 1–3, respectively.

THEOREM 5 (Weighted norm estimates for $M_{S,\mu}$). *Let $1 \le p \le q < \infty$, $0 \le \mu < n$, and let v, w be weights.*

(A) *Let S be an open starlike set, and suppose that $M_{S,\mu}$ satisfies the weak type estimate*

$$w\{x : M_{S,\mu}f(x) > s\} \le \left(\frac{B\|f\|_{p,v}}{s}\right)^q. \tag{2.1}$$

Then for every rectangle R contained in S and containing 0 on an axis, and for all cubes Q, the weights v, w satisfy

$$\begin{aligned} &|Q|^{\mu/n-1}\Big(\int_Q \delta_R w\Big)^{1/q}\Big(\int_Q (\delta_R v)^{-p'/p}\Big)^{1/p'} \le \frac{CB}{|R|}, \quad p > 1,\\ &|Q|^{\mu/n-1}\Big(\int_Q \delta_R w\Big)^{1/q} \operatorname*{ess\,sup}_Q (\delta_R v)^{-1} \le \frac{CB}{|R|}, \quad p = 1. \end{aligned} \tag{2.2}$$

Here, C is a constant independent of v, w, and R (cf. (1.2′)).

(B) *Conversely, suppose that S is a starlike set, $\{R_j\}$ is a starlike cover, and let C_j be constants such that for all cubes Q,*

$$\begin{aligned} &|Q|^{\mu/n-1}\Big(\int_Q \delta_{R_j} w\Big)^{1/q}\Big(\int_Q (\delta_{R_j} v)^{-p'/p}\Big)^{1/p'} \le \frac{C_j}{|R_j|}, \quad p > 1,\\ &|Q|^{\mu/n-1}\Big(\int_Q \delta_{R_j} w\Big)^{1/q} \operatorname*{ess\,sup}_Q (\delta_{R_j} v)^{-1} \le \frac{C_j}{|R_j|}, \quad p = 1. \end{aligned} \tag{2.3}$$

If also

$$\begin{aligned} &\sum_j C_j = b < \infty, \quad q > 1,\\ &\sum_j C_j\Big(1 + \log^+ \frac{1}{C_j}\Big) = b < \infty, \quad p = q = 1, \end{aligned} \tag{2.4}$$

then $M_{S,\mu}$ satisfies the weak type estimate (2.1), with $B \le cb$ for some constant c independent of v, w.

(C) *If $1 < p \le q < \infty$ and v, w satisfy the following strengthening of (2.3): for some $r > 1$,*

$$|Q|^{\mu/n-1/p}\Big(\int_Q \delta_{R_j} w\Big)^{1/q}\Big(\frac{1}{|Q|}\int_Q (\delta_{R_j} v)^{-rp'/p}\Big)^{1/(rp')} \le \frac{C_j}{|R_j|}$$

for all j and all cubes Q, with $\{C_j\}$ satisfying (2.4), then $M_{S,\mu}$ satisfies the strong type estimate

$$\|M_{S,\mu}f\|_{q,w} \le B\|f\|_{p,v}$$

with $B \le cb$ and c independent of w, v, and f.

We also have a corollary analogous to Corollary 1, which tells us that the necessary conditions for weak type inequalities in Theorem 5 are close to the sufficient conditions, provided the starlike set S does not branch out too fast. A similar kind of statement can be made for strong type inequalities, and we will discuss this briefly at the end of §3.

COROLLARY 3. *Let $q > 1$, let S be open, and for $t > 1$, let $S' = S_{\varrho'}$ for $\varrho'(\theta) = \varrho^t(\theta)$. If S admits a proper starlike cover $\{R_j\}$ such that*

$\#\{R_j : \operatorname{diam} R_j \le 2^k\}$ is finite for each k, $k = 0, 1, \ldots$, and if

$$\sum_{k=0}^{\infty} 2^{k(1-t)(n-\mu)} \#\{R_j : 2^{k-1} \le \operatorname{diam} R_j \le 2^k\} < \infty, \tag{2.5}$$

then the necessary condition corresponding to (2.2) for the operator $M_{S',\mu}$ is a sufficient condition for the operator $M_{S,\mu}$ to satisfy (2.3) and (2.4), and hence to satisfy the weak type estimate (2.1).

For the case $\mu = 0$, $p = q$, and $w = v = 1$, observe that (2.3) holds for $C_j = |R_j|$, so that the first line of condition (2.4) is just a requirement that S have finite measure. The second part of (2.4) shows that we require slightly more when $p = q = 1$. The extra requirement for this case can be stated in a way which contains the results for $M_{S,0}$ obtained by C. P. Calderón in [Ca], and which we present as another corollary to Theorem 5. We first introduce a small amount of notation: For $r \ge 1$, define a function $\psi_r(t) = t(1 + \log^+(1/(r^n t)))$, and define an outer measure m_r on $\mathbf{S}^{n-1}$ by

$$m_r(E) = \inf_{\bigcup B_j \supset E} \sum \psi_r(|B_j|),$$

where the sets B_j are, up to rotation, "rectangles" of the form $B = \{\theta \in \mathbf{S}^{n-1} : |\theta_k - \omega_k| < r_k,\ 1 \le k \le n\}$ for some $\omega \in \mathbf{S}^{n-1}$. Then m_r is essentially the entropy set function of R. Fefferman [F], scaled by r. Observe that $m_r(E)$ is increasing in E and decreasing in r, and that if $|E|$ is held constant, $m_r(E)$ is minimized when E is a rectangle, and consequently $m_r(E) \ge |E|(1 + \log^+(1/(r^n|E|)))$.

COROLLARY 4 (Unweighted weak type (1,1) estimates for $M_{S,0}$). *Let S be a starlike set with corresponding boundary function ϱ. Then $M_{S,0}$ satisfies the weak type estimate*

$$|\{x : M_{S,0} f(x) > s\}| \le \frac{C\|f\|_1}{s} \tag{2.6}$$

provided that S admits a starlike cover $\{R_j\}$ satisfying

$$\sum_j |R_j| \left(1 + \log^+ \frac{1}{|R_j|}\right) < \infty. \tag{2.7}$$

Also, (2.7) is equivalent to the distributional estimate

$$\int_1^{\infty} r^{n-1} m_r\{\varrho > r\}\, dr < \infty. \tag{2.8}$$

In particular, (2.6) holds if ϱ^n has finite entropy (i.e., if the integral in (2.8) remains finite with m_r replaced by m_1), and an example when this occurs is the following: up to composition with a rotation, ϱ is of the form $\varrho(\theta) = h(|(\theta_1, \ldots, \theta_k)|)$ for some k, $1 \le k \le n$, where h is nonincreasing and is such that

$$\int_{\mathbf{S}^{n-1}} \varrho^n (1 + \log^+ \log^+ \varrho) < \infty, \tag{2.9}$$

or ϱ is a finite sum of terms of this form.

We would like to thank C. Gutiérrez for pointing out that a similar result was presented by C. P. Calderón in his lectures, namely that (2.6) holds when ϱ^n has finite entropy. The example in (2.9) is also largely due to C. P. Calderón, and appears (with a weaker conclusion) in [Ca]. Also, it is shown in [ChR] using delicate orthogonality arguments that (2.6) holds provided that $\varrho \in L^n(\log L)(\mathbf{S}^{n-1})$. The corollary gives a similarly weak requirement for ϱ, which (2.9) shows is not implied by the result of [ChR]. Similarly, the result of [ChR] is not contained in the corollary since it is possible to construct functions $\varrho \in L^n(\log L)(\mathbf{S}^{n-1})$ for which (2.8) does not hold. This may be done when $n = 2$, for example, by constructing ϱ so that $\mathbf{O}_j = \{\varrho > 2^j\}$ is a union of intervals in $\mathbf{S}^1$ of total measure $2^{-2j} j^{-s}$, $s > 2$, for each j, such that $\mathbf{O}_j$ consists of N_j intervals of equal length, spaced evenly within each subinterval of $\mathbf{O}_{j-1}$. Then $\varrho \in L^2(\log L)(\mathbf{S}^1)$, since

$$\int_1^{\infty} r(\log r)|\{\varrho > r\}|\, dr \approx \sum_{j=1}^{\infty} j 2^{2j} |\{\varrho > 2^j\}| < \infty.$$

On the other hand, by choosing N_j sufficiently large for each j, we can arrange matters so that $m_{2^j}\{\mathbf{O}_j\}$ decreases as slowly as we like, so that (2.8) may fail.

THEOREM 6 (Weak type estimates for $I_{\Omega,\mu}$). *Let $1 < p \le q < \infty$, $0 < \mu < n$, and let $\Omega \ge 0$ be homogeneous of degree 0. Let $S = S_\varrho$ for $\varrho(\theta) = \Omega(\theta)^{1/(n-\mu)}$.*

If $\Omega > 0$ is even and lower semicontinuous, so that S is open and symmetric with respect to the origin, and if $I_{\Omega,\mu}$ satisfies the weak type inequality

$$w\{x : |I_{\Omega,\mu} f(x)| > s\} \le \left(\frac{B\|f\|_{p,v}}{s}\right)^q, \tag{2.10}$$

then w and v satisfy

$$\left(\int_Q \delta_R w\right)^{1/q} \left(\int_{Q^c} \frac{(\delta_R v(x))^{-p'/p}}{|x - x_Q|^{(n-\mu)p'}}\, dx\right)^{1/p'} \le \frac{CB}{|R|}, \tag{2.11}$$

for every cube Q and for every rectangle R contained in S which is centered at the origin, where x_Q is the center of Q, and C is a constant independent of Q and R.

Conversely, if $1 < p < q < \infty$, then for any Ω (not necessarily even or lower semicontinuous), if there is a starlike cover $\{R_j\}$ of S by rectangles centered at the origin, and constants C_j such that v, w satisfy

$$\Big(\int_Q \delta_{R_j} w\Big)^{1/q}\Big(\int_{Q^c}\frac{(\delta_{R_j}v(x))^{-p'/p}}{|x-x_Q|^{(n-\mu)p'}}\,dx\Big)^{1/p'} \le \frac{C_j}{|R_j|},$$

for every cube Q and every j, and if $\{C_j\}$ satisfies the summability condition (2.4), then the weak type estimate (2.10) holds, with the constant B less than a fixed multiple of the sum in (2.4).

THEOREM 7 (Strong type estimates for $I_{\Omega,\mu}$). *Let $0 < \mu < n$, $\Omega \ge 0$ be homogeneous of degree 0, $S = S_\varrho$ for $\varrho(\theta) = \Omega(\theta)^{1/(n-\mu)}$, and suppose $1 < p \le q < \infty$.*

(A) *If $I_{\Omega,\mu}$ satisfies the strong type inequality*

$$\|I_{\Omega,\mu} f\|_{q,w} \le B\|f\|_{p,v} \tag{2.12}$$

and if S is open and symmetric with respect to the origin, then for every rectangle R centered at the origin that is contained in S, and for every cube Q, we have

$$|Q|^{1-\mu/n}\Big(\int s_Q(x)^q \delta_R w(x)\,dx\Big)^{1/q} \times \Big(\int s_Q(x)^{p'}(\delta_R v(x))^{-p'/p}\,dx\Big)^{1/p'} \le \frac{cB}{|R|} \tag{2.13}$$

where $s_Q(x) = (|Q|^{1/n} + |x - x_Q|)^{\mu-n}$, and x_Q is the center of Q. Hence w, v also satisfy the weaker condition

$$|Q|^{\mu/n-1}\Big(\int_Q \delta_R w\Big)^{1/q}\Big(\int_Q (\delta_R v)^{-p'/p}\Big)^{1/p'} \le \frac{cB}{|R|}.$$

In each case, c is a constant independent of R. Conversely, for any Ω, if for some $r > 1$, there is a starlike cover $\{R_j\}$ of S by rectangles which are centered at the origin, and constants C_j satisfying (2.4) such that for every cube Q and all j, the weights v, w satisfy

$$|Q|^{\mu/n+1/q-1/p}\Big(\frac{1}{|Q|}\int_Q (\delta_{R_j} w)^r\Big)^{1/(qr)} \times \Big(\frac{1}{|Q|}\int_Q (\delta_{R_j} v)^{-rp'/p}\Big)^{1/(rp')} \le \frac{C_j}{|R_j|}, \tag{2.14}$$

then $I_{\Omega,\mu}$ satisfies the strong type inequality (2.12).

(B) *For any Ω, if $p < q$, and there is a starlike cover $\{R_j\}$ of S by rectangles which are centered at the origin, and constants C_j satisfying (2.4)*

for which

$$|Q|^{1-\mu/n}\Big(\int s_Q(x)^q \delta_{R_j} w(x)\,dx\Big)^{1/q} \times \Big(\int s_Q(x)^{p'}(\delta_{R_j} v(x))^{-p'/p}\,dx\Big)^{1/p'} \le \frac{C_j}{|R_j|},$$

for every cube Q, then (2.12) holds.

We also have an analogue of Corollary 2.

COROLLARY 5. *For p, q and μ as in Theorems 6 and 7, let $\Omega > 0$ be lower semicontinuous. Suppose that the associated starlike set S admits a proper starlike cover for which (2.5) holds for some $t > 1$, and let $\overline{\Omega} = \Omega^t$. Then*

(i) *The condition corresponding to (2.11) for $I_{\overline{\Omega},\mu}$ is sufficient for $I_{\Omega,\mu}$ to satisfy the weak type estimate (2.10). That is, the necessary condition for $I_{\overline{\Omega},\mu}$ to satisfy a weak type estimate is a sufficient condition for $I_{\Omega,\mu}$ to satisfy the same weak type estimate.*

(ii) *When $p < q$, the condition corresponding to (2.13) for $I_{\overline{\Omega},\mu}$ is sufficient for $I_{\Omega,\mu}$ to satisfy the strong type estimate (2.12). That is, when $p < q$, the necessary condition for $I_{\overline{\Omega},\mu}$ to satisfy a strong type estimate is sufficient for $I_{\Omega,\mu}$ to satisfy the same strong type estimate. In addition, if the condition corresponding to (2.14) for $I_{\overline{\Omega},\mu}$ holds with $\{C_j\}$ uniformly bounded, then (2.12) holds for $I_{\Omega,\mu}$.*

3. Basic techniques. Proofs. The proofs of Theorems 1–4 rely upon the same comparison argument. Let $k(x) = k_{\alpha,\beta}(x) = |x|^{\alpha-n+1}|x_n|^{\beta-1}$, and write $x \in \mathbb{R}^n$ as $x = (x', x_n)$, where $x' = (x_1, \ldots, x_{n-1})$. Then for $|x'| \ge |x_n|$, $k(x) \approx |x'|^{\alpha-n+1}|x_n|^{\beta-1}$, and if $|x'| \le |x_n|$, then $k(x) \approx |x_n|^{\alpha+\beta-n}$. Hence, if $c_0 > 0$ and we define

$$S^{\sim} = \{x : |x_n| \le \min\{1, c_0|x'|^{-\gamma}\}\},$$

then $S^{\sim}$ is comparable to $S = \{x : k(x) > 1\}$, in the sense that there are constants $\tau, \tau' > 0$ such that $\tau S^{\sim} \subseteq S \subseteq \tau' S^{\sim}$. Consequently, the operators obtained by replacing S with $S^{\sim}$ in our formulas are also comparable. Fix c_0 so that $Q_1 \subseteq S^{\sim}$, which also ensures that if $a \ge 1$, then the rectangle $R_a = \delta_a Q_1$ is contained in $S^{\sim}$ since $\delta_a(S^{\sim}) \subseteq S^{\sim}$ for $a \ge 1$. We can also see that $S^{\sim} \subseteq \bigcup_{k=0}^{\infty}(\kappa R_{2^k})$ for κ a geometric constant depending only on n and c_0.

This gives us the comparisons:

$$\chi_{R_a}(x/\tau) \le \chi_S(x), \tag{3.1}$$

$$(3.2)\qquad \chi_S(x) \le \sum_{k=0}^{\infty} \chi_{R_{2^k}}(x/(\tau'\kappa)).$$

Then (3.1) and (3.2) imply (1.6) and (1.7), which tell us how to compare operators involving integrals over S to operators involving the rectangles $\{R_a\}_{a\ge 1}$.

We now use the scaling argument (1.9) and (1.10) together with the following observation: for any sublinear operator T, any invertible linear transformation Λ, and any $p,q>0$, the weak type inequality

$$(3.3)\qquad w\{x : |(\Lambda^{-1}T\Lambda)f(x)| > s\} \le \left(\frac{B\|f\|_{p,v}}{s}\right)^q$$

for all $f \in L^p(v)$ and $s>0$ is equivalent to the inequality

$$(3.4)\qquad (\Lambda w)\{x : |Tf(x)| > s\} \le \left(\frac{B\|f\|_{p,\Lambda v}}{s}\right)^q$$

for all $f \in L^p(\Lambda v)$ and $s>0$, with precisely the same constant B. Similarly, the norm inequality

$$(3.5)\qquad \|\Lambda^{-1}T\Lambda f\|_{q,w} \le B'\|f\|_{p,v}$$

is equivalent to the norm inequality

$$(3.6)\qquad \|Tf\|_{q,\Lambda w} \le B'\|f\|_{p,\Lambda v}.$$

Thus for example, we see from (1.11) that any weak type inequality for $M_{a,\mu}$ (recall that $M_{a,\mu} = M_{R_a,\mu}$) is equivalent in this sense to a weak type inequality for $M_{Q_1,\mu}$ with an appropriate weak type constant.

The simplest case is that for $M_{S,\mu}$. Then by (1.6) and (1.11) we have

$$(3.8)\qquad M_{S,\mu}f(x) \ge cM_{a,\mu}f(x) = ca^{-(\gamma-n+1)}\delta_a^{-1}M_{Q_1,\mu}\delta_a f(x),$$

for $a \ge 1$, and by (1.7),

$$(3.9)\quad M_{S,\mu}f(x) \le C\sum_{k=0}^{\infty} M_{2^k,\mu}f(x) = C\sum_{k=0}^{\infty} 2^{-k(\gamma-n+1)}\delta_{2^{-k}}M_{Q_1,\mu}\delta_{2^k}f(x).$$

We can control $I_{\alpha,\beta}$ by a similar process using (0.4). If $a\ge 1$, then by (3.1) and (1.9),

$$\begin{aligned}(3.10)\qquad I_{\alpha,\beta}f(x) &\ge c\int_0^\infty t^{\alpha+\beta}A_{R_a,\tau t}f(x)\,\frac{dt}{t}\\ &\ge c'a^{-(\gamma-n+1)}\int_0^\infty t^{\alpha+\beta}\delta_a^{-1}A_{Q_1,t}\delta_a f(x)\,\frac{dt}{t}\\ &\ge c''a^{-(\gamma-n+1)}\delta_a^{-1}I_{\alpha+\beta}\delta_a f(x),\end{aligned}$$

with $I_{\alpha+\beta}$ the ordinary Riesz fractional integral of order $\alpha+\beta$. The last inequality follows by the comparability of Q_1 and the unit ball. Similarly, if $\eta = \tau'\kappa$, then

$$\begin{aligned}(3.11)\qquad I_{\alpha,\beta}f(x) &\le C\sum_{k=0}^{\infty}\int_0^\infty t^{\alpha+\beta}A_{R_{2^k},\eta t}f(x)\,\frac{dt}{t}\\ &= C'\sum_{k=0}^{\infty}2^{-k(\gamma-n+1)}\int_0^\infty t^{\alpha+\beta}\delta_{2^{-k}}A_{Q_1,t}\delta_{2^k}f(x)\,\frac{dt}{t}\\ &\le C''\sum_{k=0}^{\infty}2^{-k(\gamma-n+1)}\delta_{2^{-k}}I_{\alpha+\beta}\delta_{2^k}f(x).\end{aligned}$$

We thus reduce the study of $I_{\alpha,\beta}$ and $M_{S,\mu}$ to the study of $I_{\alpha+\beta}$ and $M_{Q_1,\mu}$, respectively, under the effects of a family of linear transformations. We do not see how to use the family of linear transformations to relate $M_{S,Q_1,\mu}$ to a standard operator. Still, we may use the first part of our comparisons. Observe that if E and E' are comparable sets and if F and F' are comparable sets, then $M_{E,F,\mu}$ and $M_{E',F',\mu}$ are comparable operators. Thus, from (3.1) and (3.2) we can write:

$$(3.12)\qquad M_{S,Q_1,\mu}f(x) \ge cM_{R_a,Q_1,\mu}f(x),\quad a\ge 1,$$

$$(3.13)\qquad M_{S,Q_1,\mu}f(x) \le C\sum_{k=0}^{\infty} M_{R_{2^k},Q_1,\mu}f(x).$$

Proof of Theorem 1. The first two parts of the theorem will follow from (3.8), (3.9) and the following standard result, which may be proven using techniques developed by Muckenhoupt [M].

THEOREM M. *If $1\le p\le q<\infty$, $0\le\mu<n$, and v,w are weights, then $M_{Q_1,\mu}$ satisfies the weak type inequality*

$$w\{x : M_{Q_1,\mu}f(x) > s\} \le \left(\frac{B_0\|f\|_{p,v}}{s}\right)^q$$

if and only if v,w satisfy the $A^{\mu}_{p,q}$ condition: i.e., if and only if there is a constant B_1 such that

$$|Q|^{\mu/n-1}\Big(\int_Q w\Big)^{1/q}\Big(\int_Q v^{-p'/p}\Big)^{1/p'} \le B_1,\quad p>1,$$

$$|Q|^{\mu/n-1}\Big(\int_Q w\Big)^{1/q}\operatorname*{ess\,sup}_Q v(x)^{-1} \le B_1,\quad p=1,$$

for every cube Q. The constants B_0 and B_1 are comparable, with constants of comparability independent of v,w.

To prove part (A) of Theorem 1, first note that by (3.8), if (1.1) holds then
$$w\{x:\delta_a^{-1}M_{Q_1,\mu}\delta_a f(x)>s\}\le\left(\frac{cBa^{\gamma-n+1}\|f\|_{p,v}}{s}\right)^q,$$
or by the equivalence between (3.3) and (3.4),
$$\delta_a w\{x:M_{Q_1,\mu}f(x)>s\}\le\left(\frac{cBa^{\gamma-n+1}\|f\|_{p,\delta_a v}}{s}\right)^q.$$
Thus, by Theorem M, we obtain (1.2′), which is equivalent to (1.2).

For $q>1$, $L^{q,\infty}(w)$ is a Banach space, so by (3.9),
$$\|M_{S,\mu}f\|_{L^{q,\infty}(w)}\le C\sum_{k=0}^{\infty}2^{-k(\gamma-n+1)}\|\delta_{2^{-k}}M_{Q_1,\mu}\delta_{2^k}f\|_{L^{q,\infty}(w)}.$$
By Theorem M, if (1.3′) holds, then by the equivalence between (3.3) and (3.4),
$$\|\delta_{2^{-k}}M_{Q_1,\mu}\delta_{2^k}f\|_{L^{q,\infty}(w)}\le c2^{k(\gamma-n+1)}C(2^k)\|f\|_{L^p(v)}.$$
Combining these estimates with (1.4) gives (B) of Theorem 1 for the case $q>1$. The case $q=1$ is handled similarly, but instead of using norm inequalities, we apply Lemma 2.3 of E. Stein and N. Weiss [StWe] on summing weak type (1, 1) estimates, which requires the slightly stronger convergence condition on $C(a)$.

The strong type result (C) follows in a similar fashion from the following result for $M_{Q_1,\mu}$.

THEOREM P (Perez, [P]). *Let $1<p\le q<\infty$, $0<\mu<n$, and suppose for some $r>1$ that v,w satisfy the following strengthening of the $A^{\mu}_{p,q}$ condition:*
$$|Q|^{\mu/n-1/p}\Big(\int_Q w\Big)^{1/q}\Big(\frac{1}{|Q|}\int_Q v^{-rp'/p}\Big)^{1/(rp')}\le B_1,$$
for all cubes Q. Then $M_{Q_1,\mu}$ satisfies the strong type inequality
$$\|M_{Q_1,\mu}f\|_{q,w}\le B_0\|f\|_{p,v}$$
with $B_0\le CB_1$, for some constant C independent of v,w.

It is easy to see that (C) follows from Theorem P in the same way that (B) follows from Theorem M. ∎

This same method of proof may be used to obtain weak or strong type estimates for $I_{\alpha,\beta}$ from the corresponding estimates for $I_{\alpha+\beta}$. The necessary requirement in this procedure is that we have precise control over the operator norm of $I_{\alpha+\beta}$ in terms of constants appearing in the conditions on the weights. The proofs of Theorems 2 and 3 follow from the following results in this manner.

THEOREM GK (Gabidzashvili–Kokilashvili, [GK]). *Let $1<p<q<\infty$, $0<\mu<n$. Then I_μ satisfies the weak type inequality*
$$w\{x:|I_\mu f(x)|>s\}\le\left(\frac{C_0\|f\|_{p,v}}{s}\right)^q$$
if and only if w and v satisfy
$$\Big(\int_Q w\Big)^{1/q}\Big(\int_{Q^c}\frac{v(x)^{-p'/p}}{|x-x_Q|^{(n-\mu)p'}}\Big)^{1/p'}\le C_1$$
for every cube Q, where x_Q is the center of Q. Furthermore, the constants C_0 and C_1 are comparable.

THEOREM SW (Sawyer–Wheeden, [SaWh]).

(A) *If $1<p\le q<\infty$, $0<\mu<n$, and I_μ satisfies the strong type inequality*

(3.14) $$\|I_\mu f\|_{q,w}\le C_0\|f\|_{p,v}$$

then w,v satisfy the condition

(3.15) $$|Q|^{1-\mu/n}\Big(\int s_{Q,\mu}(x)^q w(x)\,dx\Big)^{1/q}\Big(\int s_{Q,\mu}(x)^{p'}v(x)^{-p'/p}\,dx\Big)^{1/p'}\le C_1$$

for every cube Q, where $s_{Q,\mu}(x)=(|Q|^{1/n}+|x-x_Q|)^{\mu-n}$, and x_Q is the center of Q. Hence w,v also satisfy the weaker $A^{\mu}_{p,q}$ condition
$$|Q|^{\mu/n-1}\Big(\int_Q w\Big)^{1/q}\Big(\int_Q v^{-p'/p}\Big)^{1/p'}\le C_1.$$
In each of these conditions, we have $C_1\le\text{const}\cdot C_0$. Conversely, if for some $r>1$ the weights w,v satisfy the following strengthening of the $A^{\mu}_{p,q}$ condition:
$$|Q|^{\mu/n+1/q-1/p}\Big(\frac{1}{|Q|}\int_Q w^r\Big)^{1/(qr)}\Big(\frac{1}{|Q|}\int_Q v^{-rp'/p}\Big)^{1/(rp')}\le C_r$$
for every cube Q, then I_μ satisfies the strong type inequality (3.14), with $C_0\le\text{const}\cdot C_r$.

(B) *If $p<q$, then (3.14) holds if and only if (3.15) holds, and C_0, C_1 are comparable.*

As noted in [SaWh], part (B) follows by combining the results in [GK] and [Sa]. Applying these theorems gives:

Proofs of Theorems 2 and 3. Theorem 2 follows from the theorem of Gabidzashvili–Kokilashvili, (3.10) and (3.11) in precisely the same manner that Theorem 1 was proven from Theorem M. For Theorem 3, the necessary condition in part (A) follows from the corresponding necessary condition for $I_{\alpha+\beta}$ in Theorem SW, using (3.10). Likewise, the sufficient conditions in parts (A) and (B) follow from the corresponding sufficient conditions for $I_{\alpha+\beta}$ in Theorem SW using (3.11), in the same way the sufficient conditions were proved for $M_{S,\mu}$ in Theorem 1.

Proofs of Corollaries 1 and 2. Recall that the family of rectangles $\bigcup\{\mathfrak{B}(R_a) : a \geq 1\}$ in Theorems 1–3 is actually independent of $\gamma > 0$, and that it is only the parametrization of these rectangles which varies with γ. The proof of the corollaries consists of keeping track of the parametrizations. Let $\overline{\delta}_a x = (ax_1, \ldots, ax_{n-1}, a^{-\overline{\gamma}} x_n)$ be the dilation δ_a with γ replaced by $\overline{\gamma}$. Observe that $\overline{\delta}_a x = r\delta_b x$ for r, b given by

$$a^{\overline{\gamma}+1} = b^{\gamma+1}, \quad r = a^{(\gamma-\overline{\gamma})/(\gamma+1)}.$$

Thus a rectangle of the form $R = \overline{\delta}_a Q$ for some cube Q may also be written in the form $R = \delta_b Q^{\sim}$ for $Q^{\sim} = rQ$, so the necessary condition corresponding to (1.2) for $M_{S',\mu}$ and $p > 1$ may be rewritten in terms of the original parametrization in the form

$$|R|^{\mu/n-1}\Big(\int_R w\Big)^{1/q}\Big(\int_R v^{-p'/p}\Big)^{1/p'} \leq Cb^{-\varepsilon} b^{(\gamma-n+1)(1-\mu/n)}$$

for all $R \in \mathfrak{B}(R_b)$, $b \geq 1$, where $\varepsilon = (\gamma - \overline{\gamma})(n-\mu)/(\overline{\gamma}+1) > 0$. The corresponding formula for $p = 1$ may be written similarly. Thus the summability requirement (1.4) for $M_{S,\mu}$ is satisfied with $C(a) = a^{-\varepsilon}$, yielding the weak type estimate (1.1).

Corollary 2 may be proven similarly. For example, the condition corresponding to (1.13) for $I_{\overline{\alpha},\overline{\beta}}$ is

$$(\overline{1.13}) \qquad \Big(\int_Q \overline{\delta}_a w\Big)^{1/q}\Big(\int_{Q^c} \frac{(\overline{\delta}_a v(x))^{-p'/p}}{|x-x_Q|^{(n-\overline{\alpha}-\overline{\beta})p'}}\,dx\Big)^{1/p'} \leq CBa^{\overline{\gamma}-n+1}.$$

Then for r, b as given above, $(\overline{\delta}_a w)(x) = r^n(\delta_b w)(rx)$, and similarly for $\overline{\delta}_a v$. Rewriting $(\overline{1.13})$ in terms of δ_b, using $\overline{\alpha}+\overline{\beta} = \alpha+\beta$ and making the substitution $y = rx$, we get

$$\Big(\int_{rQ} \delta_b w\Big)^{1/q}\Big(r^{-p'(\alpha+\beta)}\int_{(rQ)^c} \frac{(\delta_b v(y))^{-p'/p}}{|y-rx_Q|^{(n-\alpha-\beta)p'}}\,dy\Big)^{1/p'} \leq Cb^{(\overline{\gamma}-n+1)\frac{\gamma+1}{\overline{\gamma}+1}}$$

for all cubes Q and all $b \geq 1$. Since rx_Q is the center of rQ, upon relabeling cubes and writing r in terms of b, we get

$$\Big(\int_Q \delta_b w\Big)^{1/q}\Big(\int_{Q^c} \frac{(\delta_b v(y))^{-p'/p}}{|y-x_Q|^{(n-\alpha-\beta)p'}}\,dy\Big)^{1/p'} \leq Cb^{-\varepsilon} b^{\gamma-n+1},$$

for all cubes Q and all $b \geq 1$, where $\varepsilon = (\gamma-\overline{\gamma})(n-\alpha-\beta)/(\overline{\gamma}+1) = (\gamma-\overline{\gamma})(1-\overline{\beta}) > 0$, so by Theorem 2 we see that v, w satisfy a sufficient condition for (1.12) to hold. The remaining part of Corollary 2 may be proven similarly using the observation that $s_Q(x/r) = r^{n-\alpha-\beta} s_{rQ}(x)$.

Proof of Theorem 4. To prove the necessity statement (A), first note that if (1.18) holds, then by (3.12), we see (1.18) holds with $M_{R_a,Q_1,\mu} f(x)$ replacing $M_{S,Q_1,\mu} f(x)$ for $a \geq 1$, with weak type constant independent of a. Suppose then that (1.18) holds, and let $(R, R^*) \in \mathfrak{B}(R_a, R_a^*)$. Suppose first that $p > 1$, and let $f = v^{-p'/p}\chi_R$. If ξ denotes the center of R, R^*, then for some $t > 0$ we have

$$R = \xi + tR_a, \quad R^* = \xi + tR_a^*.$$

If $x \in R^*$, we can easily find $y \in tQ_1$ so that $z = x + y$ lies within R, so that R is contained in the rectangle $z - 2tR_a$. Then since $M_{S,Q_1,\mu} f \geq M_{R_a,Q_1,\mu} f$, and $x \in z - tQ_1 \subseteq z - 2tQ_1$, $R \subseteq z - 2tR_a$, we have

$$M_{S,Q_1,\mu} f(x) \geq (2t)^{\mu-n}\Big(\int_R v(y)^{-p'/p}\,dy\Big)\chi_{R^*}(x).$$

Substituting this into (1.18), using $t^n = |R|/|R_a|$ and

$$\|f\|_{p,v} = \Big(\int_R v(y)^{-p'/p}\,dy\Big)^{1/p},$$

we get (1.19), assuming $\int_R v^{-p'/p}\,dy \neq 0, \infty$. If this integral is 0, there is nothing to prove, and the case for which it is infinite can be treated by a standard limiting argument. For $p = 1$, let $A = \operatorname{ess\,inf}\{v(x) : x \in R\}$, so that if $\varepsilon > 0$, then $E = \{x \in R : v(x) \leq A + \varepsilon\}$ has positive measure. Letting $f = \chi_E$, we get $M_{S,Q_1,\mu} f(x) \geq (2t)^{\mu-n}|E|\chi_{R^*}(x)$. Also, $\|f\|_{1,v} \leq (A+\varepsilon)|E|$. Substituting this into (1.18), cancelling $|E|$, and letting $\varepsilon \to 0$, we get (1.19).

To prove (B) of Theorem 4, we will show that for each $a \geq 1$, (1.20) implies the weak type estimate

$$(3.16) \qquad w\{x : M_{R_a,Q_1,\mu} f(x) > s\} \leq \Big(\frac{DC(a)\|f\|_{p,v}}{s}\Big)^q,$$

with D a constant independent of a. From this, the weak type estimate for $M_{S,Q_1,\mu}$ follows by much the same argument used for the centered operator, which for $q > 1$ is that

$$\|M_{S,Q_1,\mu} f\|_{L^{q,\infty}(w)} \leq C\sum_{k=0}^{\infty} \|M_{2^k,Q_1,\mu} f\|_{L^{q,\infty}(w)} \leq C'\sum_{k=0}^{\infty} C(2^k)\|f\|_{p,v} \leq C''\|f\|_{p,v},$$

where the first inequality uses (3.13) and the fact that $L^{q,\infty}(w)$ is a Banach space for $q>1$, the second inequality uses (3.16), and the final inequality uses (1.4). Similarly, the case $q=1$ uses the theorem of N. Weiss and E. Stein on adding weak type bounds. In order to prove that (1.20) implies (3.16) when $p>1$, let $f\ge 0$, $f\in L^p(v)$, and let $s>0$. Let $E=\{x: M_{R_a,Q_1,\mu}f(x)>s\}$. For each $x\in E$ we can find $\xi\in\mathbb{R}^n$ and $t>0$ such that $x\in\xi-tQ_1$ and

$$t^{\mu-n}\int_{R(x)} f(y)\,dy>s\,,$$

where $R(x)=\xi-tR_a$. Writing $R^*(x)=\xi+tR_a^*$, since $t^n=|R|/|R_a|$, by Hölder's inequality and (1.20) we get

$$\begin{aligned} s&<\left(\frac{|R|}{|R_a|}\right)^{\mu/n-1}\int_{R(x)} f(y)\,dy\\ &\le (a^{\gamma-n+1}|R|)^{\mu/n-1}\Big(\int_{R(x)} f^p v\Big)^{1/p}\Big(\int_{R(x)} v^{-p'/p}\Big)^{1/p'}\\ &\le C(a)\Big(\int_{R(x)} f^p v\Big)^{1/p}\Big(\int_{R^*(x)} w\Big)^{-1/q}, \end{aligned}$$

yielding

$$s^q\Big(\int_{R^*(x)} w\Big)<C(a)^q\Big(\int_{R(x)} f^p v\Big)^{q/p}.$$

We may now apply the Besicovitch covering lemma to E and the cover $\{R^*(x): x\in E\}$ to choose a countable subcollection $\{R_j^*\}$ which covers E and which has bounded overlap, and in fact we may assume that the collection overlaps at most N times for N a constant independent of a. (The proof of the standard Besicovitch lemma for cubes remains unchanged upon a proportional rescaling of each of the edgelengths, so that one may replace cubes by translates and dilates of some fixed rectangle without any change in the covering argument or the overlap constant.) If we let R_j be the corresponding rectangle so that $(R_j,R_j^*)\in\mathfrak{B}(R_a,R_a^*)$, then since $q\ge p$, we have

$$\begin{aligned} s^q w\{x: M_{R_a,Q_1,\mu}f(x)>s\}&\le s^q\sum_j\int_{R_j^*} w\le C(a)^q\sum_j\Big(\int_{R_j} f^p v\Big)^{q/p}\\ &\le C(a)^q\Big(\sum_j\int_{R_j} f^p v\Big)^{q/p}\le NC(a)^q\|f\|_{p,v}^q\,, \end{aligned}$$

giving the desired result. The argument for $p=1$ is similar.

We now consider the stronger necessary conditions mentioned after Theorems 2 and 3. We begin by verifying (1.17). From the definitions of s_Q and $\widetilde{s}_R$, $R=\delta_a Q$, and assuming as we may that $x_Q=0$, it is enough to show that

$$\frac{1}{(|Q|^{1/n}+|\delta_a^{-1}y|)^{n-\alpha-\beta}}\le C\frac{1}{(a|Q|^{1/n}+|y|)^{n-1-\alpha}(a^{-\gamma}|Q|^{1/n}+|y_n|)^{1-\beta}}\,,$$

with C independent of $a\ge 1$ and Q. Replacing y by $\delta_a x$ and $|Q|^{1/n}$ by h, we must show that

$$(ah+|\delta_a x|)^{n-1-\alpha}(a^{-\gamma}h+a^{-\gamma}|x_n|)^{1-\beta}\le C(h+|x|)^{n-\alpha-\beta}\tag{3.17}$$

for all $a\ge 1$, $h>0$. If we write $x=(x',x_n)$, then $|\delta_a x|\approx a|x'|+a^{-\gamma}|x_n|$. Recall that $n-1-\alpha=\gamma(1-\beta)$, and consider separately the cases $a^{-\gamma}|x_n|\le ah+a|x'|$, and $a^{-\gamma}|x_n|\ge ah+a|x'|$. In the first case, the left side of (3.17) is comparable to

$$\begin{aligned} (ah+a|x'|)^{n-1-\alpha}(a^{-\gamma}h+a^{-\gamma}|x_n|)^{1-\beta}&=(h+|x'|)^{n-1-\alpha}(h+|x_n|)^{1-\beta}\\ &\le (h+|x|)^{n-1-\alpha}(h+|x|)^{1-\beta}\\ &=(h+|x|)^{n-\alpha-\beta}\,. \end{aligned}$$

In the second case, the left side of (3.17) is comparable to

$$\begin{aligned} (a^{-\gamma}|x_n|)^{n-1-\alpha}&(a^{-\gamma}h+a^{-\gamma}|x_n|)^{1-\beta}\\ &=a^{-\gamma(n-\alpha-\beta)}|x_n|^{n-1-\alpha}(h+|x_n|)^{1-\beta}\\ &\le a^{-\gamma(n-\alpha-\beta)}(h+|x|)^{n-1-\alpha}(h+|x|)^{1-\beta}\\ &\le (h+|x|)^{n-\alpha-\beta}\,, \end{aligned}$$

the last inequality holding because $a\ge 1$. This proves (3.17) in both cases, and so also proves (1.17).

We next show that (1.15′) is necessary for (1.14) if $1<p\le q<\infty$. Fix $R\in\mathfrak{B}(R_a)$, and pick $f=\widetilde{s}_R^{\,p'/p}v^{-p'/p}$ in (1.14), noting that

$$\|f\|_{p,v}=\Big(\int\widetilde{s}_R(y)^{p'}v(y)^{-p'/p}\,dy\Big)^{1/p},$$

and

$$I_{\alpha,\beta}f(x)=\int\widetilde{s}_R(y)^{p'/p}v(y)^{-p'/p}k_{\alpha,\beta}(x-y)\,dy\,.\tag{3.18}$$

We claim that

$$\left(\frac{|R|}{|R_a|}\right)^{1-(\alpha+\beta)/n}\widetilde{s}_R(x)\widetilde{s}_R(y)\le k_{\alpha,\beta}(x-y)\,.\tag{3.19}$$

To prove (3.19), we write R as the product of an $(n-1)$-dimensional cube with edgelength ah and a one-dimensional interval of length $a^{-\gamma}h$, so that we have $|R|/|R_a| = h^n$. Then

$$\tilde{s}_R(y)^{-1} = (ah + |y - x_R|)^{n-1-\alpha}(a^{-\gamma}h + |y_n - x_{R,n}|)^{1-\beta},$$

and (3.19) is equivalent to

$$|x-y|^{n-1-\alpha}|x_n - y_n|^{1-\beta} \le h^{\alpha+\beta-n}\tilde{s}_R(x)^{-1}\tilde{s}_R(y)^{-1}. \tag{3.20}$$

For $\sigma, \tau > 0$, we have (cf. [SaWh])

$$|x-y| \le \sigma^{-1}(\sigma + |x - x_R|)(\sigma + |y - x_R|),$$
$$|x_n - y_n| \le \tau^{-1}(\tau + |x_n - x_{R,n}|)(\tau + |y_n - x_{R,n}|).$$

Choosing $\sigma = ah$ and $\tau = a^{-\gamma}h$, (3.20) then follows by taking the product of the appropriate powers of the last two inequalities and using the relation $n - 1 - \alpha = \gamma(1-\beta)$.

We see from (3.18) and (3.19) that

$$\left(\frac{|R|}{|R_a|}\right)^{1-(\alpha+\beta)/n}\Big(\int \tilde{s}_R(y)^{p'}v(y)^{-p'/p}\,dy\Big)\tilde{s}_R(x) \le I_{\alpha,\beta}f(x), \tag{3.21}$$

and therefore (1.14) implies that

$$\left(\frac{|R|}{|R_a|}\right)^{1-(\alpha+\beta)/n}\Big(\int \tilde{s}_R(y)^{p'}v(y)^{-p'/p}\,dy\Big)\Big(\int \tilde{s}_R(x)^{q}w(x)\,dx\Big)^{1/q}$$
$$\le C\Big(\int \tilde{s}_R(y)^{p'}v(y)^{-p'/p}\,dy\Big)^{1/p}.$$

From this we obtain (1.15′) by dividing by the expression on the right side, provided this expression is neither 0 nor ∞. In case the right side is 0 for one choice of R, it is 0 for all R, and (1.15′) is obvious. The case when the right side is ∞ can be treated by a limit argument which we shall omit.

Finally, the fact that (1.13′) is necessary for (1.12) can be obtained by slightly modifying the argument above. In fact, for R and f as above, and for λ equal to the integral appearing in parentheses in (3.21), we have

$$\left(\frac{|R|}{|R_a|}\right)^{1-(\alpha+\beta)/n}\lambda\tilde{s}_R(x) \le I_{\alpha,\beta}f(x).$$

Since for $a \ge 1$ and $x \in R$, we have

$$\tilde{s}_R(x)^{-1} \le c(ah)^{n-1-\alpha}(a^{-\gamma}h)^{1-\beta} = ch^{n-\alpha-\beta} = c\left(\frac{|R|}{|R_a|}\right)^{1-(\alpha+\beta)/n},$$

it follows that $\lambda \le cI_{\alpha,\beta}f(x)$ for $x \in R$, and we obtain (1.13′) by applying (1.12) to f and λ.

Proof of Lemma 1. To prove (A), write $S = \{r\theta : \theta \in \mathbf{S}^{n-1},\ 0 \le r < \varrho(\theta)\}$, and let $\mathbf{O}_j = \{\theta \in \mathbf{S}^{n-1} : \varrho(\theta) > 2^{j-1}\}$, $j \in \mathbb{Z}$. For each j, cover $\mathbf{O}_j$ by discs $D_{jk} = \{\theta \in \mathbf{S}^{n-1} : |\theta - \theta_{jk}| < \varepsilon_{jk}\}$ so that $\sum_k |D_{jk}| \approx |\mathbf{O}_j|$,

and to each disc D_{jk} assign a rectangle R_{jk} with major axis in the direction θ_{jk}, so that R_{jk} is the smallest rectangle containing the cone $\{r\theta : \theta \in D_{jk},\ -2^j < r < 2^j\}$. Note that R_{jk} is unique up to rotation about its major axis, and any choice will do. Then for each j, $|R_{jk}| \approx 2^{nj}|D_{jk}|$, with comparability constants depending only on n, giving

$$\sum_j\sum_k |R_{jk}| \approx \sum_j 2^{nj}|\mathbf{O}_j| = \int_0^\infty t^{n-1}|\{\theta : \varrho(\theta) > t\}|\,dt \approx \frac{1}{n}\|\varrho\|_n = c_n|S|.$$

To complete the proof of (A), note that

$$S \subseteq \bigcup_j \{r\theta : \theta \in \mathbf{O}_j,\ 0 \le r < 2^j\} \subseteq \bigcup_j R_{jk}.$$

To prove (B), since S is open, it follows that $\varrho \ge c > 0$, and ϱ is lower semicontinuous. For convenience, let us suppose that $c = 1$. Form the sets $\mathbf{O}_j$ as in the proof of (A), except this time we consider only $j \ge 0$, noting that $\mathbf{O}_j$ is open for $j > 0$, and $\mathbf{O}_0 = \mathbf{S}^{n-1}$. With $\mathbf{O}_0$, we associate the rectangle $R_{01} = (1/\sqrt{n})Q_1$, and with $\mathbf{O}_j$ for $j > 0$, we associate a possibly infinite collection of rectangles R_{jk} as follows.

First, write $\mathbf{O}_j$ as the union of discs $D(\omega, r) = \{\theta \in \mathbf{S}^{n-1} : |\theta - \omega| < r\}$ with $\omega \in \mathbf{O}_j$ and $r = 2^{-j}\operatorname{dist}(\omega, \partial\mathbf{O}_j)$. By a standard selection process we can choose from this collection disjoint discs $D_{jk} = D(\theta_{jk}, r_{jk})$ such that $\mathbf{O}_j \subseteq \bigcup_k D(\theta_{jk}, 5r_{jk})$. Let C_{jk} be the smallest cylinder with axis parallel to θ_{jk} containing the cone $\{r\theta : \theta \in D_{jk},\ 0 \le r < 2^{j-1}\}$, and choose $R_{jk}^{\sim}$ to be one of the largest rectangles contained in C_{jk} with maximum radial length from the origin $\le 2^{j-1}$. Then $R_{jk}^{\sim}$ contains the origin on the intersection of one face with the major axis, and $R_{jk}^{\sim}$ is contained in S since we assume that S contains the unit ball and we constructed the discs D_{jk} so that they can be enlarged by a factor of 2^j and still be contained in $\mathbf{O}_j$.

To understand this point, it is helpful to visualize S as containing the union of the unit ball centered at the origin and a cone of length 2^{j-1} with vertex at the origin; then the intersection of the cone with $\mathbf{S}^{n-1}$ is a disc with a diameter that is roughly 2^{-j} times the diameter of the base of the cone. We can extend $R_{jk}^{\sim}$ slightly in the direction of $-\theta_{jk}$ to obtain a rectangle (again denoted by $R_{jk}^{\sim}$) which properly contains the origin and is contained in S, and if S is symmetric with respect to the origin, we may actually extend $R_{jk}^{\sim}$ so that it is centered at the origin.

Observe that $|R_{jk}^{\sim}| \approx 2^{nj}|D_{jk}|$, and that $|\mathbf{O}_j| \approx \sum_k |D_{jk}|$. Also, there is a positive constant $c_0 < 1$ depending only on n such that $D(\theta_{jk}, c_0 r_{jk})$ is contained in the projection of the base of $R_{jk}^{\sim}$ onto the unit sphere, so if $C = 5/c_0$, then $S \cap \{2^{j-1} < |x| < 2^j\} \subseteq \bigcup_k CR_{jk}^{\sim}$. Thus the collection $\{R_{jk}\}_{j,k}$ for $R_{jk} = CR_{jk}^{\sim}$ is our desired proper starlike cover.

It should be pointed out that although this proof demonstrates the existence of starlike covers, it does not give an optimal starlike cover. For example, S may admit a starlike cover $\{R_j\}$ such that $\#\{R_j : \operatorname{diam} R_j \le 2^k\} < \infty$ for each k, yet this may fail for the cover obtained in our proof.

Proofs of Theorems 5–7. From Lemma 1, we see that if S is starlike with respect to the origin and $\{R_j\}$ is a starlike cover as guaranteed by Lemma 1, then

$$\chi_S(x) \le \sum_{j=0}^{\infty} \chi_{R_j}(x), \tag{3.22}$$

with the integrals of the two sides being comparable. Further, Lemma 1 shows that if S is open, we can choose the cover to additionally satisfy

$$\chi_S(x) \ge \chi_{R_j}(x/c), \tag{3.23}$$

where c is the constant such that $cR_j \subseteq S$ for all j.

In the parts of Theorems 5–7 that deal with necessary conditions, we require that S be symmetric with respect to the origin in addition to being open, so that S admits a proper starlike cover by rectangles centered at the origin. The reason we make this requirement is that if R is a rectangle centered at the origin and if δ_R, Q_R are respectively the corresponding linear transformation and cube of unit edgelength such that $R = \delta_R Q_R$, then Q_R is also centered at the origin and hence is comparable to Q_1 and the unit ball, with comparability constants that do not depend on R.

In the parts of Theorems 5–7 that deal with sufficient conditions, we do not require S to be either open or symmetric with respect to the origin. We obtain sufficient conditions any time we can cover S by rectangles containing the origin (but the smaller the cover, the better the sufficient conditions), and we do not require the rectangles to be centered at the origin in this case, because if R is a rectangle containing the origin with associated δ_R, Q_R, then Q_R contains the origin, so that $Q_R \subseteq CQ_1$ with C a constant depending only on n, which allows us to dominate operators involving averages over Q_R by the corresponding operators involving averages over Q_1. A similar statement holds with Q_1 replaced with the unit ball centered at the origin. The opposite set comparison fails since the origin may be arbitrarily close to the boundary of Q_R.

With this in mind, Theorems 5–7 follow much as do Theorems 1–3, because when (3.22) holds, then by (1.10) (recalling that $|\det \delta_R| = |R|$, and letting $Q_j = Q_R$ for $R = R_j$) we have

$$M_{S,\mu} f(x) \le C \sum_{j=0}^{\infty} M_{R_j,\mu} f(x) = C \sum_{j=0}^{\infty} |R_j| \delta_{R_j}^{-1} M_{Q_j,\mu} \delta_{R_j} f(x) \tag{3.24}$$

$$\le C' \sum_{j=0}^{\infty} |R_j| \delta_{R_j}^{-1} M_{Q_1,\mu} \delta_{R_j} f(x).$$

Similarly, when (3.22) holds for the starlike set S associated with $I_{\Omega,\mu}$, then

$$\begin{aligned} I_{\Omega,\mu} f(x) &\le (n-\mu) \sum_j \int_0^{\infty} t^{\mu} A_{R_j,t} f(x) \frac{dt}{t} \\ &= (n-\mu) \sum_j |R_j| \int_0^{\infty} t^{\mu} \delta_{R_j}^{-1} A_{Q_{R_j},t} \delta_{R_j} f(x) \frac{dt}{t} \\ &\le C \sum_j |R_j| \delta_{R_j}^{-1} I_{\mu} \delta_{R_j} f(x). \end{aligned} \tag{3.25}$$

The last inequality in each of (3.24) and (3.25) follows by replacing Q_R with the fixed multiple of Q_1 or the unit ball that contains Q_R for all j. Conversely, if R is a rectangle centered at the origin contained in S, then

$$\begin{aligned} M_{S,\mu} f(x) &\ge c M_{R,\mu} f(x) = c|R| \delta_R^{-1} M_{Q_R,\mu} \delta_R f(x) \\ &\ge c|R| \delta_R^{-1} M_{Q_1,\mu} \delta_R f(x), \end{aligned} \tag{3.26}$$

where the last inequality uses the fact that Q_1 and Q_R are comparable if R is centered at the origin, with constants of comparability depending only on n. Similarly,

$$I_{\Omega,\mu} f(x) \ge (n-\mu) \int_0^{\infty} t^{\mu} A_{R,t} f(x) \frac{dt}{t} \ge c|R| \delta_R^{-1} I_{\mu} \delta_R f(x), \tag{3.27}$$

for any rectangle R centered at the origin which is contained in the starlike set associated with $I_{\Omega,\mu}$, and the last inequality follows by replacing Q_R by the multiple of the unit ball contained in Q_R (using the fact that R is centered at the origin), and c is independent of R.

Arguing as in Theorem 1, Theorem 5 for $M_{S,\mu}$ then follows from (3.24) and (3.26), using the equivalence of (3.3) and (3.4), and of (3.5) and (3.6). In a similar fashion, Theorems 6 and 7 follow via (3.25) and (3.27) from the corresponding estimates for I_{μ}.

Proof of Corollary 4. That (2.6) follows from (2.7) is a direct result of Theorem 5, observing that (2.3) holds with $C_j = |R_j|$, so that (2.7) is only a special case of the summability requirement (2.4). The equivalence of (2.7) and (2.8) may be proven by adapting the part of the argument of Lemma 1 showing $|S| \approx \sum |R_j|$ to the measures in Corollary 4. That is, in the language of the proof of Lemma 1, cover $\mathbf{O}_j$ by a collection $\{B_{jk}\}$ for which

$$\sum_k \psi_{2^j}(|B_{jk}|) \ge m_{2^j}(\mathbf{O}_j) \ge C \sum_k \psi_{2^j}(|B_{jk}|),$$

for some $C<1$ independent of j, and construct rectangles accordingly.

Finally, the claims for the example may be proven using a modification of an argument in [Ca]. In the case that the sum which defines ϱ consists of only one term, the set $\{\varrho>r\}$ is comparable to a single "rectangle" in $\mathbf{S}^{n-1}$, so that

$$m_r\{\varrho>r\}\le C|\{\varrho>r\}|\left(1+\log^+\frac{1}{r^n|\{\varrho>r\}|}\right).$$

To show that (2.8) holds, we decompose the integral in (2.8) into an integral over the set

$$X=\left\{r\ge 1:|\{\varrho>r\}|\le\frac{1}{r^n(\log r)^2}\right\}$$

and an integral over $[1,\infty)-X$. To see that the first integral is finite, observe that ψ_r is monotone so that for $r\in X$,

$$m_r\{\varrho>r\}\le C\frac{1+2\log^+\log r}{r^n(\log r)^2},$$

and therefore

$$\int\limits_X r^{n-1}m_r\{\varrho>r\}\,dr\le C\int\limits_1^\infty\frac{1+2\log^+\log r}{r(\log r)^2}\,dr<\infty.$$

For the second integral, we note that if $r\in[1,\infty)-X$, then $m_r\{\varrho>r\}\le C|\{\varrho>r\}|(1+2\log^+\log r)$, giving

$$\int\limits_{[1,\infty)-X} r^{n-1}m_r\{\varrho>r\}\,dr\le C\int\limits_1^\infty r^{n-1}(1+2\log^+\log r)|\{\varrho>r\}|\,dr$$
$$\le C\int\limits_{\mathbf{S}^{n-1}}\varrho^n(1+\log^+\log^+\varrho)<\infty.$$

Proofs of Corollaries 3 and 5. Suppose $S=S_\varrho$ is open and let $S'=S_{\varrho'}$ for $\varrho'(\theta)=\varrho(\theta)^t$ for some $t>1$. It is easy to see that S' is open. Further, we see that if $\{R_j\}$ is a proper starlike cover of S, and if we let $R_j'=(\operatorname{diam}R_j)^{t-1}R_j$, then $\{R_j'\}$ is a proper starlike cover for S'. If we choose $Q_{R_j'}$ and $\delta_{R_j'}$ so that $Q_{R_j'}$ has edgelength 1 and $\delta_{R_j'}Q_{R_j'}=R_j'$, then as linear transformations, $\delta_{R_j'}=(\operatorname{diam}R_j)^{t-1}\delta_{R_j}$, giving $\det\delta_{R_j'}=(\operatorname{diam}R_j)^{n(t-1)}\det\delta_{R_j}$.

Let us prove Corollary 3. Since $\{R_j'\}$ is a proper cover for S', we see that the necessary conditions for $M_{S',\mu}$ to satisfy (2.1) for $p>1$ are

$$|Q|^{\mu/n-1}\Big(\int\limits_Q\delta_{R_j'}w\Big)^{1/q}\Big(\int\limits_Q(\delta_{R_j'}v)^{-p'/p}\Big)^{1/p'}\le\frac{C}{|R_j'|},$$

for all cubes Q. Letting $\eta=\operatorname{diam}R_j$, a substitution in each integral shows that this may be rewritten as

$$|Q|^{\mu/n-1}\Big(\int\limits_{\eta^{t-1}Q}\delta_{R_j}w\Big)^{1/q}\Big(\int\limits_{\eta^{t-1}Q}(\delta_{R_j}v)^{-p'/p}\Big)^{1/p'}\le\frac{C}{|R_j|},$$

for all cubes Q (recall cQ denotes the cube $\{cx:x\in Q\}$), and relabeling cubes shows that this in turn is equivalent to

$$(3.28)\qquad |Q|^{\mu/n-1}\Big(\int\limits_Q\delta_{R_j}w\Big)^{1/q}\Big(\int\limits_Q(\delta_{R_j}v)^{-p'/p}\Big)^{1/p'}\le\frac{C(\operatorname{diam}R_j)^{(1-t)(n-\mu)}}{|R_j|}.$$

Letting $C_j=(\operatorname{diam}R_j)^{(1-t)(n-\mu)}$, we see that the condition $\sum_j C_j<\infty$ is equivalent to (2.5), and therefore by part B of Theorem 5, if (2.5) holds then the sequence of necessary conditions (3.28) for the operator $M_{S',\mu}$ to satisfy (2.1) are in fact sufficient for $M_{S,\mu}$ to satisfy (2.1). The argument for Corollary 5 is similar, once one observes that $|x/r-x_Q|=(1/r)|x-x_{rQ}|$, and $s_Q(x/r)=r^{n-\mu}s_{rQ}(x)$.

Corollary 3 has an analogue for strong type estimates, as mentioned after the statement of Theorem 5. To describe it, using the notation above, we assume that $1<p\le q<\infty$ and that v,w satisfy

$$|Q|^{\mu/n-1/p}\Big(\int\limits_Q\delta_{R_j'}w\Big)^{1/q}\Big(\frac{1}{|Q|}\int\limits_Q(\delta_{R_j'}v)^{-rp'/p}\Big)^{1/(rp')}\le\frac{c}{|R_j'|}$$

for all j, all cubes Q, and some $r>1$. We then obtain as above the following analogue of (3.28):

$$|Q|^{\mu/n-1/p}\Big(\int\limits_Q\delta_{R_j}w\Big)^{1/q}\Big(\frac{1}{|Q|}\int\limits_Q(\delta_{R_j}v)^{-rp'/p}\Big)^{1/(rp')}\le\frac{c(\operatorname{diam}R_j)^{(1-t)(n-\mu)}}{|R_j|}.$$

From this, if (2.5) holds, we see from the strong type result of part (C) of Theorem 5 that

$$\|M_{S,\mu}f\|_{q,w}\le B\|f\|_{p,v}.$$

Finally, we will verify the remark made after Theorem 1. As before, if $1<p,q<\infty$ and $0\le\mu<n$, we say that the ordered pair $(w,v)\in A_{p,q}^\mu$ if

$$|Q|^{\mu/n-1}\Big(\int\limits_Q w\Big)^{1/q}\Big(\int\limits_Q v^{-p'/p}\Big)^{1/p'}\le c.$$

It follows easily by applying Hölder's inequality to the integral involving v that $A^{\mu/\delta}_{p\delta,q\delta}$ is nested in δ, i.e., if $0<\delta<1$ then $A^{\mu/\delta}_{p\delta,q\delta}\subseteq A^{\mu}_{p,q}$. Similarly, by applying Hölder's inequality to the integral involving v in (1.3), it is not difficult to check that both (1.3) and (1.4) hold if $(w,v)\in A^{\mu/\delta}_{p\delta,q\delta}$ for some δ satisfying $\delta<1-(n-\mu)/(\gamma+1)$. In the special case when $w=v$, $\mu=0$ and $p=q$, note that the $A^{\mu}_{p,q}$ condition reduces to the A_p condition defined in [M], i.e., $(w,w)\in A^{0}_{p,p}$ iff $w\in A_p$. Consequently, if $w\in A_{p\delta}$ for some $\delta<1-n/(\gamma+1)$, and if $\gamma+1>n$, $p\delta>1$, then by Theorem 1, $M_{S,0}$ satisfies the weak type estimate $\|M_{S,0}f\|_{L^{p,\infty}(w)}\le c\|f\|_{p,w}$. Finally, using the fact from [M] that a weight in A_p is also in $A_{p\pm\varepsilon}$ for small ε, it follows that if $w\in A_{p(1-n/(\gamma+1))}$, $p\big(1-\frac{n}{\gamma+1}\big)>1$ and $\gamma+1>n$, then $\|M_{S,0}f\|_{L^{r,\infty}(w)}\le c\|f\|_{r,w}$ for all r in an open interval containing p. Using the Marcinkiewicz interpolation theorem, we see that these weak type estimates imply the corresponding strong type estimate at $r=p$, thereby proving the first part of the remark after Theorem 1.

The second part of the remark can be derived by similar reasoning, after noting that $(w,v)\in A^{\mu}_{p,q}$ is the same as $(v^{-p'/p},w^{-q'/q})\in A^{\mu}_{q',p'}$, and using this to show that both (1.3) and (1.4) hold if $(v^{-p'/p},w^{-q'/q})\in A^{\mu/\delta}_{q'\delta,p'\delta}$ for some $\delta<1-(n-\mu)/(\gamma+1)$.

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