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A partial differential operator which is surjective on Gevrey classes $\Gamma^d(\mathbb{R}^3)$ with $1 \le d < 2$ and $d \ge 6$ but not for $2 \le d < 6$

by

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Abstract. It is shown that the partial differential operator $P(D) = \partial^4/\partial x^4 - \partial^2/\partial y^2 + i\partial/\partial z : \Gamma^d(\mathbb{R}^3) \to \Gamma^d(\mathbb{R}^3)$ is surjective if $1 \le d < 2$ or $d \ge 6$ and not surjective for $2 \le d < 6$.

1. Introduction. Following a conjecture of De Giorgi and Cattabriga [7], it has been shown by Piccinini [10] that the heat operator is not surjective on the space of all real-analytic functions on $\mathbb{R}^3$. Hörmander [8] then characterized the surjective partial differential operators with constant coefficients on the space of all real-analytic functions on a convex open set in $\mathbb{R}^N$. The condition is whether or not on the variety of the corresponding polynomial an analogue of the classical Phragmén–Lindelöf principle holds. Concerning this problem in nonquasianalytic Gevrey classes Γ^d, Cattabriga [5], [6], has investigated the heat equation, and he [5] and Zampieri [11] have given sufficient conditions for surjectivity. A characterization of the surjectivity, which is again expressed in terms of a Phragmén–Lindelöf condition, has been established by Braun, Meise, and Vogt [3], [4].

Cattabriga's result [5], [6] for the heat operator is that it is surjective for $d \ge 2$, but not for $1 < d < 2$. In [4], it is shown that other operators, including the Schrödinger operator, show a similar effect. Switching of the behavior of a Phragmén–Lindelöf condition when changing the Gevrey exponent d has also been noted in the context of right inverses. The existence of a continuous linear right inverse has been shown by Meise, Taylor, and Vogt [9] to be equivalent to some Phragmén–Lindelöf condition. In this case, it may happen that $P(D)$ has a right inverse for small d, but not for the others. In all these cases, the behavior may change only once. In contrast to this, we present here a partial differential operator that is surjective for small as well as for large d, but not for exponents in between. To be precise,

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we show that the operator

$$P(D) = \frac{\partial^4}{\partial x^4} - \frac{\partial^2}{\partial y^2} + i\frac{\partial}{\partial z} : \Gamma^d(\mathbb{R}^3) \to \Gamma^d(\mathbb{R}^3)$$

is onto if and only if $d \in [1,2[\cup [6,\infty[$. This answers the question whether there are operators with more than one switch, that Professor Rauch posed during a discussion in Ann Arbor in 1988.

The proof relies on the characterization of Braun, Meise, and Vogt [3], [4]. First, we derive from this two more tractable conditions, one of them necessary and one sufficient, which we then apply to the operator $P(D)$ above.

2. Notation. We denote by $\mathbb{N} = \{1, 2, \ldots\}$ the natural numbers, by $|x|$ the Euclidean norm of $x \in \mathbb{C}^N$, and by $U_r(x)$ the open neighborhood of x with radius r. For a polynomial $P \in \mathbb{C}[Z_1, \ldots, Z_N]$ we denote by $V = P^{-1}(0)$ its variety and by $P(D)$, $D = -i\partial$, the corresponding differential operator. We say that a function on an open subset G of the variety V is plurisubharmonic if it is plurisubharmonic in the regular points of G and locally bounded everywhere. We denote by $\mathrm{PSH}(G)$ the set of all plurisubharmonic functions on G.

We consider the same classes of ultradifferentiable functions as Braun, Meise, and Taylor [2].

3. Definition. A continuous function $\omega : \mathbb{C}^N \to [0,\infty[$ depending only on $|z|$ is called a *weight function* if it satisfies

(a) $\omega(2t) = O(\omega(t))$, (b) $\int_0^\infty \frac{\omega(t)}{1+t^2}\,dt < \infty$,

(c) $\log = o(\omega)$, (d) $\varphi : t \mapsto \omega(e^t)$ is convex on $\mathbb{R}$.

We define the Young conjugate φ^* of φ by

$$\varphi^*(y) = \sup_{x\ge 0} xy - \varphi(x).$$

4. Definition. For $K > 0$ we define

$$\mathcal{E}_{\{\omega\},K} = \left\{ f \in C^\infty(\mathbb{R}^N) \;\middle|\; \sup_{\substack{|x|\le K\\ \alpha\in\mathbb{N}_0^N}} |f^{(\alpha)}(x)| \exp\left(-\frac{1}{m}\varphi^*(|\alpha|m)\right) < \infty \right.$$
$$\left.\text{for some } m \in \mathbb{N}\right\}.$$

We endow $\mathcal{E}_{\{\omega\},K}$ with the inductive limit topology, and we set

$$\mathcal{E}_{\{\omega\}}(\mathbb{R}^N) = \operatorname*{proj}_{\leftarrow K} \mathcal{E}_{\{\omega\},K}.$$

5. Remark. For $\omega(z) = |z|^{1/d}$, $d \ge 1$, the space $\mathcal{E}_{\{\omega\}}$ is the classical Gevrey class Γ^d of order d.

Once a weight function ω is fixed, we set for $\mu > 0$

$$\mathrm{PSH}_\mu(V) = \{\varphi \in \mathrm{PSH}(V) \mid \varphi(z) - \mu|\mathrm{Im}\, z| = o(\omega(z)) \text{ as } |z| \to \infty, z \in V\}.$$

Braun, Meise, and Vogt [3], [4], have given the following characterization of the surjective partial differential operators on $\mathcal{E}_{\{\omega\}}(\mathbb{R}^N)$.

6. Theorem. *The operator $P(D) : \mathcal{E}_{\{\omega\}}(\mathbb{R}^N) \to \mathcal{E}_{\{\omega\}}(\mathbb{R}^N)$ is surjective if and only if, for all $\mu > 0$, there are $\delta, k > 0$ such that, for all $L, \varepsilon > 0$, there are $\eta, C > 0$ such that, for all $\varphi \in \mathrm{PSH}_\mu(V)$ satisfying (α) and (β), also (γ) holds, where*

(α) *for all $z \in V$, $\varphi(z) \le \mu|\mathrm{Im}\, z| + \delta\omega(z)$,*
(β) *for all $z \in V$, $\varphi(z) \le L|\mathrm{Im}\, z| + \eta\omega(z)$,*
(γ) *for all $z \in V$, $\varphi(z) \le k|\mathrm{Im}\, z| + \varepsilon\omega(z) + C$.*

7. Proposition. *The following condition implies the surjectivity of $P(D) : \mathcal{E}_{\{\omega\}}(\mathbb{R}^N) \to \mathcal{E}_{\{\omega\}}(\mathbb{R}^N)$:*

(S) *There are $K, A > 0$ such that, for $a > 0$, there is $R > 0$ such that, for all $z \in V$ with $|z| \ge R$ and $a\omega(z) \le |\mathrm{Im}\, z| < A\omega(z)$, there are $D, t > 0$ and (for some k) a k-valued holomorphic function $f : \{\zeta \in \mathbb{C} \mid |\zeta| < D, \mathrm{Im}\,\zeta > 0\} \to \mathbb{C}^N$ with*

- (S1) *if $|\zeta| < D$, $\mathrm{Im}\,\zeta > 0$, and $w \in f(\zeta)$, then $w \in V$ and $|w - z| < \omega(z)$,*
- (S2) *for all sequences $(\zeta_n)_n$ in $\{\zeta \in \mathbb{C} \mid |\zeta| < D, \mathrm{Im}\,\zeta > 0\}$ with $\lim_{n\to\infty} \zeta_n \in]-D, D[$ we have $\limsup_{n\to\infty} |\mathrm{Im}\, f(\zeta_n)| \le a\omega(z)$, where $|\mathrm{Im}\, f(\zeta_n)|$ means the largest of the k possible values,*
- (S3) *$z \in f(it)$,*
- (S4) *$t\omega(z) \le D(K|\mathrm{Im}\, z| + a\omega(z))$.*

Proof. Let K, A be as in (S). Since we may enlarge K, we can assume that $KA \ge 1$. We fix $\mu \ge 1$, set $\delta = 1$ and $k = (4/\pi)(\mu(A+1) + \delta C_\omega)K$, where C_ω is a constant with $\omega(2z) \le C_\omega\omega(z)$ for $z \ge 2$. Now let L and ε be given. We apply (S) with $a = (\varepsilon\pi/8)(\mu(A+1) + \delta C_\omega + L)^{-1}$ and choose $\eta = \varepsilon(2C_\omega)^{-1}$. Let $\varphi \in \mathrm{PSH}_\mu(V)$ with (α) and (β) be given. We prove (γ) for $z \in V$ with $|z| > R$, where $R \ge 2$ is the constant appearing in (S). The other z are taken care of by choosing an appropriate C.

If $|\mathrm{Im}\, z| \ge A\omega(z)$, then by (α),

$$\varphi(z) \le \mu|\mathrm{Im}\, z| + \delta\omega(z) \le \left(\mu + \frac{\delta}{A}\right)|\mathrm{Im}\, z| \le k|\mathrm{Im}\, z|.$$

If $|\mathrm{Im}\, z| < a\omega(z)$, then by (β),

$$\varphi(z) \le L|\mathrm{Im}\, z| + \eta\omega(z) \le (La + \eta)\omega(z) \le \varepsilon\omega(z).$$

Thus we may assume $a\omega(z) \le |\operatorname{Im} z| < A\omega(z)$. Let D, t, and f be as in (S). Because of (S1), we can define a function by

$$\psi(\zeta) = \max_{w \in f(\zeta)} \varphi(w), \qquad |\zeta| < D,\ \operatorname{Im}\zeta > 0\,.$$

This function is upper semicontinuous and subharmonic at those points where the number of distinct values of $f(\zeta)$ is maximal. Thus by [8], 4.4, it is subharmonic. By (S1) we have, for $|\zeta| < D$,

$$|\operatorname{Im} f(\zeta)| \le |f(\zeta) - z| + |\operatorname{Im} z| \le \omega(z) + |\operatorname{Im} z|$$

and

$$\omega(f(\zeta)) \le \omega(|z| + \omega(z)) \le C_\omega \omega(z)\,,$$

provided R is reasonably large. Thus (α) implies

$$\begin{aligned}\psi(\zeta) &\le \mu|\operatorname{Im} f(\zeta)| + \delta\omega(f(\zeta)) \le \mu|\operatorname{Im} z| + (\mu + \delta C_\omega)\omega(z)\\ &\le (\mu(A+1) + \delta C_\omega)\omega(z)\,.\end{aligned}$$

For a sequence $(\zeta_n)_n$ tending to a point in $]-D, D[$, we have, by (β) and (S2),

$$\limsup_{n\to\infty} \psi(\zeta_n) \le La\omega(z) + \eta C_\omega \omega(z)\,.$$

Thus by a classical estimate for the harmonic measure of a half disk (see [1], proof of Theorem 3–4)

$$\psi(it) \le \frac{4}{\pi}\frac{\mu(A+1) + \delta C_\omega}{D}\omega(z)t + (La + C_\omega\eta)\omega(z)\,.$$

Because of (S3), (S4), and the choices of k, η, and a,

$$\begin{aligned}\varphi(z) \le \psi(it) &\le \frac{4}{\pi}(\mu(A+1) + \delta C_\omega)K|\operatorname{Im} z|\\ &\quad + \Big(\frac{4}{\pi}(\mu(A+1) + \delta C_\omega + L)a + C_\omega\eta\Big)\omega(z)\\ &= k|\operatorname{Im} z| + \varepsilon\omega(z)\,.\end{aligned}$$

Our necessary condition relies on pasting techniques of Hörmander [8], proof of 2.1. There, a variant of the following well known fact is used. It can be proved by solving a Dirichlet problem in one variable.

8. Lemma. *There are $c > 0$ and a plurisubharmonic function $H : D = \{z \in \mathbb{C}^N \mid |z| < 1\} \to \mathbb{R}$, continuous up to the boundary, such that*

$$H(z) \le |\operatorname{Im} z| \quad \text{for all } z \in D\,, \qquad H(z) \le |\operatorname{Im} z| - c \quad \text{for } |z| = 1\,,$$
$$H(iy) \ge 0 \quad \text{for } y \in D \cap \mathbb{R}^N\,.$$

9. Proposition. *Assume that V has the following property: For all $\varrho > 0$ there is a sequence $(\theta_n)_{n\in\mathbb{N}}$ in V with $\lim_{n\to\infty}|\theta_n| = \infty$ and*

(i) $0 < \liminf\limits_{n\to\infty} \dfrac{|\operatorname{Im}\theta_n|}{\omega(\theta_n)} \le \limsup\limits_{n\to\infty} \dfrac{|\operatorname{Im}\theta_n|}{\omega(\theta_n)} < \varrho$,

(ii) *for each $n \in \mathbb{N}$ there is an irreducible component V_n of $V \cap U_{\omega(\theta_n)}(\theta_n)$ that contains θ_n satisfying*

$$\liminf_{n\to\infty} \frac{\inf_{z\in V_n}|\operatorname{Im} z|}{|\operatorname{Im}\theta_n|} > 0\,.$$

Then the operator $P(D) : \mathcal{E}_{\{\omega\}}(\mathbb{R}^N) \to \mathcal{E}_{\{\omega\}}(\mathbb{R}^N)$ is not surjective.

Proof. Let c denote the constant of Lemma 8, and let C_ω be a constant with $\omega(2z) \le C_\omega\omega(z)$ for sufficiently large z. Set $\mu = 1$ and let δ and k' be given. For $\varrho = c(8k')^{-1}$ there are θ_n as in the hypothesis. A subsequence is enough, so we may assume the existence of

$$b := \lim_{n\to\infty} \frac{|\operatorname{Im}\theta_n|}{\omega(\theta_n)} \in\,]0, \varrho]\,.$$

We let $k = c(8b)^{-1}$. As k is larger than k', it suffices to disprove (γ) for k instead of k'. Let $2a$ denote the inferior limit in (ii). We choose $L = 4k/a + 1$ and $\varepsilon = kb/2$. Let $C > 0$ be arbitrary. There is n with $\omega(\theta_n) > C/\varepsilon$ and

$$\frac{1}{2}b\omega(\theta_n) < |\operatorname{Im}\theta_n| < 2b\omega(\theta_n), \tag{1}$$

$$\inf_{z\in V_n} |\operatorname{Im} z| \ge a|\operatorname{Im}\theta_n|\,. \tag{2}$$

We define $\psi \in \operatorname{PSH}(V \cap U_{\omega(\theta_n)}(\theta_n))$ by $\psi|V_n \equiv 4k|\operatorname{Im}\theta_n|$ and $\psi \equiv 0$ on the other components of $V \cap U_{\omega(\theta_n)}(\theta_n)$, if there are any. We let now for $z \in V$,

$$\varphi(z) = \begin{cases} \max\Big(|\operatorname{Im} z|, \psi(z) + \omega(\theta_n)H\Big(\dfrac{z - \operatorname{Re}\theta_n}{\omega(\theta_n)}\Big)\Big), & |z - \theta_n| < \omega(\theta_n),\\ |\operatorname{Im} z|, & |z - \theta_n| \ge \omega(\theta_n). \end{cases}$$

For $|z - \omega(\theta_n)| = \omega(\theta_n)$ we have

$$\psi(z) + \omega(\theta_n)H\Big(\frac{z - \operatorname{Re}\theta_n}{\omega(\theta_n)}\Big) \le 4k|\operatorname{Im}\theta_n| + (|\operatorname{Im} z| - c\omega(\theta_n)) < |\operatorname{Im} z|\,.$$

Thus $\varphi \in \operatorname{PSH}_\mu(V)$. For $|z - \theta_n| < \omega(\theta_n)$ we have

$$\begin{aligned}\psi(z) + \omega(\theta_n)H\Big(\frac{z - \operatorname{Re}\theta_n}{\omega(\theta_n)}\Big) &\le 4k|\operatorname{Im}\theta_n| + |\operatorname{Im} z| \le 8kb\omega(\theta_n) + |\operatorname{Im} z|\\ &\le \mu|\operatorname{Im} z| + \varepsilon\omega(z)\end{aligned}$$

and

$$\psi(z) + \omega(\theta_n)H\Big(\frac{z - \operatorname{Re}\theta_n}{\omega(\theta_n)}\Big) \le 4k|\operatorname{Im}\theta_n| + |\operatorname{Im} z| \le \Big(\frac{4k}{a} + 1\Big)|\operatorname{Im} z|\,.$$

So φ satisfies (α) and (β). However, it does not satisfy (γ), as the following estimate shows:

$$\varphi(\theta_n) = 4k|\operatorname{Im}\theta_n| + \omega(\theta_n)H\left(\frac{i\operatorname{Im}\theta_n}{\omega(\theta_n)}\right) \geq 4k|\operatorname{Im}\theta_n| > k|\operatorname{Im}\theta_n| + kb\omega(\theta_n)$$
$$= k|\operatorname{Im}\theta_n| + 2\varepsilon\omega(\theta_n) \geq k|\operatorname{Im}\theta_n| + \varepsilon\omega(\theta_n) + C\,.$$

10. THEOREM. *The operator*

$$(3)\qquad P(D) = \frac{\partial^4}{\partial x^4} - \frac{\partial^2}{\partial y^2} + i\frac{\partial}{\partial z} : \mathcal{E}_{\{\omega\}}(\mathbb{R}^3) \to \mathcal{E}_{\{\omega\}}(\mathbb{R}^3)$$

is surjective for weight functions ω with

(a) $\sqrt{\cdot} = o(\omega)$, (b) $\omega = O(\sqrt[6]{\cdot})$.

It is not surjective for weights ω with

(c) $\sqrt[4]{\cdot} = o(\omega)$ *and* $\omega = O(\sqrt{\cdot})$, (d) $\sqrt[6]{\cdot} = o(\omega)$ *and* $\omega = O(\sqrt[4]{\cdot})$.

11. Remark. For $d = 1$, i.e., on the space of all analytic functions, $P(D)$ is also surjective, since the hypothesis of Hörmander [8], 6.5, is easily verified.

12. COROLLARY. *The operator* $P(D) : \Gamma^d(\mathbb{R}^3) \to \Gamma^d(\mathbb{R}^3)$ *from* (3) *is surjective for* $d \in [1,2[\,\cup\,[6,\infty[$ *and not surjective for* $d \in [2,6[$.

The proof of Theorem 10 will be carried out in four steps.

13. Case $\sqrt{\cdot} = o(\omega)$. We verify condition (S) of Proposition 7. Let $A = 1/6$ and $K = 1/A$. Let $a > 0$ be arbitrary. Let $\theta \in V$ with large $|\theta|$ and $a\omega(\theta) \leq |\operatorname{Im}\theta| < A\omega(\theta)$ be given. We let $\theta = (x,y,z)$ and $D = A\omega(\theta)$, and for $|\zeta| < D$ we set

$$y(\zeta) = \operatorname{Re} y + \zeta\frac{\operatorname{Im} y}{1+|\operatorname{Im}(y,z)|}, \qquad z(\zeta) = \operatorname{Re} z + \zeta\frac{\operatorname{Im} z}{1+|\operatorname{Im}(y,z)|},$$
$$t = 1 + |\operatorname{Im}(y,z)|\,.$$

For large $|\theta|$, we have (S4) because of

$$t\omega(\theta) = D\left(\frac{1}{A}|\operatorname{Im}(y,z)| + \frac{1}{A}\right) \leq D(K|\operatorname{Im}\theta| + a\omega(\theta))\,.$$

Case 1: $|\operatorname{Re} y| < 8A\omega(\theta)$. Then for $|\zeta| < D$ and large $|\theta|$,

$$(4)\qquad |z(\zeta) - y(\zeta)^2| \leq |\theta| + A\omega(\theta) + (|\operatorname{Re} y| + A\omega(\theta))^2$$
$$\leq A^2\omega(\theta)^2 + A\omega(\theta) + 81A^2\omega(\theta)^2 \leq 83A^2\omega(\theta)^2\,.$$

We define the following 4-valued holomorphic functions:

$$(5)\qquad x(\zeta) = \sqrt[4]{z(\zeta) - y(\zeta)^2}, \qquad f(\zeta) = (x(\zeta), y(\zeta), z(\zeta))\,.$$

Note that (4) implies $|x(\zeta)| \leq a\omega(\theta)$ for large $|\theta|$. We have

$$|f(\zeta) - \theta| < 2a\omega(\theta) + 4A\omega(\theta) \leq \omega(\theta)\,.$$

This is (S1). For $\xi \in\,]-D, D[$, (S2) follows from $|\operatorname{Im} f(\xi)| \leq |x(\xi)| \leq a\omega(\theta)$. (S3) is obvious.

Case 2: $|\operatorname{Re} y| \geq 8A\omega(\theta)$. Then for $|\zeta| < D$ and large $|\theta|$,

$$(6)\qquad |y(\zeta)^2 - z(\zeta)| \geq (\operatorname{Re} y - A\omega(\theta))^2 - |\operatorname{Re} z| - A\omega(\theta)$$
$$\geq 49A^2\omega(\theta)^2 - |\theta| - A\omega(\theta) \geq 47A^2\omega(\theta)^2\,.$$

In particular, we can define the (single-valued) holomorphic functions

$$x(\zeta) = \sqrt[4]{z(\zeta) - y(\zeta)^2}, \quad f(\zeta) = (x(\zeta), y(\zeta), z(\zeta)), \quad \text{for } |\zeta| < D\,,$$

where we choose the branch with $x(it) = x$. We have seen in (6) that $|x(\zeta)| \geq \sqrt{6A\omega(\theta)}$. Also, for large $|\theta|$,

$$|x(\zeta)| > \sqrt[4]{(\operatorname{Re} y)^2 - 2|\operatorname{Re} y|A\omega(\theta) - A^2\omega(\theta)^2 - |\theta| - A\omega(\theta)}$$
$$\geq \sqrt[4]{(\operatorname{Re} y)^2\left(1 - \frac{2}{8} - \frac{1}{64} - \frac{1}{64}\right)} \geq \sqrt{\frac{|\operatorname{Re} y|}{2}}\,.$$

For $|\tau| < D$,

$$x'(\tau) = \frac{1}{4}x(\tau)^{-3}\left(\frac{\operatorname{Im} z}{1+|\operatorname{Im}(y,z)|} - 2(\operatorname{Re} y)\frac{\operatorname{Im} y}{1+|\operatorname{Im}(y,z)|} - 2\tau\left(\frac{\operatorname{Im} y}{1+|\operatorname{Im}(y,z)|}\right)^2\right),$$

$$|x'(\tau)| \leq \frac{1}{4}\left(1 + 2\frac{|\operatorname{Re} y|}{(|\operatorname{Re} y|/2)^{3/2}} + \frac{2A\omega(\theta)}{8(A\omega(\theta))^{3/2}}\right) \leq 1\,.$$

This shows $|x(\zeta) - x| \leq |\zeta - it| \leq 2A\omega(\theta)$ for $|\zeta| < D$. Thus (S1) holds. For $\xi \in\,]-D, D[$, we have $|\operatorname{Im} x(\xi)| \leq |x(\xi)| \leq \sqrt[4]{|\theta| + A\omega(\theta) + (|\theta| + A\omega(\theta))^2} = o(\omega(\theta))$, which shows (S2). Again, (S3) is clear.

14. Case $\omega = O(\sqrt[6]{\cdot})$. We will establish the validity of condition (S) of 7. Choose C_1, C_2 with

$$\omega(2t) \leq C_1\omega(t), \quad \omega(t) \leq C_2\sqrt[6]{t} \quad \text{for all } t \geq 2\,.$$

Let $A = \min(1/50, (72C_2)^{-1})$, set $K = 1/A$. Let $a > 0$ be given. Let $R > 0$ be large enough and let $\theta \in V$ with $|\theta| = R$ and $a\omega(R) < |\operatorname{Im}\theta| < A\omega(R)$ be given. We let $\theta = (x,y,z)$. At least one coordinate has modulus greater than $R/\sqrt{3}$. This is not x, since otherwise $|(y,z)| > R$. We show first that it is not y either, because then $\operatorname{Re}(y^2 - z) > R^2/4$ and thus $\operatorname{Re} x^4 < -R^2/4$. But now $|x| > \sqrt{R/2}$, while $|\operatorname{Im} x| < A\omega(R) = O(\sqrt[6]{R})$, thus $|\operatorname{Im} x| < |\operatorname{Re} x|/3$.

Then

$$\begin{aligned} -R^2/4 > \operatorname{Re} x^4 &= ((\operatorname{Re} x)^2 - (\operatorname{Im} x)^2)^2 - 4(\operatorname{Re} x)^2(\operatorname{Im} x)^2 \\ &> \frac{64}{81}(\operatorname{Re} x)^4 - \frac{4}{9}(\operatorname{Re} x)^4 > 0 . \end{aligned}$$

This is a contradiction, so $|y| < R/\sqrt{3}$. Thus, for large R,

$$\operatorname{Re} z \geq R/2 . \tag{7}$$

Set $D = A\omega(R)$ and $B = \{\zeta \in \mathbb{C} \mid |\zeta| < D, \operatorname{Im} \zeta > 0\}$. We distinguish two cases:

Case $|\operatorname{Re} y| > |\operatorname{Re} x|^3$. Then from (7) we get

$$\begin{aligned} (\operatorname{Re} y)^2 &= \operatorname{Re} y^2 + (\operatorname{Im} y)^2 = \operatorname{Re} z - \operatorname{Re} x^4 + (\operatorname{Im} y)^2 \\ &= \operatorname{Re} z - (\operatorname{Re} x)^4 + 6(\operatorname{Re} x)^2(\operatorname{Im} x)^2 - (\operatorname{Im} x)^4 + (\operatorname{Im} y)^2 \\ &> R/2 - |\operatorname{Re} y|^{4/3} - A^4\omega(R)^4 . \end{aligned}$$

This implies, for large R,

$$|\operatorname{Re} y| \geq \sqrt{R}/2 . \tag{8}$$

If x, z were real, then y would be either real or purely imaginary. The latter is impossible because of (8), the first alternative contradicts $a\omega(R) < |\operatorname{Im} \theta|$. So we have $\operatorname{Im}(x, z) \neq 0$ and for $\zeta \in B$ we define

$$x(\zeta) = \operatorname{Re} x + \zeta \frac{\operatorname{Im} x}{|\operatorname{Im}(x,z)|}, \quad z(\zeta) = \operatorname{Re} z + \zeta \frac{\operatorname{Im} z}{|\operatorname{Im}(x,z)|}, \quad t = |\operatorname{Im}(x,z)| .$$

We have, applying (8) in the last step but one,

$$\begin{aligned} &|(z(\zeta) - x(\zeta)^4) - (z - x^4)| \\ &\quad \leq |z(\zeta) - z| + |x(\zeta) - x||x(\zeta)^3 + x(\zeta)^2 x + x(\zeta)x^2 + x^3| \\ &\quad \leq 2A\omega(R) + 8A\omega(R)(|\operatorname{Re} x| + A\omega(R))^3 \\ &\quad \leq A\omega(R)(2 + 64\max(|\operatorname{Re} x|^3, A^3\omega(R)^3)) \\ &\quad \leq A\omega(R)(2 + 64\max(|\operatorname{Re} y|, A^3C_2^3\sqrt{R})) \\ &\quad = A\omega(R)(2 + 64|\operatorname{Re} y|) \leq 65A\omega(R)|\operatorname{Re} y| . \end{aligned} \tag{9}$$

Hence, for large R,

$$\begin{aligned} |z(\zeta) - x(\zeta)^4| &\geq |z - x^4| - 65A\omega(R)|\operatorname{Re} y| = |y|^2 - 65A\omega(R)|\operatorname{Re} y| \\ &\geq |\operatorname{Re} y|(|\operatorname{Re} y| - 65A\omega(R)) \geq |\operatorname{Re} y|^2/2 . \end{aligned} \tag{10}$$

In particular, $z(\zeta) - x(\zeta)^4 \neq 0$ for $\zeta \in B$. We define

$$y(\zeta) = \sqrt{z(\zeta) - x(\zeta)^4}, \quad f(\zeta) = (x(\zeta), y(\zeta), z(\zeta)) ,$$

where that branch of the root is chosen for which $y(it) = y$. The mean value theorem, (9), and (10) imply

$$\begin{aligned} |y(\zeta) - y| &\leq \frac{1}{\sqrt{2}|\operatorname{Re} y|} 65A\omega(R)|\operatorname{Re} y| \leq 46A\omega(R) , \\ |f(\zeta) - \theta| &\leq |x(\zeta) - x| + |y(\zeta) - y| + |z(\zeta) - z| \leq 50A\omega(R) \leq \omega(R) . \end{aligned}$$

This is (S1), while (S3) and (S4) are obvious. For all $\xi \in\,]-D, D[$, the radicand $z(\xi) - x(\xi)^4$ is real. On the other hand, it is close to y^2, thus positive. This implies $f(\xi) \in \mathbb{R}^3$, and finally (S2).

Case $|\operatorname{Re} y| \leq |\operatorname{Re} x|^3$. We claim that for large R,

$$|\operatorname{Re} x| \geq \sqrt[6]{R}/2 . \tag{11}$$

Assume $|\operatorname{Re} x| < \sqrt[6]{R}/2$; then $|\operatorname{Re} y| < \sqrt{R}/8$ and, by (7), we get the following contradiction:

$$\begin{aligned} R/2 \leq \operatorname{Re} z &= \operatorname{Re} y^2 + \operatorname{Re} x^4 \\ &= (\operatorname{Re} y)^2 - (\operatorname{Im} y)^2 + (\operatorname{Re} x)^4 - 6(\operatorname{Re} x)^2(\operatorname{Im} x)^2 + (\operatorname{Im} x)^4 \\ &< R/64 + R^{2/3}/16 + A^4\omega(R)^4 . \end{aligned}$$

If y, z were real, then $x^4 \in \mathbb{R}$ and thus x is either real or of the form $\pm|x|\sqrt{i}$, since by (11) it cannot be purely imaginary. But in the latter case $|\operatorname{Im} x| = |\operatorname{Re} x| \geq \sqrt[6]{R}/2 \geq \omega(R)(2C_2)^{-1} > A\omega(R)$, thus this possibility as well as the one that x should be real contradict the assumption that $a\omega(R) < |\operatorname{Im} \theta| < A\omega(R)$. Thus we have $\operatorname{Im}(y, z) \neq 0$ and we define

$$y(\zeta) = \operatorname{Re} y + \zeta \frac{\operatorname{Im} y}{|\operatorname{Im}(y,z)|}, \quad z(\zeta) = \operatorname{Re} z + \zeta \frac{\operatorname{Im} z}{|\operatorname{Im}(y,z)|}, \quad t = |\operatorname{Im}(y,z)| .$$

Then

$$\begin{aligned} |(z(\zeta) - y(\zeta)^2) - (z - y^2)| &\leq |z(\zeta) - z| + |y - y(\zeta)||y + y(\zeta)| \\ &\leq 2A\omega(R) + 4A\omega(R)(|\operatorname{Re} y| + A\omega(R)) \\ &\leq A\omega(R)(2 + 8\max(|\operatorname{Re} y|, A\omega(R))) \\ &\leq A\omega(R)(2 + 8\max(|\operatorname{Re} x|^3, AC_2\sqrt[6]{R})) \\ &\leq 9A\omega(R)|\operatorname{Re} x|^3 . \end{aligned} \tag{12}$$

Applying (11), this gives, for large R,

$$\begin{aligned} |z(\zeta) - y(\zeta)^2| &\geq |z - y^2| - 9A\omega(R)|\operatorname{Re} x|^3 = |x|^4 - 9A\omega(R)|\operatorname{Re} x|^3 \\ &\geq |\operatorname{Re} x|^3(|\operatorname{Re} x|/2 - 9A\omega(R)) \\ &\geq |\operatorname{Re} x|^3(|\operatorname{Re} x|/2 - 9C_2A\sqrt[6]{R}) \\ &\geq |\operatorname{Re} x|^3(|\operatorname{Re} x|/2 - 18C_2A|\operatorname{Re} x|) \geq |\operatorname{Re} x|^4/4 . \end{aligned} \tag{13}$$

Therefore, for $\zeta \in B$, we can define

$$x(\zeta) = \sqrt[4]{z(\zeta) - y(\zeta)^2}, \qquad f(\zeta) = (x(\zeta), y(\zeta), z(\zeta)),$$

where we choose that branch of $\sqrt[4]{\cdot}$ for which $x(it) = x$. The mean value theorem implies, using (13) and (12),

$$(14) \qquad |x(\zeta) - x| \le \frac{4^{3/4}}{4|\operatorname{Re} x|^3} 9A\omega(R)|\operatorname{Re} x|^3 \le 7A\omega(R),$$

$$|f(\zeta) - \theta| < 4A\omega(R) + 7A\omega(R) \le \omega(\theta).$$

Thus (S1) is shown. Let now $\xi \in \,]-D, D[$ be given. Then, by (11) and (14),

$$(15) \qquad |\operatorname{Re} x(\xi)| \ge |\operatorname{Re} x| - |x(\xi) - x| \ge \sqrt[6]{R}/2 - 7A\omega(R) \ge \sqrt[6]{R}/4.$$

On the other hand, using (14) and $|\operatorname{Im} x| \le |\operatorname{Im} \theta| < A\omega(R)$,

$$(16) \qquad |\operatorname{Im} x(\xi)| \le |\operatorname{Im} x| + |x(\xi) - x| < 8A\omega(R) \le 8C_2 A\sqrt[6]{R}.$$

But $x(\xi)^4$ is real, so also $x(\xi)$ must be real, because otherwise it would be either purely imaginary, which contradicts (15), or $|\operatorname{Re} x(\xi)| = |\operatorname{Im} x(\xi)|$, which contradicts (15) and (16). This proves (S2), while (S3) and (S4) are clear.

15. Case $\sqrt[4]{\cdot} = o(\omega)$ and $\omega = O(\sqrt{\cdot})$. We verify the hypotheses of Proposition 9. By $\sqrt{\cdot}, \sqrt[4]{\cdot} : \mathbb{C} \setminus \,]-\infty, 0] \to \mathbb{C}$ we denote those branches that map positive reals to positive reals. There are $C_1, C_2 > 0$ with

$$\omega(2t) \le C_1\omega(t) \quad \text{and} \quad \omega(t) \le C_2\sqrt{t} \quad \text{for all } t > 0.$$

Let $\delta > 0$ be given. We may assume $\delta \le 2^{-2}C_1^{-2}C_2^{-1}$. Define $\varepsilon = 2^{-8}C_1^{-4}C_2^{-2}$. By our hypothesis, there is $R > 1$ with

$$(17) \qquad 4\delta^4\omega(r)^4 \ge r/\varepsilon \quad \text{for all } r \ge R.$$

Fix any $y \ge R$. We claim the existence of $z \in \,]0, y^2[$ with

$$(18) \qquad \sqrt[4]{y^2 - z} = \sqrt{2}\delta\omega(\sqrt[4]{y^2 - z}, y, z).$$

For $z = 0$, the left hand side is $\sqrt{y}$, while the right hand side is equal to

$$\sqrt{2}\delta\omega(\sqrt{y}, y, 0) \le \sqrt{2}\delta\omega(2y) \le \sqrt{2}\delta C_1\omega(y) \le \sqrt{2}\delta C_1 C_2\sqrt{y} < \sqrt{y}.$$

For $z = y^2$ the left hand side of (18) vanishes, while the right hand side does not. Thus our claim follows by a continuity argument. We define

$$\theta = (\sqrt{i}\sqrt[4]{y^2 - z}, y, z).$$

Then $\theta \in V$ and, by (18), $|\operatorname{Im} \theta| = \delta\omega(\theta)$.

From (18) we get $y^2 - z = 4\delta^4\omega(\theta)^4$ and $y^2 \ge 4\delta^4\omega(\theta)^4 \ge 4\delta^4\omega(z)^4$. This implies, by (17),

$$(19) \qquad z \le \varepsilon y^2.$$

Since $\varepsilon \le 1/2$, we get

$$(20) \qquad \sqrt[4]{y^2 - z} \ge \sqrt[4]{y^2 - \varepsilon y^2} \ge \sqrt[4]{y^2/2} \ge \sqrt{y/2}.$$

By (18), this implies

$$\sqrt{y} \le \sqrt{2}\sqrt[4]{y^2 - z} = 2\delta\omega(\theta) \le 2\delta\omega(\sqrt{y}, y, z)$$
$$\le 2\delta C_1^2 \max(\omega(y), \omega(z)) \le 2\delta C_1^2 C_2 \max(\sqrt{y}, \sqrt{z}) \le \tfrac{1}{2}\max(\sqrt{y}, \sqrt{z}).$$

This implies $y \le z/4$, and hence, applying (19),

$$(21) \qquad \omega(\theta) \le C_1^2\omega(z) \le C_1^2 C_2\sqrt{z} \le C_1^2 C_2\sqrt{\varepsilon}y.$$

Let now $\Xi = (\xi, \zeta, \eta) \in V$ with $|\Xi - \theta| < \omega(\theta)$ be given. We have, by (21) and the choice of ε,

$$|(\eta^2 - \zeta) - (y^2 - z)| \le |\eta - y||\eta + y| + |\zeta - z| \le \omega(\theta)(2y + \omega(\theta)) + \omega(\theta)$$
$$\le C_1^2 C_2\sqrt{\varepsilon}y(2y + C_1^2 C_2\sqrt{\varepsilon}y + 1)$$
$$\le 2^{-4}y(2y + 2^{-4}y + 1) \le y^2/4.$$

This implies, using (20),

$$|\eta^2 - \zeta| \ge (y^2 - z) - y^2/4 \ge y^2/4.$$

We apply the mean value theorem to get

$$|\sqrt[4]{y^2 - z} - \sqrt[4]{\eta^2 - \zeta}| \le \frac{1}{4}(y^2/4)^{-3/4}y^2/4 = 2^{-5/2}\sqrt{y}.$$

This gives the next two estimates, if we apply (18) and (20) several times:

$$|\operatorname{Im} \sqrt[4]{\eta^2 - \zeta}| \le 2^{-5/2}\sqrt{y} \le 2^{-2}\sqrt[4]{y^2 - z} \le 2^{-3/2}\delta\omega(\theta),$$
$$|\operatorname{Re} \sqrt[4]{\eta^2 - \zeta}| \ge \sqrt[4]{y^2 - z} - 2^{-5/2}\sqrt{y} \ge \sqrt{y}(2^{-1/2} - 2^{-5/2})$$
$$\ge 2^{-3/2}\sqrt{y} \ge 2^{-3/2}\sqrt[4]{y^2 - z} = \frac{\delta}{2}\omega(\theta).$$

For some $n \in \mathbb{N}$, we have $\xi = i^{n+1/2}\sqrt[4]{\eta^2 - \zeta}$. Thus

$$|\operatorname{Im} \xi| \ge |\operatorname{Im} i^{n+1/2}||\operatorname{Re} \sqrt[4]{\eta^2 - \zeta}| - |\operatorname{Re} i^{n+1/2}||\operatorname{Im} \sqrt[4]{\eta^2 - \zeta}|$$
$$\ge \frac{1}{\sqrt{2}}\left(\frac{\delta}{2}\omega(\theta) - \frac{\delta}{\sqrt{8}}\omega(\theta)\right) \ge \frac{\delta}{8}\omega(\theta).$$

We finally get

$$\frac{|\operatorname{Im} \Xi|}{|\operatorname{Im} \theta|} \ge \frac{(\delta/8)\omega(\theta)}{\delta\omega(\theta)} = \frac{1}{8}.$$

16. Case $\sqrt[6]{\cdot} = o(\omega)$ and $\omega = O(\sqrt[4]{\cdot})$. We verify the hypotheses of Proposition 9. There are C_1, C_2 with

$$\omega(2t) \le C_1\omega(t) \quad \text{and} \quad \omega(t) \le C_2\sqrt[4]{t} \quad \text{for all } t.$$

Let $\delta > 0$ be given. We may assume $\delta < (2C_2)^{-1}$. For sufficiently large R, we have $\delta^4\omega(R)^4 < R/2$, thus we can define

$$\theta := (x, y, z) := (i\delta\omega(R), \sqrt{R - \delta^4\omega(R)^4}, R).$$

Note that $\omega(R) \le \omega(\theta) \le C_1\omega(R)$ for large R. For $(\xi, \eta, \zeta) \in U_{\omega(\theta)}(\theta)$ we have

$$(22) \qquad \begin{aligned} |(\zeta - \eta^2) - (z - y^2)| &\le |\zeta - z| + |\eta - y||\eta + y| \\ &\le \omega(\theta)(1 + 2\sqrt{R} + \omega(\theta)) \le 3\sqrt{R}\omega(\theta). \end{aligned}$$

Since $\sqrt{\cdot} = o(\omega^3)$, this means, for large R,

$$(23) \qquad \begin{aligned} |\zeta - \eta^2| &\ge |z - y^2| - 3\sqrt{R}\omega(\theta) \\ &\ge \delta^4\omega(R)^4 - 3C_1\sqrt{R}\omega(R) \ge \delta^4\omega(R)^4/2. \end{aligned}$$

In particular, on $U_{\omega(\theta)}(\theta)$ there is a function f with $f(\xi, \eta, \zeta) = \sqrt[4]{\zeta - \eta^2}$ and $f(\theta) = x$. The set $W := \{(\xi, \eta, \zeta) \in U_{\omega(\theta)}(\theta) \mid \xi = f(\xi, \eta, \zeta)\}$ is a component of $U_{\omega(\theta)}(\theta) \cap V$. For $(\xi, \eta, \zeta) \in W$, we have, applying (22), (23), and the mean value theorem,

$$\begin{aligned} |\operatorname{Im}(\xi, \eta, \zeta)| &\ge |\operatorname{Im} f(\xi, \eta, \zeta)| \ge \delta\omega(R) - |f(\theta) - f(\xi, \eta, \zeta)| \\ &\ge \delta\omega(R) - \frac{2^{3/4}}{4\delta^3\omega(R)^3} 3\sqrt{R}\omega(\theta) \ge \frac{\delta}{2}\omega(R) = \frac{1}{2}|\operatorname{Im}\theta| \end{aligned}$$

if R is large enough.

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