[O] S. Ogawa, *The stochastic integral of noncausal type as an extension of the symmetric integrals*, Japan J. Appl. Math. 2 (1985), 229–240.

[R] S. Ropela, *Spline bases in Besov spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 24 (1976), 319–325.

[Ro] B. Roynette, *Mouvement brownien et espaces de Besov*, Stochastics and Stochastics Rep. (1993), à paraître.

[S] P. Sjögren, *Riemann sums for stochastic integrals and L^p moduli of continuity*, Z. Wahrsch. Verw. Gebiete 59 (1982), 411–424.

[T] M. Talagrand, *Sur l'intégrabilité des vecteurs gaussiens*, ibid. 68 (1984), 1–8.

INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
ABRAHAMA 18
81-825 SOPOT, POLAND
E-mail: MATZC@HALINA.UNIV.GDA.PL

FACULTÉ DE MATHÉMATIQUES
ET INFORMATIQUE
UNIVERSITÉ DE PICARDIE
33, RUE SAINT LEU
80039 AMIENS CEDEX 1, FRANCE

DÉPARTEMENT DE MATHÉMATIQUES
UNIVERSITÉ DE NANCY
B.P. 239
54506 VANDŒUVRE-LÈS-NANCY CEDEX, FRANCE

Received January 4, 1993
Revised version April 14, 1993 (3045)

STUDIA MATHEMATICA 107 (2) (1993)

On first integrals for polynomial differential equations on the line

by

HENRYK ŻOŁĄDEK (Warszawa)

Abstract. We show that any equation $dy/dx = P(x,y)$ with P a polynomial has a global (on $\mathbb{R}^2$) smooth first integral nonconstant on any open domain. We also present an example of an equation without an analytic primitive first integral.

1. Introduction. The first integrals for equations

$$\frac{dy}{dx} = f(x,y), \qquad (x,y) \in \mathbb{R}^2, \tag{1}$$

were studied by T. Ważewski [8], Z. Szmydtówna [7] and by J. Szarski [6]. It was shown that even when f is infinitely differentiable there can be no differentiable first integral $F(x,y)$ different from a constant.

Recently K. Krzyżewski (University of Warsaw) asked about the situation with polynomial right hand side of (1). The answer is contained in this paper.

Let me give a little explanation of the origin of nondifferentiability of first integrals. If $y = \phi(x, y_0)$ is the solution of (1) with the initial condition $\phi(0, y_0) = y_0$ on the line $L_0 = \{x = 0\}$ then solving the latter equation with respect to y_0 we get the first integral

$$F(x,y) = y_0(x,y).$$

F is defined in the domain $U_0 = \{(x, \psi(x)) \colon \psi$ a solution of (1), 0 in the domain of $\psi\}$. We want to extend it to the whole $\mathbb{R}^2$ in a smooth way. The obstacles to such extension lie in the escaping of the solutions of (1) to infinity in finite time (x).

If a trajectory γ_0 from the boundary ∂U_0 of U_0 tends to infinity ($+\infty$ or $-\infty$) and does not intersect the line L_0 then we can extend F to a neighbourhood of γ_0 by choosing a section T_0 transversal to γ_0, extending $F|_{T_0 \cap U_0}$ to T_0 and then extending F from $T_0 \setminus U_0$ along the trajectories

1991 *Mathematics Subject Classification*: Primary 34C05.

Supported by Polish KBN grant No 2 1090 91 01.

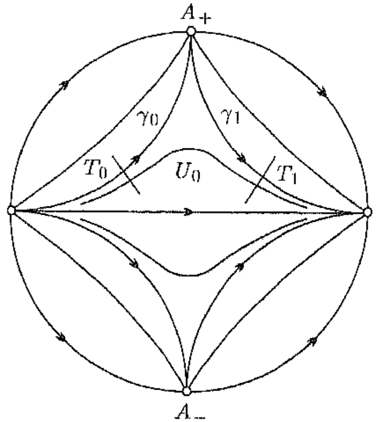

Fig. 1

of (1) to some domain U_1. (See Figure 1 where the interior of the disc represents $\mathbb{R}^2$). The problem appears when ∂U_0 contains another trajectory γ_1 going from infinity (the same which γ_0 tends to) separating U_0 from some domain U_2 and such that the trajectories going through $T_0 \cap U_0$ near $T_0 \cap \gamma_0$ run also near γ_1. In this case U_0 near infinity forms the so called *hyperbolic sector at infinity*; we denote it by S. Let T_1 be a section transversal to γ_1. Then there is the *correspondence map* $\Delta_S : T_0 \cap U_0 \to T_1 \cap U_0$ such that

$$F(p) = F(\Delta_S(p)), \qquad p \in T \cap U_0 .$$

The correspondence map may have very bad properties but if the equation (1) has analytic right hand side then Δ_S is analytic outside $T_0 \cap \gamma_0$. So, F is analytic in $U_0 \cup U_1$ but often cannot be continued analytically to $\gamma_0 \cup \gamma_1$.

We see that F can be continued analytically to $\mathbb{R}^2 \setminus \bigcup \gamma_i$, the sum over all *separatrices-boundaries* γ_i of hyperbolic sectors at infinity. Our task is to modify F in such a way that it becomes smooth on $\bigcup \gamma_i$.

We note that Kaplan also studied first integrals for analytic and rational systems. In particular, he proved in [3] that any analytic or rational nonautonomous system in $\mathbb{C}^n$ has a complete system of first integrals which are analytic outside a set of Lebesgue measure zero. (This, however, does not imply the results of the present work because he does not exclude the possibility of the set of nonanalyticity of the integral being dense in $\mathbb{R}^2$). In [4] he found some conditions ensuring regularity of the correspondence map Δ_S of a hyperbolic sector.

2. The result. We begin with some definitions.

We recall that a *first integral* for the equation (1) is a function $F : \mathbb{R}^2 \to \mathbb{R}P^1 = \mathbb{R}^1 \cup \{\infty\}$ which is constant on the integral curves $\{(x, \phi(x))\}$ of (1).

In what follows we assume that (1) is analytic and we only consider first integrals which are continuous and *piecewise smooth*. The latter means that $\mathbb{R}^2 = \bigcup_{j=1}^N \overline{U}_j$, $N \in \mathbb{N} \cup \{\infty\}$, where each U_j is open with piecewise smooth boundary $\partial U_j = \bigcup_{k=1}^{m_j} \gamma_{jk}$, $m_j \le \infty$, with γ_{jk} integral curves of (1), $\overline{U}_j = U_j \cup \partial U_j$, and F restricted to each U_j is smooth. The values $f_{ij} = F(\gamma_{ij})$ are called the *critical values* of F.

Of course, the first integral F is not unique. The functions $\Phi \circ F$, where $\Phi : \mathbb{R}P^1 \to \mathbb{R}P^1$, are also first integrals. We say that a first integral G is *induced* from F iff $G = \Phi \circ F$ for a piecewise smooth map Φ. We say that F is *primitive* iff any integral H from which F is induced ($F = \Psi \circ H$) is also induced from F, i.e. Ψ is a piecewise smooth homeomorphism and H is equivalent to F. (Primitive integrals are maximal elements in the partial ordering defined by the induction relation).

THEOREM 1. (a) *Every equation*

$$\frac{dy}{dx} = P(x, y) \tag{2}$$

with P a polynomial has a smooth first integral which is not constant on any open domain.

(b) *There are polynomial equations for which no primitive first integral is of class C^1.*

Remark 1. A desirable property of the first integral is $\partial F/\partial y \ne 0$ and

$$\frac{\partial F}{\partial x} + f(x, y)\frac{\partial F}{\partial y} = 0 ,$$

F of class C^1 (see [6]–[8]). The example from the next section (showing part (b) of Theorem 1) also shows that there are polynomial equations which do not have integrals with this property.

Remark 2. Part (a) of Theorem 1 is not true when we replace the polynomial right hand side by a smooth right hand side. The reason is that there are smooth equations for which any primitive first integral has the set of critical values dense in $\mathbb{R}P^1$. That was also the main idea in the papers [6]–[8].

The author thinks that there should exist analogous examples with analytic right hand side. Probably the technique from [8] combined with some approximation theorems would give the result. We shall not attempt this here.

3. Example of a nonsmooth primitive integral

Proof of Theorem 1(b). The equation is quartic:

$$\frac{dy}{dx}=\frac{y^2(xy+2\lambda-1)}{\lambda(\lambda-1)},\qquad 0<\lambda<1,$$

λ irrational. This equation has a first integral of Darboux type,

$$H=y|xy+\lambda-1|^{-\lambda}|xy+\lambda|^{\lambda-1}.$$

From this and from Figure 1 it is seen that the correspondence map $\Delta: T_0\cap U_0\to T_1\cap U_0$ is of the form

$$s\to s^{\lambda/(1-\lambda)}(1+o(1)).$$

Any primitive first integral has a nondifferentiable singularity on $\gamma_0\subset\{xy+\lambda=0\}$ or on $\gamma_1\subset\{xy+\lambda-1=0\}$. This proves part (b) of Theorem 1.

Remark 3. There are examples of equations (2) of degree 8 whose phase portraits contain a hyperbolic sector at infinity with flat correspondence map, $s\to e^{-1/s}$.

4. The Poincaré compactification. Integral curves of the rational equation $dy/dx=P(x,y)/N(x,y)$ are phase curves of the polynomial system

(3) $$\dot{x}=N(x,y),\qquad \dot{y}=P(x,y).$$

Equation (3) defines a field of directions in $\mathbb{R}^2$. This field of directions extends analytically to $\mathbb{R}P^2=\mathbb{R}^2\cup\mathbb{R}P^1$, where $\mathbb{R}P^2=\{[x,y,z]\}$ with $\mathbb{R}^2$ as $\{[x,y,1]\}$ and $\mathbb{R}P^1=\{z=0\}$ is the line at infinity. Near $z=0$ we have the projective coordinates $z=1/y$, $u=x/y$ and $\widetilde{z}=1/x$, $\widetilde{u}=y/x$ and the system (3) changes to

$$\dot{z}=-z^2P(u/z,1/z),\qquad \dot{u}=z[N(u/z,1/z)-uP(u/z,1/z)]$$

(we get a similar system in the other coordinates). Multiplying it by a suitable power of z (or $\widetilde{z}$) we get again a polynomial vector field.

The projective plane $\mathbb{R}P^2$ has the sphere S^2 as the double covering with the natural antipodal map $\pi: S^2\to\mathbb{R}P^2$. The field of directions from $\mathbb{R}P^2$ transforms to S^2. It is just the *Poincaré compactification*. S^2 consists of two half-planes with the same phase portraits and hence it is convenient to draw the Poincaré compactification in the form of a disc with the circle boundary as the line at infinity.

For the vector field

(4) $$\dot{x}=1,\quad \dot{y}=P(x,y),\quad \deg P=n,$$

we get

(5) $$\dot{z}=-z\widetilde{P}(z,u),\quad \dot{u}=z^n-u\widetilde{P}(z,u),\quad \widetilde{P}(z,u)=z^nP\left(\frac{u}{z},\frac{1}{z}\right).$$

If $n\neq 0$ then the singular points at infinity are given by $z=0, u\widetilde{P}(0,u)=0$ (and $\widetilde{z}=\widetilde{u}=0$ ($u=\infty$) if $\widetilde{P}(0,0)=0$).

Of course, for $n=0,1$ equation (2) is integrable with an analytic first integral. Therefore we shall always assume that $n\geq 2$.

Notice that the point

$$A: z=u=0$$

is always singular, independently of $P(x,y)$. It represents two singular points A_+ and A_- in the Poincaré plane (see Figure 1).

Only the point A is important for the analysis of the first integral. This is because the asymptotic behaviour of (4) near any singular point $z=0, u=u_0\neq 0$ (including $u_0=\infty$) is such that $\dot{x}/\dot{y}\to u_0$ as $|x|,|y|\to\infty$, $x/y\to u_0$. Then dy/dx is bounded and the definition of the first integral described in the introduction works without obstacles (no escaping to infinity in finite time).

5. The resolution of singularity. For the first time the theorem about resolution of a singular point of a vector field appeared in the paper of Bendixson [2], but the final proof was given in the 80s by Van den Essen (see [5]).

THEOREM 2 (Resolution of singularity). *Let $0\in\mathbb{R}^2$ be an isolated singular point of an analytic vector field V defined in a neighbourhood of 0. Then there exists a manifold M with distinguished divisors $E_j\simeq\mathbb{R}P^1$, $j=1,\ldots,r$, a vector field $\widetilde{V}$ on M and a map $\Pi: M\to\mathbb{R}^2$ such that*

(i) *$\Pi(\bigcup E_j)=\{0\}$, $\Pi|_{M\setminus\bigcup E_j}$ is an analytic diffeomorphism, and $\Pi(M\setminus\bigcup E_j)=\mathbb{R}^2\setminus\{0\}$;*

(ii) *if $E_i\cap E_j\neq\emptyset$ then they intersect transversally only at one point;*

(iii) *$\Pi_*\widetilde{V}=f\cdot(V\circ\Pi)$, where $f=\prod\phi_j^{d_j}$, ϕ_j are linear functions such that $\{\phi_j=0\}=E_j$ and d_j are integers;*

(iv) *either E_i is transversal to the phase curves of $\widetilde{V}$ or E_i is an invariant curve for $\widetilde{V}$ with singular points on $E_i\cap E_j$ or/and at a finite number of other points;*

(v) *all singular points are elementary (i.e. with at least one nonzero eigenvalue).*

Applying the resolution (also called the σ-process) of the singularity A allows us to control the behaviour of the integral curves of (4) near infinity. In particular, we get the following.

COROLLARY 1. *The plane $\mathbb{R}^2$ can be divided into a finite number of domains $\overline{U}_j$, $j=1,\ldots,N$, with boundaries $\partial U_j=\bigcup\gamma_{jk}$ consisting of trajectories γ_{jk}, $k=1,\ldots,m_j<\infty$, forming the separatrices-boundaries of* *hyperbolic sectors at A_+ or A_-. In particular, there are only a finite number of hyperbolic sectors.*

Remark 4. For the hyperbolic singular point $\dot{x} = x(1 + \ldots)$, $\dot{y} = y(-\lambda + \ldots)$ the correspondence map between transversals to the separatrices is of the form $s \to s^\lambda + \sum a_{jl} s^{\nu_j} (\ln s)^l$ (Dulac series). For the elementary nonhyperbolic singular point $\dot{x} = x(x^k + \ldots)$, $\dot{y} = y(\lambda + \ldots)$ the correspondence map is flat $s \to e^{-1/h}$, where $h = ks^k/\lambda + \ldots$ is a Dulac series (see [1, Chapter 6.4]). The correspondence map Δ_S of a hyperbolic sector at infinity is a composition of maps which are either of one of the above two forms or are their inverses.

6. Proof of Theorem 1(a). Using Corollary 1 we can define some first integral F_0. We divide the set $\{\gamma_{jk}\}$ of separatrices into equivalence classes by means of the equivalence relation generated by equivalence of separatrices of hyperbolic sectors at infinity. F_0 is a function with the following properties:

- it is nonconstant analytic in each U_j and constant on the trajectories of (1),
- it is continuous on $\mathbb{R}^2$,
- it has nonzero gradient on one of γ_{jk} from each equivalence class.

The differentiability properties may fail on other trajectories separating two domains U_i. Let $f_{jk} = F_0(\gamma_{jk})$ be the critical values of F_0.

Let $\Phi : \mathbb{R}P^1 \to \mathbb{R}P^1$ be a smooth map which:

(i) keeps f_{jk} fixed;
(ii) is a diffeomorphism of $\mathbb{R}P^1 \setminus \{f_{jk}\}$;
(iii) near f_{jk} is of the form

$$f \to f_{jk} + \Psi^M (f - f_{jk}),$$

where $\Psi^M = \Psi \circ \ldots \circ \Psi$ (M times), $\Psi(x) = \exp[-\exp(1/x)]$ and M is the number of singular points in $\bigcup E_j$ of the resolution of the singularity of the system (4) at A.

Of course, $\Psi(x) \ll \exp(-\mu x^{-k})$. This property shows that the first integral

$$F = \Phi \circ F_0$$

is flat on the separatrices γ_{jk}. Hence, F is smooth on $\mathbb{R}^2$.

Acknowledgements. The author thanks Professor K. Krzyżewski for introducing him into the problem and providing him with the literature references and to Professor J.-M. Strelcyn for showing him the works of Kaplan.

References

[1] V. I. Arnold and S. Yu. Ilyashenko, *Ordinary Differential Equations, Dynamical Systems I*, Encyclopedia Math. Sci. 1, Springer, Berlin, 1988. (Russian original: Fundamental Directions, Dynamical Systems I, VINITI, Moscow, 1985.)

[2] I. Bendixson, *Sur les courbes définies par des équations différentielles*, Acta Math. 25 (1901), 1–88.

[3] W. Kaplan, *Analytic first integrals of ordinary differential equations*, Comment. Math. Helv. 47 (1972), 205–212.

[4] —, *On partial differential equations of first order*, in: Topics in Analysis, Jyväskylä 1970, O. Lehto *et al.* (eds.), Lecture Notes in Math. 419, Springer, Berlin, 1974, 221–231.

[5] J.-F. Mattei et R. M. Moussu, *Holonomie et intégrales premières*, Ann. Sci. Ecole Norm. Sup. (4) 13 (1980), 469–513.

[6] J. Szarski, *Sur un problème de caractère intégral relatif à l'équation* : $\frac{\partial z}{\partial x} + Q(x,y)\frac{\partial z}{\partial y} = 0$ *définie dans le plan tout entier*, Ann. Soc. Polon. Math. 19 (1946), 106–132.

[7] Z. Szmydtówna, *Sur les intégrales premières de l'équation* $y' = f(x,y)$, ibid. 23 (1950), 167–182.

[8] T. Ważewski, *Sur un problème de caractère intégral relatif à l'équation* $\frac{\partial z}{\partial x} + Q(x,y)\frac{\partial z}{\partial y} = 0$, Mathematica (Cluj) 8 (1933), 103–116.

INSTITUTE OF MATHEMATICS
UNIVERSITY OF WARSAW
BANACHA 2
02-097 WARSZAWA, POLAND

Received January 19, 1993
Revised version May 19, 1993
(3050)