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Rearrangement of series in nonnuclear spaces

by

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Abstract. It is proved that if a metrizable locally convex space is not nuclear, then it does not satisfy the Lévy–Steinitz theorem on rearrangement of series.

Let $\sum_{n=1}^{\infty} u_n$ be a convergent series in a topological vector space E (all vector spaces are assumed to be real). Its sum will be denoted by the same symbol $\sum_{n=1}^{\infty} u_n$, that is

$$\sum_{n=1}^{\infty} u_n = \lim_{N\to\infty} \sum_{n=1}^{N} u_n .$$

We shall frequently write $\sum u_n$ instead of $\sum_{n=1}^{\infty} u_n$ to denote both the series and its sum.

The set of sums of the series $\sum u_n$, denoted by $\mathfrak{S}(\sum u_n)$, is defined in the following way: a vector $v \in E$ belongs to $\mathfrak{S}(\sum u_n)$ if there is a permutation π of $\mathbb{N}$ such that the series $\sum_{n=1}^{\infty} u_{\pi(n)}$ converges to v. We also define

$$\Gamma\Big(\sum u_n\Big) = \Big\{ f \in E' : \sum_{n=1}^{\infty} |f(u_n)| < \infty \Big\},$$

$$\Gamma_0\Big(\sum u_n\Big) = \Big\{ v \in E : f(v) = 0 \text{ for all } f \in \Gamma\Big(\sum u_n\Big)\Big\}.$$

It is obvious that $\mathfrak{S}(\sum u_n) \subset \Gamma_0(\sum u_n) + \sum u_n$. The Lévy–Steinitz theorem asserts that if $\sum u_n$ is a convergent series in $\mathbb{R}^n$, then

$$(*) \qquad \mathfrak{S}\Big(\sum u_n\Big) = \Gamma_0\Big(\sum u_n\Big) + \sum u_n .$$

It was proved in [2] that $\mathbb{R}^n$ may be replaced here by an arbitrary metrizable nuclear locally convex space. The aim of this paper is to prove the following converse of that result:

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THEOREM. *If a metrizable locally convex space is not nuclear, then it contains a convergent series* $\sum u_n$ *such that* $(*)$ *does not hold.*

The assumption of metrizability is essential: each convergent series in a locally convex direct sum of real lines lies in a finite-dimensional subspace.

Let p be a norm on a vector space X. Let Y be a subspace of X and Z a subspace of Y. By Y_p and $(Y/Z)_p$ we shall denote the spaces Y and Y/Z endowed, respectively, with the norms induced by p. The Banach–Mazur distance between a normed space X and $l_2^{\dim X}$ will be denoted by d_X.

LEMMA 1. *To each* $\vartheta \in (0,1)$ *there corresponds a constant* $c > 0$ *satisfying the following condition*:

(i) *given arbitrary norms* p, q *on a finite-dimensional vector space* X, *one can find a subspace* Y *of* X *and a subspace* Z *of* Y *such that*

$$\dim(Y/Z) > \vartheta \dim X, \tag{1}$$

$$d_{(Y/Z)_p} \le c, \tag{2}$$

$$d_{(Y/Z)_q} \le c. \tag{3}$$

Proof. Fix ϑ. According to the Milman quotient subspace theorem, there exists a constant $c > 0$ satisfying the following condition:

(ii) every finite-dimensional normed space X contains a subspace Y and another subspace $Z \subset Y$ such that $\dim(Y/Z) > \sqrt{\vartheta} \dim X$ and $d_{Y/Z} \le c$.

We shall prove that c satisfies (i). So, let p, q be two norms on an n-dimensional vector space X. By (ii), we can find a subspace Y_1 of X and a subspace Z_1 of Y_1 such that

$$\dim(Y_1/Z_1) > \sqrt{\vartheta} n, \tag{4}$$

$$d_{(Y_1/Z_1)_p} \le c. \tag{5}$$

Applying (ii) once again, we find a subspace Y_2 of Y_1/Z_1 and a subspace Z_2 of Y_2 such that

$$\dim(Y_2/Z_2) > \sqrt{\vartheta} \dim(Y_1/Z_1), \tag{6}$$

$$d_{(Y_2/Z_2)_q} \le c. \tag{7}$$

Let $\psi : Y_1 \to Y_1/Z_1$ be the canonical projection. Set $Y = \psi^{-1}(Y_2)$ and $Z = \psi^{-1}(Z_2)$. From (6) and (4) we obtain (1) because $\dim(Y/Z) = \dim(Y_2/Z_2)$.

If F is a subspace of a normed space E, then, naturally, $d_F, d_{E/F} \le d_E$. Hence

$$d_{(Y_2/Z_2)_p} \le d_{(Y_2)_p} \le d_{(Y_1/Z_1)_p} \tag{8}$$

because $(Y_2/Z_2)_p$ is a quotient space of $(Y_2)_p$ which, in turn, is a subspace of $(Y_1/Z_1)_p$. Since the canonical isomorphism $\varphi : Y/Z \to Y_2/Z_2$, defined in the obvious way, is an isometry of $(Y/Z)_p$ onto $(Y_2/Z_2)_p$, from (8) and (5) we obtain (2). Finally, (7) implies (3) because φ is also an isometry of $(Y/Z)_q$ onto $(Y_2/Z_2)_q$. ∎

Let U, W be two symmetric convex bodies in an n-dimensional vector space X. Their volume ratio will be denoted by $\frac{|U|}{|W|}$. More precisely, we define

$$\frac{|U|}{|W|} = \frac{\lambda(T(U))}{\lambda(T(W))}$$

where λ is the Lebesgue measure on $\mathbb{R}^n$ and $T : X \to \mathbb{R}^n$ a linear isomorphism.

LEMMA 2. *Let* U, W *be two symmetric convex bodies in an* n-*dimensional vector space* N, *with* $U \subset W$. *Let* M *be an* m-*dimensional subspace of* N *and* $\pi : N \to M$ *an arbitrary projection. Then*

(a) $$\frac{|U \cap M|}{|W \cap M|} \ge \frac{m!}{n!} \frac{|U|}{|W|},$$

(b) $$\frac{|\pi(U)|}{|\pi(W)|} \ge \frac{m!}{n!} \frac{|U|}{|W|}.$$

For (a), see [1], Lemma 3. The proof of (b) is similar; we leave it to the reader.

LEMMA 3. *Let* r *be an integer greater than* 1 *and let* E_{r-1} *be an* $(r-1)$-*dimensional euclidean space. There exist vectors* $u_1, \ldots, u_r \in E_{r-1}$ *with norms* ≤ 1 *such that*

$$u_1 + \ldots + u_r = 0 \tag{9}$$

and the following condition is satisfied:

(iii) *given arbitrary* $k_1, \ldots, k_r \in \mathbb{Z}$, *denoting by* σ *the fractional part of* $\frac{1}{r}\sum_{i=1}^r k_i$, *one has*

$$\Big\| \sum_{i=1}^r k_i u_i \Big\| \ge r^{1/2}[\sigma(1-\sigma)]^{1/2}. \tag{10}$$

Proof. Let E_r be an r-dimensional euclidean space and $e_1, \ldots, e_r$ an orthonormal basis in E_r. We may assume that E_{r-1} is the orthogonal complement of the vector $e_1 + \ldots + e_r$. Let $\pi : E_r \to E_{r-1}$ be the orthogonal projection. An easy calculation shows that the vectors $u_i = \pi(e_i)$, $i = 1, \ldots, r$, satisfy the desired conditions. ∎

The closed unit ball of a normed space E will be denoted by B_E. Let $T : E \to F$ be a bounded linear operator acting between normed spaces. For each $n = 1, 2, \ldots$, we define

$$v_n(T) = \sup_M \left(\frac{|T(B_E \cap M)|}{|B_F \cap T(M)|} \right)^{1/n}$$

where the supremum is taken over all linear subspaces M of E with $\dim M = \dim T(M) = n$. If $\operatorname{rank} T < n$, then we define $v_n(T) = 0$.

Let p be a seminorm on a vector space E. The pseudometric on E induced by p will be denoted by d_p. We write $B_p = \{u \in E : p(u) \leq 1\}$. By E_p we denote the space $E/p^{-1}(0)$ with its natural norm. The canonical projection $E \to E_p$ is denoted by ψ_p. Thus $\|\psi_p(u)\| = p(u)$ for $u \in E$. Let $q \leq p$ be another seminorm on E. The canonical operator $E_p \to E_q$ is denoted by T_{pq}. We write

$$v_n(p,q) = v_n(T_{pq}) \quad (n = 1, 2, \ldots).$$

Given a subset A of a vector space E, by $\operatorname{gp} A$ we shall denote the additive subgroup of E generated by A.

LEMMA 4. *Let p, q be two norms on a vector space E such that $p \geq q$ and*

(11) $$\limsup_{n\to\infty} n^{0.1} v_n(p,q) > 1.$$

(a) *Take any $r_0 \in \mathbb{N}$ and $\delta > 0$. Then we can find some $r, s \in \mathbb{N}$ with $r_0 \leq r < s$ and some $w_1, \ldots, w_s \in B_p$ with*

(12) $$w_1 + \ldots + w_s = 0$$

such that the following condition is satisfied:

(iv) *given arbitrary $k_1, \ldots, k_s \in \mathbb{Z}$, denoting by σ the fractional part of $\frac{1}{r}\sum_{i=1}^{r} k_i$, one has*

(13) $$q\Big(\sum_{i=1}^{s} k_i w_i\Big) \geq \delta^{-1}[\sigma(1-\sigma)]^{1/2}.$$

(b) *Let K be a finitely generated subgroup of E. Take arbitrary $u \in K$, $a \in \operatorname{span} K$ and $\varepsilon > 0$. Then we can find some $v_1, \ldots, v_s \in B_p$ such that*

(14) $$v_1 + \ldots + v_s = u,$$

(15) $$d_q(a, K + \mathbb{Z}v_1 + \ldots + \mathbb{Z}v_s) \geq d_q(a, K) - \varepsilon.$$

(c) *Let K be a finitely generated subgroup of E. Take arbitrary $a \in \operatorname{span} K$ and $\varepsilon > 0$. Then we can find another finitely generated subgroup K' of E with $K \subset K'$ such that*

(16) $$K' = \operatorname{gp}(K' \cap B_p),$$

(17) $$d_q(a, K') \geq d_q(a, K) - \varepsilon.$$

Proof. (a) Let c be the constant corresponding to the constant $\vartheta = 0.95$ due to Lemma 1. By (11) and the definition of $v(p,q)$, we can find an n-dimensional subspace L of E such that $p_{|L}$, $q_{|L}$ are norms on L and

(18) $$0.475n - 1 \geq \max(r_0, 2^{10}c^{40}\delta^{-10}),$$

(19) $$\frac{|B_p \cap L|}{|B_q \cap L|} > n^{-0.1n}.$$

According to our definition of c, we can find a subspace M of L and a subspace N of M such that

(20) $$m := \dim(M/N) > 0.95n,$$

(21) $$d_{(M/N)_p} \leq c,$$

(22) $$d_{(M/N)_q} \leq c.$$

Let $\varphi : M \to M/N$ be the canonical projection. Set $l = \dim M$. Applying Lemma 2(a) and then (b), we see that

$$\frac{|\varphi(B_p \cap M)|}{|\varphi(B_q \cap M)|} \geq \frac{m!}{l!}\frac{|B_p \cap M|}{|B_q \cap M|} \geq \frac{m!}{l!}\frac{l!}{n!}\frac{|B_p \cap L|}{|B_q \cap L|}.$$

Hence, by (19) and (20), after easy calculations we derive

(23) $$\frac{|\varphi(B_p \cap M)|}{|\varphi(B_q \cap M)|} > m^{-0.2m}.$$

It follows from (21) that there is an m-dimensional o-symmetric ellipsoid U in M/N such that

(24) $$U \subset \varphi(B_p \cap M) \subset cU.$$

Similarly, (22) implies that there is an m-dimensional o-symmetric ellipsoid W in M/N such that

(25) $$c^{-1}W \subset \varphi(B_q \cap M) \subset W.$$

From (23)–(25) we get

(26) $$\frac{|U|}{|W|} > c^{-2m}m^{-0.2m}.$$

We may assume that $M/N = \mathbb{R}^m$ and

$$U = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_1^2 + \ldots + x_m^2 \leq 1\}.$$

Let $\mu_1 \leq \ldots \leq \mu_m$ be the principal semiaxes of W. We may assume that

$$W = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_1^2/\mu_1^2 + \ldots + x_m^2/\mu_m^2 \leq 1\}.$$

Let r be the integer part of $m/2$. From (20) we get

(27) $$r > 0.475n - 1,$$

whence $r \geq r_0$ due to (18). As $B_p \subset B_q$, from (24) and (25) we have $U \subset W$. Therefore $\mu_1, \ldots, \mu_m \geq 1$, so that

$$\mu_{r-1}^{m-r} \leq \mu_{r-1} \ldots \mu_m \leq \mu_1 \ldots \mu_m = \frac{|W|}{|U|}.$$

Hence, by (26), after easy calculations we get

$$\mu_{r-1} < 2c^4 r^{0.4}. \tag{28}$$

Let $\| \ \|$ denote the euclidean norm on the subspace

$$\mathbb{R}^{r-1} := \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_i = 0 \text{ for } i \geq r\}.$$

According to Lemma 3, there exist some $u_1, \ldots, u_r \in U \cap \mathbb{R}^{r-1}$ satisfying (9) and (iii). In view of (24), for each $i = 1, \ldots, r$ we can find some $w_i \in B_p \cap M$ with $\varphi(w_i) = u_i$. Then $\sum_{i=1}^r w_i \in N$ because, by (9),

$$\varphi\Big(\sum_{i=1}^r w_i\Big) = \sum_{i=1}^r \varphi(w_i) = \sum_{i=1}^r u_i = 0.$$

Therefore we can find some $w_{r+1}, \ldots, w_s \in B_p \cap N$ such that (12) is satisfied.

To prove (iv), take arbitrary $k_1, \ldots, k_s \in \mathbb{Z}$ and let σ be the fractional part of $\frac{1}{r}\sum_{i=1}^r k_i$. Then (10) is satisfied. From (25) and the inclusion $W \cap \mathbb{R}^{r-1} \subset \mu_{r-1} U$ it follows easily that $q(u) \geq \mu_{r-1}^{-1} \|\varphi(u)\|$ for each $u \in \varphi^{-1}(\mathbb{R}^{r-1})$. Thus

$$q\Big(\sum_{i=1}^s k_i w_i\Big) \geq \mu_{r-1}^{-1} \Big\|\varphi\Big(\sum_{i=1}^s k_i w_i\Big)\Big\| = \mu_{r-1}^{-1} \Big\|\sum_{i=1}^r k_i u_i\Big\|.$$

Hence, by (10) and (28), we obtain

$$q\Big(\sum_{i=1}^s k_i w_i\Big) \geq 2^{-1} c^{-4} r^{0.1} [\sigma(1-\sigma)]^{1/2},$$

which, by (27) and (18), yields (13). This proves (a).

(b) Choose $r_0 \geq 2p(u)$ and $\delta \in (0, 1/2)$ such that

$$(1-\delta)[d_q(a, K) - \delta q(u)] \geq d_q(a, K) - \varepsilon, \tag{29}$$

$$\frac{1}{6}\delta^{-1}[\delta(1-\delta)]^{1/2} \geq d_q(a, K) - \varepsilon. \tag{30}$$

A standard argument allows us to find a subspace F of E with $\operatorname{codim} F < \infty$ such that

$$q(x+y) \geq (1-\delta) q(x) \quad (x \in \operatorname{span} K,\ y \in F). \tag{31}$$

This implies that

$$q(x+y) \geq \frac{1-\delta}{2-\delta} > \frac{1}{3} q(y) \quad (x \in \operatorname{span} K,\ y \in F). \tag{32}$$

It follows easily from (11) and Lemma 2(a) that

$$\limsup_{n\to\infty} n^{0.1} v_n(p_{|F}, q_{|F}) > 1.$$

So, by (a), we can find some $r, s \in \mathbb{N}$ with $r_0 \leq r < s$ and some $w_1, \ldots, w_s \in B_p \cap F$ such that (12) and (iv) are satisfied. Set $v_i = \frac{1}{2}w_i + \frac{1}{r}u$ for $i = 1, \ldots, r$ and $v_i = \frac{1}{2}w_i$ for $i = r+1, \ldots, s$. Then (12) implies (14). For each $i = 1, \ldots, s$, we have

$$p(v_i) \leq \frac{1}{2}p(w_i) + \frac{1}{r}p(u) \leq \frac{1}{2} + \frac{1}{2} = 1$$

because $w_i \in B_p$ and $r \geq r_0 \geq 2p(u)$. Thus $v_1, \ldots, v_s \in B_p$.

To prove (15), take any $b \in K$ and $k_1, \ldots, k_s \in \mathbb{Z}$. We are to show that

$$\varrho := q\Big(a - \Big(b + \sum_{i=1}^s k_i v_i\Big)\Big) \geq d_q(a, K) - \varepsilon. \tag{33}$$

According to the definition of v_i, we have

$$\varrho = q\Big(a - b - \frac{1}{r}\Big(\sum_{i=1}^r k_i\Big) u - \frac{1}{2}\sum_{i=1}^s k_i w_i\Big). \tag{34}$$

Let σ be the fractional part of $\frac{1}{r}\sum_{i=1}^r k_i$. We shall consider three cases.

I. $\sigma \in [0, \delta)$. Let m be the integer part of $\frac{1}{r}\sum_{i=1}^r k_i$. Then $b + mu \in K$. From (34) and (31) we get

$$\varrho \geq (1-\delta) q\Big(a - b - \frac{1}{r}\Big(\sum_{i=1}^r k_i\Big)u\Big) \geq (1-\delta)[q(a-b-mu) - q(\sigma u)]$$
$$\geq (1-\delta)[d_q(a, K) - \delta q(u)].$$

Hence, by (29), we obtain (33).

II. $\sigma \in [\delta, 1-\delta)$. From (iv) we have (13). Applying (34), (32) and (13), we get

$$\varrho \geq \frac{1}{3} q\Big(\frac{1}{2}\sum_{i=1}^s k_i w_i\Big) \geq \frac{1}{6}\delta^{-1}[\delta(1-\delta)]^{1/2}.$$

Hence, by (30), we obtain (33).

III. $\sigma \in [1-\delta, 1)$. Let m be the integer part of $1 + \sum_{i=1}^r k_i$. As in I, we have

$$\varrho \geq (1-\delta) q\Big(a - b - \frac{1}{r}\Big(\sum_{i=1}^r k_i\Big)u\Big) = (1-\delta)[q(a-b-mu) - (1-\sigma)q(u)]$$
$$\geq (1-\delta)[d_q(a, K) - \delta q(u)] \geq d_q(a, K) - \varepsilon.$$

(c) Let $\{u_j\}_{j=1}^l$ be a set of generators of K. By (b), we can find some $v_1^1, \ldots, v_{s_1}^1 \in B_p$ such that $v_1^1 + \ldots + v_{s_1}^1 = u_1$ and

$$d_q(a, K + \mathbb{Z}v_1^1 + \ldots + \mathbb{Z}v_{s_1}^1) \geq d_q(a, K) - \frac{\varepsilon}{l}.$$

Applying (b) once again, we find some $v_1^2, \ldots, v_{s_2}^2 \in B_p$ such that $v_1^2 + \ldots + v_{s_2}^2 = u_2$ and

$$d_q(a, K + \mathbb{Z}v_1^1 + \ldots + \mathbb{Z}v_{s_1}^1 + \mathbb{Z}v_1^2 + \ldots + \mathbb{Z}v_{s_2}^2) \geq d_q(a, K + \mathbb{Z}v_1^1 + \ldots + \mathbb{Z}v_{s_1}^1) - \frac{\varepsilon}{l}.$$

Then we proceed by induction. In the kth step, having constructed vectors v_i^j for all $j = 1, \ldots, k-1$ and $i = 1, \ldots, s_j$, we can find vectors $v_1^k, \ldots, v_{s_k}^k \in B_p$ such that $v_1^k + \ldots + v_{s_k}^k = u_k$ and

$$d_q(a, K + \mathrm{gp}\{v_i^j\}_{j=1,i=1}^{k\ \ s_j}) \geq d_q(a, K + \mathrm{gp}\{v_i^j\}_{j=1,i=1}^{k-1\, s_j}) - \frac{\varepsilon}{l}.$$

After l such steps we shall obtain some vectors v_i^j for $j = 1, \ldots, l$ and $i = 1, \ldots, s_j$. Set $K' = \mathrm{gp}\{v_i^j\}_{j=1,i=1}^{l\ \ s_j}$. It follows directly from our construction that $K \subset K'$, and that (16) and (17) are satisfied. This proves (c). ∎

LEMMA 5. *Let E be a locally convex space. Suppose that there exists an $\varepsilon > 0$ with the following property: to each continuous seminorm q on E there corresponds another continuous seminorm $p \geq q$ such that $\upsilon_n(p, q) = o(n^{-\varepsilon})$. Then E is nuclear.*

This is Lemma 2 of [1].

A topological group G is said to be *locally generated* if $\mathrm{gp}\, U = G$ for each neighbourhood U of the neutral element.

LEMMA 6. *Let E be a metrizable locally convex space. If E is not nuclear, then it contains a locally generated subgroup K such that $(\frac{1}{2}K) \setminus \overline{K} \neq \emptyset$.*

Proof. We can find a sequence $p_0 \leq p_1 \leq p_2 \leq \ldots$ of continuous seminorms on E such that $\{B_{p_k}\}_{k=1}^{\infty}$ is a basis of neighbourhoods of zero in E. If E is not nuclear, then, according to Lemma 5, there is an index k_0 such that

$$\limsup_{n\to\infty} n^{0.1}\upsilon_n(p_k, p_{k_0}) > 1 \quad \text{for all } k \geq k_0. \tag{35}$$

We may assume that $k_0 = 0$.

Choose some $a \in E$ with $p_0(a) = 2$ and set $K_0 = 2\mathbb{Z}a$. Then $d_{p_0}(a, K) = 2$. By (35) and Lemma 4(c), we can find some finitely generated subgroup $K_1 \supset K_0$ of E such that $K_1 = \mathrm{gp}(K_1 \cap B_{p_1})$ and $d_{p_0}(a, K_1) \geq d_{p_0}(a, K_0) - 1/2$. Next, we can find a finitely generated subgroup $K_2 \supset K_1$ of E such that $K_2 = \mathrm{gp}(K_2 \cap B_{p_2})$ and $d_{p_0}(a, K_2) \geq d_{p_0}(a, K_1) - 1/4$. Then we proceed by induction. In the kth step, having constructed K_{k-1}, we find, applying (35) and Lemma 4(c), a finitely generated subgroup $K_k \supset K_{k-1}$ of E such that $K_k = \mathrm{gp}(K_k \cap B_{p_k})$ and $d_{p_0}(a, K_k) \geq d_{p_0}(a, K_{k-1}) - 2^{-k}$.

Set $K = \bigcup_{k=0}^{\infty} K_k$. We have $a \in \frac{1}{2}K_0 \subset \frac{1}{2}K$. It follows directly from our construction that K is locally generated and

$$d_{p_0}(a, K) \geq d_{p_0}(a, K_0) - \sum_{k=1}^{\infty} 2^{-k} = 2 - 1 = 1.$$

Thus $a \notin \overline{K}$, so that $(\frac{1}{2}K) \setminus \overline{K} \neq \emptyset$. ∎

Remark 1. If K is a locally generated subgroup of a nuclear space E, then $\overline{K}$ is a linear subspace of E (see [3]).

Let $\sum u_n$ be a convergent series in a topological vector space E. For each $m \in \mathbb{N}$, let Z_m denote the closure of the set of all points of the form $\sum_{n\in N} u_n$ where N is a finite subset of $\{m, m+1, \ldots\}$. We define $\mathfrak{A}(\sum u_n) = \bigcap_{m=1}^{\infty} Z_m$. The set $\mathfrak{A}(\sum u_n)$ is a closed additive subgroup of E (see e.g. [2], Lemma 5).

LEMMA 7. *Let E be a metrizable vector space and K a locally generated subgroup of E such that $(\frac{1}{2}K) \setminus \overline{K} \neq \emptyset$. Then there exists in E a convergent series $\sum u_n$ such that $\mathfrak{A}(\sum u_n)$ is not a linear subspace of E.*

Proof. Choose some $a \in (\frac{1}{2}K) \setminus \overline{K}$. Let $U_1 \supset U_2 \supset \ldots$ be a basis of neighbourhoods of zero in E. For each $m = 1, 2, \ldots$, we can find some vectors $w_1^m, \ldots, w_{k_m}^m \in U_m \cap K$ with $w_1^m + \ldots + w_{k_m}^m = 2a$ because K is locally generated. Consider the following series:

$$\sum_{n=1}^{\infty} u_n = w_1^1 - w_1^1 + w_2^1 - w_2^1 + - \ldots + w_{k_1}^1 - w_{k_1}^1 + w_1^2 - w_1^2 + w_2^2 - w_2^2 + - \ldots + w_{k_2}^2 - w_{k_2}^2 + - \ldots$$

It is clear that $\mathfrak{A}(\sum u_n)$ contains the subgroup $2\mathbb{Z}a$. On the other hand, $a \notin \mathfrak{A}(\sum u_n)$ because, evidently, $\mathfrak{A}(\sum u_n) \subset \overline{K}$. ∎

Remark 2. If $\sum u_n$ is a convergent series in a nuclear space E, then $\mathfrak{A}(\sum u_n)$ is a closed linear subspace of E; see e.g. [2], Lemma 6.

Proof of Theorem. Let E be a metrizable locally convex space which is not nuclear. According to Lemmas 6 and 7, there exists a convergent series $\sum u_n$ in E such that $\mathfrak{A}(\sum u_n)$ is not a linear subspace. The definition of $\mathfrak{A}(\sum u_n)$ implies immediately that $\mathfrak{A}(\sum u_n) \subset \Gamma_0(\sum u_n)$. It is a standard fact that $\mathfrak{S}(\sum u_n) \subset \mathfrak{A}(\sum u_n) + \sum u_n$ (see e.g. [2], Lemma 5). Thus $\mathfrak{S}(\sum u_n) \neq \Gamma_0(\sum u_n) + \sum u_n$. ∎

Remark 3. The author does not know whether the following statement is true: if a metrizable locally convex space is not nuclear, then it contains a convergent series $\sum u_n$ such that $\mathfrak{S}(\sum u_n)$ is not a linear manifold.

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Some integral and maximal operators related to starlike sets

by

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Abstract. We prove two-weight norm estimates for fractional integrals and fractional maximal functions associated with starlike sets in Euclidean space. This is seen to include general positive homogeneous fractional integrals and fractional integrals on product spaces. We consider both weak type and strong type results, and we show that the conditions imposed on the weight functions are fairly sharp.

0. Introduction. This paper is concerned with studying weighted norm inequalities for certain generalizations of the Riesz fractional integral operators and associated maximal operators. One such operator is the following: on $\mathbb{R}^n$, $n > 1$, define

$$I_{\alpha,\beta} f(x) = f * k_{\alpha,\beta}(x) ,$$

where

$$k_{\alpha,\beta}(x) = \frac{1}{|x|^{n-1-\alpha}|x_n|^{1-\beta}} , \qquad (0.1)$$

for $x = (x_1, \ldots, x_n)$. Here, $-\beta < \alpha < n-1$ and $0 < \beta < 1$. We may think of these operators as interpolating between an n-dimensional Riesz fractional integral operator and a 1-dimensional Riesz fractional integral operator in the last coordinate.

We shall derive results for our operators from corresponding results for more standard operators. For example, we derive weak and strong type estimates for $I_{\alpha,\beta}$ from corresponding results for the ordinary fractional integral $I_{\alpha+\beta}$. The necessary requirement for this derivation is that we have precise control over the operator norms of the standard operators in terms of the constants appearing in the conditions on the weights.

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