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Estimation of the position of intermediate spaces for a Banach couple

by

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Abstract. The position of intermediate spaces for a Banach couple is estimated with the help of its fundamental function and co-function. We study the completeness of the collection of all such functions, and the methods of calculating and estimating them for different couples. Finally, these functions are used to compare the position of spaces obtained under the action of some interpolation functors.

1. Introduction. A basic concept in interpolation theory of linear operators in Banach spaces is that of a couple $\vec{A} = (A_0, A_1)$, where the Banach spaces A_0, A_1 are continuously imbedded in a common Hausdorff topological vector space. Without loss of generality one may assume that the spaces of the couple consist of elements of the same nature with natural definitions of sums and equalities. Each couple $\vec{A}$ generates a collection of spaces lying between $\Delta(\vec{A}) = A_0 \cap A_1$ and $\Sigma(\vec{A}) = A_0 + A_1$ and called intermediate spaces. Among them the interpolation spaces, i.e. those invariant under any linear operator $T : \Sigma(\vec{A}) \to \Sigma(\vec{A})$ bounded in A_0 and in A_1, are the most studied ones. In the last 10–15 years certain fine results were obtained which describe all or the greater part of interpolation spaces for different Banach couples. They use orbit methods [O2], interpolation functors [BK2], minimal and maximal interpolation methods [J], isolate spaces with important special properties [Cw], [S] etc.

There are considerably fewer papers which deal with objects more general than interpolation spaces (we mention the rather old work [D], the recent work [MM] and the author's papers [P1, 3–5]). However, there are many situations when almost nothing is known about a given space except its being intermediate in certain Banach couples. As a consequence of this, the instruments and methods for investigation of such spaces are not elaborated

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so far. As a bit of help one can take the imbedding constants of a space A in the couple $\vec{A}$, i.e. the minimal numbers C, D for which

$$\|x\|_A \le C\|x\|_{\Delta(\vec{A})} \quad (\forall x \in \Delta(\vec{A})), \qquad \|x\|_{\Sigma(\vec{A})} \le D\|x\|_A \quad (\forall x \in A).$$

To make even more use of these constants it makes sense to consider the imbedding of A in the couples $(\alpha A_0, \beta A_1)$ for all possible $\alpha, \beta > 0$ and to use, as a characteristic of the position of A in the couple $\vec{A}$, the constants

$$C(\alpha,\beta) = \sup_{x\in\Delta(\vec{A})} \frac{\|x\|_A}{\|x\|_{\alpha A_0 \cap \beta A_1}}, \qquad D(\alpha,\beta) = \sup_{x\in A} \frac{\|x\|_{\alpha A_0 + \beta A_1}}{\|x\|_A},$$

considered as functions of α, β. These functions are homogeneous of degree one, hence one of their arguments may be taken fixed. In particular, $C(1, 1/t)$ and $D(t, 1)$ coincide with the fundamental function $\varphi_A(t, \vec{A})$ and fundamental co-function $\psi_A(t, \vec{A})$ defined in this paper (the reasons for such choice will be given below).

The present paper continues the study of the collection of all intermediate spaces of an arbitrary Banach couple with the help of fundamental functions and co-functions, initiated in the author's previous works [P1, 3–5]. In Section 2 the definitions and properties of the above-mentioned functions are given. In Section 3 these functions are applied to comparison of different intermediate spaces, and the following "completeness" theorem is proved: *if there is an intermediate space A for the couple $\vec{A}$ with a quasi-power fundamental function $\varphi_A(t, \vec{A})$ then each quasi-concave function is equivalent to the fundamental function of some intermediate space for this couple.*

Some examples of computing fundamental functions and co-functions are given in Section 4. We prove there a theorem on conditions under which two quasi-concave functions φ, ψ are equivalent when they take equivalent values on some sequence of positive numbers $t = w_n$. This allows us to describe the weighted sequence spaces which do not have a "complete" collection of intermediate spaces.

The main results of the paper are contained in Section 5 where the fundamental functions and co-functions are used for comparison of interpolation functors for concrete Banach couples. For real interpolation functors, the position of the space $\vec{A}^K_E$ in the couple $(\vec{A}^K_{E_0}, \vec{A}^K_{E_1})$ only depends on the position of E in (E_0, E_1) but not on the couple $\vec{A}$. For the Calderón–Lozanovskiĭ functor the dependence on $\vec{A}$ does exist, and it is estimated in the cases of weighted $L_p(w)$ spaces and Orlicz spaces.

For the reader's convenience we formulate some previous results from not translated or not easily accessible Russian literature which relate to the subject and are used in the basic text.

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2. The fundamental function and co-function of an intermediate Banach space. The Banach space A is called *intermediate* for the couple $\vec{A} = (A_0, A_1)$ if $\Delta(\vec{A}) \subset A \subset \Sigma(\vec{A})$, where $\Delta(\vec{A}) = A_0 \cap A_1, \Sigma(\vec{A}) = A_0 + A_1$ and all imbeddings are considered to be continuous. In the case of an imbedded couple, i.e. when $A_0 \subset A_1$, we obtain $A_0 \subset A \subset A_1$, i.e. the usual notion of being intermediate for spaces ordered by imbedding. The collection of all intermediate spaces for the given couple $\vec{A}$ will be denoted by $\pi(\vec{A})$.

It is desirable to characterize the position of intermediate spaces more precisely with the help of some additional parameters, which would allow one to estimate their proximity to the "extreme" spaces of the couple. Such parameters also allow one to compare specific intermediate spaces, to isolate classes of spaces which are easier to study, and to approximate the other ones (e.g. for the assessment of interpolation properties). As a model, one may consider the collection of rearrangement invariant spaces intermediate for the couple (L_∞, L_1). The position of a space E with respect to this couple is characterized by the fundamental function $\varphi_E(t) = \|\chi_D\|_E$, $\operatorname{mes} D = t$. Among all the spaces with a given fundamental function the narrowest and widest ones are determined, which are called Lorentz and Marcinkiewicz spaces respectively and characterize the whole structure of rearrangement invariant spaces (see e.g. [KPS]).

The abstract analog of the above-mentioned function is defined by

$$\varphi_A(t) = \varphi_A(t, \vec{A}) = \sup_{\|x\|_{A_0} \le 1,\ \|x\|_{A_1} \le t} \|x\|_A, \qquad 0 < t < \infty,$$

and still called the *fundamental function*. It is not difficult to verify that this agrees with the previous definition for each rearrangement invariant intermediate space E for the couple $\vec{A} = (L_\infty, L_1)$. The following inequality is valid:

$$\|x\|_A \le \|x\|_{A_0} \varphi_A(\|x\|_{A_1}/\|x\|_{A_0}), \qquad x \in \Delta(\vec{A}),$$

where φ_A cannot be replaced by any other quasi-concave function φ distinct from φ_A at least at one point. This function was first introduced by A. Dmitriev [D] and further extensively used in [P1], [DKO], [M] etc.

Let us note some properties of fundamental functions:

1) $\varphi_A = \varphi_{A^\circ}$, where A° is the closure of $\Delta(\vec{A})$ in A; in particular, $\varphi_{\overline{A}^B} = \varphi_B$ for $A, B \in \pi(\vec{A})$;

2) if $A \subset B$, then $\varphi_A(t) \le C\varphi_B(t)$, where C is the imbedding constant;

3) $\varphi_{A\cap B}(t) = \max(\varphi_A(t), \varphi_B(t))$, $\varphi_{A+B}(t) \le \min(\varphi_A(t), \varphi_B(t))$;

4) for each nonimbedded couple $\vec{A}$ with $\Delta(\vec{A})$ not closed in A_0, A_1 we have $\varphi_{A_0}(t) = 1$, $\varphi_{A_1}(t) = t$, $\varphi_{\Delta(\vec{A})}(t) = \max(1,t)$, and $k\min(1,t) \le \varphi_{\Sigma(\vec{A})}(t) \le \min(1,t)$, where

$$k = \sup \|x\|_{\Sigma(\vec{A})}/\|x\|_{\Delta(\vec{A})};$$

5) if $A_2, A_3 \in \pi(\vec{A})$, $A \in \pi(A_2, A_3)$, $\varphi_A(t) = \varphi_A(t, \vec{A})$, $\overline{\varphi}_A(t) = \varphi_A(t, A_2, A_3)$, then

$$\varphi_A(t) \le \varphi_{A_2}(t)\overline{\varphi}_A(\varphi_{A_3}(t)/\varphi_{A_2}(t));$$

6) $\varphi_A(t, A_1, A_0) = t\varphi_A(1/t, A_0, A_1)$;

7) if $A_0^\circ \subset A_1$ with imbedding constant C, then $\varphi_A(t) = \text{const}\ (= \varphi_A(C))$ for $t > C$ and all $A \in \pi(\vec{A})$;

8) if $A_1^\circ \subset A_0$ with imbedding constant C, then $\varphi_A(t) = tC\varphi_A(1/C)$ for $t < 1/C$ and all $A \in \pi(\vec{A})$.

The fundamental function can be expressed through the Peetre functional $J(t,x) = \max(\|x\|_{A_0}, t\|x\|_{A_1})$ by the formula

$$\varphi_A(t) = \sup_{x\in\Delta(\vec{A})} \|x\|_A/J(1/t,x),$$

i.e. $\varphi_A(t) = \inf \varphi(t)$ over all quasi-concave functions φ for which $\|x\|_A \le \varphi(t)J(1/t,x)$ for all $x \in \Delta(\vec{A})$, $t > 0$.

Unlike for rearrangement invariant spaces, the abstract fundamental function is not sufficient for characterizing the position of an intermediate space since it estimates the norm $\|x\|_A$ from one side only. Another characteristic is obtained by passing to the dual space:

$$\varphi_{A'}(t, \vec{A}') = \sup_{\|f\|_{A_0'} \le t,\ \|f\|_{A_1'} \le 1} \|f\|_{A'}, \qquad 0 < t < \infty.$$

In the case of a nondense couple $\vec{A}$ the adjoint spaces do not form a Banach couple, hence the use of another form of duality is essential (see [P2]). In order that φ_A be independent of the choice of the dual couple (it automatically does not depend on the choice of A'), it is sufficient that the couple $\vec{A}'$ be *normative dual* for $\vec{A}$, i.e. that

$$\|x\|_{A_1+tA_0} = \sup\{|\langle x,y\rangle| : \|y\|_{A_1'\cap(tA_0)'} \le 1\}$$

for each $t > 0$ (note that the conjugate and preconjugate spaces as well as the dual spaces from [BK2] satisfy this condition). In this case $\varphi_{A'}(t)$ is expressed through the Peetre functional $K(t,x) = \inf\{\|x_0\|_{A_0} + t\|x_1\|_{A_1} : x_0 \in A_0,\ x_1 \in A_1,\ x_0 + x_1 = x\}$ by the formula

$$\varphi_{A'}(t) = \sup_{x\in A} tK(1/t,x)/\|x\|_A.$$

We shall denote the right-hand side of this formula by $\psi_A(t) = \psi_A(t, \vec{A})$ and call it the *fundamental co-function*, whether the dual couple exists or not. Then $\psi_A(t) = \inf \psi(t)$ over all quasi-concave functions ψ for which $\|x\|_A \ge tK(1/t,x)/\psi(t)$ for all $x \in A$, $t > 0$.

As above, in the case of a nondense couple, A_i° is the closure of $\Delta(\vec{A})$ in A_i, $i = 0,1$. Is it possible to replace, in the definition of $\varphi_A(t)$, $\psi_A(t)$, the couple $\vec{A}$ by $\vec{A}^\circ$? The answer is affirmative if $A \in \pi(\vec{A}^\circ)$. In the other case $\varphi_A(t, \vec{A}) = \varphi_{A^\circ}(t, \vec{A}^\circ)$, but for the co-function it is only possible to assert that $\psi_A(t, \vec{A}) \ge \psi_{A^\circ}(t, \vec{A}^\circ)$.

Let us record some properties of fundamental co-functions, by analogy to those of fundamental functions:

1) if $A \subset B$, then $\psi_A(t) \le C\psi_B(t)$, where C is the imbedding constant;

2) $\psi_{A\cap B}(t) \le \min(\psi_A(t), \psi_B(t))$, $\psi_{A+B}(t) \ge \max(\psi_A(t), \psi_B(t))$, and the second inequality turns into equality if $\vec{A}$ has a normative dual couple or $A, B \in \pi(\vec{A}^\circ)$;

3) for each nonimbedded couple $\vec{A}$ with $\Delta(\vec{A})$ not closed in A_0, A_1 we have $\psi_{A_0}(t) = t$, $\psi_{A_1}(t) = 1$, $\psi_{\Sigma(\vec{A})}(t) = \max(1,t)$, and $k\min(1,t) \le \psi_{\Delta(\vec{A})}(t) \le \min(1,t)$ (with the same k as for $\varphi_A(t)$);

4) if $A_0 \subset A_1$ with imbedding constant C, then $\psi_A(t) = \text{const}\ (= \psi_A(C))$ for $t \ge C$ and each $A \in \pi(\vec{A})$;

5) if $A_1 \subset A_0$ with imbedding constant C, then $\psi_A(t) = tC\psi_A(1/C)$ for $t \le 1/C$ and each $A \in \pi(\vec{A})$.

For a rearrangement invariant space A intermediate for the couple (L_∞, L_1) the equality $\psi_A(t) = \varphi_A^*(t) = t/\varphi_A(t)$ is valid, hence to characterize such a space it is sufficient to know the fundamental function. In general, there is no connection between φ_A and ψ_A; some connections appear under additional restrictions on the whole couple $\vec{A}$ or on the space A [P3], [D].

PROPOSITION 1. *Let A be an interpolation space for one-dimensional operators acting in the couple $\vec{A}$. Then $\psi_A(t) \le C\varphi_A^*(t)$, where C is the interpolation constant.*

PROPOSITION 2. *Let the Peetre functionals for the couple $\vec{A}$ satisfy*

$$\inf_t \sup_{x\in\Delta(\vec{A})} \frac{K(t,x,\vec{A})}{J(t,x,\vec{A})} = \gamma > 0. \tag{1}$$

Then $\psi_A(t) \ge \gamma\varphi_A^*(t)$ for each intermediate space A.

Remark. The converse to Proposition 1 is also valid: the inequality $\psi_A(t) \le C\varphi_A^*(t)$ implies that A is an interpolation space for one-dimensional operators.

3. Comparison of intermediate spaces. Now let us show how the fundamental functions and co-functions of intermediate spaces can be used to estimate their relative position. For this purpose we define a generalization of Lorentz and Marcinkiewicz spaces playing the role of "marks" in $\pi(\vec{A})$ by analogy to the case of rearrangement invariant spaces. Namely, for every quasi-concave function φ let $\Lambda_\varphi(\vec{A})$ be the intersection of all spaces in $\pi(\vec{A})$ for which $\varphi_A(t,\vec{A}) \le \varphi(t)$, and let $M_\varphi(\vec{A})$ be the sum of all spaces in $\pi(\vec{A})$ for which $\psi_A(t,\vec{A}) \le \varphi^*(t)$. Evidently, $\Lambda_\varphi(\vec{A}) \subset M_\varphi(\vec{A})$ and $\Lambda_{\varphi_A}(\vec{A}) \subset A \subset M_{\psi_A^*}(\vec{A})$ for each $A \in \pi(\vec{A})$. For these spaces to be instrumental it is especially important that for some indices the inverse imbeddings also hold. As was proved in [P1], the inequality

$$\int_0^\infty \frac{\psi(t)}{t\varphi(t)}\,dt < \infty$$

is sufficient for the imbedding $M_\varphi \subset \Lambda_\psi$. From this we deduce immediately the following

PROPOSITION 3. *Let $A, B \in \pi(\vec{A})$ and*

$$C = \int_0^\infty \frac{\varphi_A(t)\psi_B(t)}{t^2}\,dt < \infty\,. \tag{2}$$

Then $B \subset A$ with imbedding constant depending only on C.

Note that the requirements on φ_A, ψ_B cannot be weakened in general; this follows e.g. from the equivalence of the norm in Λ_φ and of the integral

$$I = \int_0^\infty K(t,x)\frac{\varphi(t)}{t^2}\,dt$$

for quasi-power φ (i.e. $\varphi(t) \sim t^\alpha\theta(t^{\beta-\alpha})$ for $0<\alpha<\beta<1$ and some quasi-concave θ). The spaces $\Lambda_\varphi(\vec{A})$ and $M_\varphi(\vec{A})$ are interpolation spaces for the couple $\vec{A}$ and even generate interpolation functors in the category of Banach couples. Hence if $A \in \pi(\vec{A})$, $B \in \pi(\vec{B})$, $\varphi_A(t) = \varphi_A(t,\vec{A})$, $\varphi_B(t) = \varphi_B(t,\vec{B})$, then (2) implies that the triple B_0, B_1, B is interpolating with respect to the triple A_0, A_1, A.

The functors $\Lambda_\varphi, M_\varphi$ are particular cases of real interpolation functors because they are expressed through the Peetre functionals $J(t,x)$ and $K(t,x)$ [P1]. As was shown in [BK1], for description of such functors, in fact, the K-functional alone is sufficient. We only formulate here part of this statement in a form suitable for our purposes.

PROPOSITION 4. *Let $x \in \Sigma(\vec{A})$, $y \in \Sigma(\vec{B})$ and suppose $K(t,x,\vec{A}) \le K(t,y,\vec{B})$. Then for each quasi-concave φ,*

$$\|x\|_{\Lambda_\varphi(\vec{A})} \le C\|y\|_{\Lambda_\varphi(\vec{B})}, \qquad \|x\|_{M_\varphi(\vec{A})} \le C\|y\|_{M_\varphi(\vec{B})}\,.$$

This means that interpolation from $\Lambda_\varphi(\vec{B})$ to $\Lambda_\varphi(\vec{A})$ (as well as from $M_\varphi(\vec{B})$ to $M_\varphi(\vec{A})$) is K-monotonic. It follows that the above-mentioned interpolation from the triple B_0, B_1, B to A_0, A_1, A under condition (2) is also K-monotonic, i.e. the K-orbit of each $x \in B$ (see [O2]) is contained in A.

Using the partial ordering of intermediate spaces induced by imbedding we may say that each intermediate space is bounded above and below by some spaces Λ_φ and M_ψ, with possibly different indices. To have an imbedding $\Lambda_\varphi \subset A \subset M_\varphi$ with equal indices it is necessary and sufficient that A be an interpolation space for one-dimensional operators acting in the couple $\vec{A}$; all the more, such an imbedding holds for any space which is an interpolation space in the general sense. For φ, we can take any quasi-concave function φ satisfying $C_1\varphi_A(t) \le \varphi(t) \le C_2\psi_A^*(t)$ with some constants $C_1, C_2 > 0$, hence it is not unique. In order that all the admissible φ coincide up to equivalence it is sufficient, for instance, that the condition (1) be satisfied.

Another question: To what extent an inverse correspondence can be established such that for each quasi-concave function $\varphi(t)$ there exists a space $A \in \pi(\vec{A})$ with $\varphi_A(t,\vec{A}) \sim \varphi(t)$? Evidently, in the case of nonequivalent $\varphi_A(t)$, $\varphi_B(t)$ the norms in A and B are not equivalent on $A \cap B$ (we shall write $A \ne B$). Consequently, the norms in $\Lambda_{\varphi_A(t)}(\vec{A})$, $\Lambda_{\varphi_B(t)}(\vec{A})$ are also nonequivalent. Thus the store of intermediate spaces for the couple $\vec{A}$ with nonequivalent fundamental functions can be estimated by that of $\Lambda_\varphi(\vec{A})$ spaces. Moreover, one can use the duality between Λ_φ and M_{φ^*} to estimate the store of M_φ spaces. In the next section it will be shown by examples of concrete couples how this store may be incomplete, but now we will present some general results (partially proved in [P4]). Recall that the couple $\vec{A}$ is called *trivial* if $\Delta(\vec{A})$ is closed in $\Sigma(\vec{A})$, and the function $\varphi(t)$ is called *regular at zero* (*infinity*) if $\lim_{t\to0}\varphi(t) = 0$ ($\lim_{t\to\infty}\varphi(t) = \infty$).

PROPOSITION 5. *Let the couple $\vec{A}$ be nontrivial, the functions φ, ψ, θ be quasi-concave and θ^* be regular at zero and at infinity. If one of the following conditions is satisfied:*

1) *$\Lambda_\varphi(\vec{A})$ is not imbedded in A_0, and $\psi(t) \ge \theta(\varphi(t))$ for $t < 1$,*
2) *$\Lambda_\varphi(\vec{A})$ is not imbedded in A_1, and $\psi^*(t) \le \theta(\varphi^*(t))$ for $t > 1$,*

then the spaces Λ_φ and Λ_ψ are nonequivalent.

If one of the following conditions is satisfied:

1) $M_\varphi(\vec{A})$ *does not contain* A_0, *and* $\psi(t) \le \theta(\varphi(t))$ *for* $t > 1$,

2) $M_\varphi(\vec{A})$ *does not contain* A_1, *and* $\psi^*(t) \ge \theta(\varphi^*(t))$ *for* $t < 1$,

then the spaces M_φ *and* M_ψ *are nonequivalent.*

In particular, it follows that the generalized Lorentz and Marcinkiewicz spaces with numerical indices are nonequivalent for different indices, since these indices are, in fact, power functions with different exponents.

As follows from the definition of M_φ spaces, their store (and so that of Λ_φ spaces in the dual couple) is connected with the store of K-functionals for the couple $\vec{A}$. The latter is called *complete* (or *abundant*) if for each quasi-concave φ there exists $x \in A_0 + A_1$ such that $K(t, x, \vec{A}) \sim \varphi(t)$ with a universal equivalence constant. It was shown recently [BK2] that for $\vec{A}$ to have an abundance of K-functionals, it is necessary and sufficient that there exists at least one $x \in A_0 + A_1$ with a quasi-power K-functional. Again referring to [P4] we can compare the generalized Marcinkiewicz spaces in this situation.

PROPOSITION 6. *Let the couple* $\vec{A}$ *have an abundance of* K*-functionals. Then* $M_\varphi(\vec{A}) = M_\psi(\vec{A})$ *iff* $\varphi \sim \psi$. *Moreover, for* φ, ψ *nonequivalent* $M_\varphi(\vec{A})$ *cannot be densely imbedded in* $M_\psi(\vec{A})$.

Using some basic ideas of the above-mentioned proof from [BK2] we now prove the following general assertion.

THEOREM 1. *Suppose that, for a couple* $\vec{A}$, *there is an intermediate space* A *with a quasi-power fundamental function. Then, for any given quasi-concave function* φ, *there exists an intermediate space for this couple whose fundamental function is equivalent to* φ *(with a universal equivalence constant).*

Proof. We show that the spaces $\Lambda_\varphi(\vec{A})$ have the required property. First we consider the space $\Lambda_{\varphi_A}(\vec{A})$. It is not difficult to see that its fundamental function is precisely φ_A. Since we shall only deal with generalized Lorentz spaces, we may suppose without loss of generality that $\vec{A}$ is a dense couple, and introduce the conjugate couple $\vec{A}^*$. Assuming $(A_i^*)' = A_i$, $i = 0, 1$, we get the normative duality of the couple $\vec{A}$ for the couple $\vec{A}^*$, and therefore $\psi_{M_{\varphi_A^*}}(t, \vec{A}^*) = \varphi_{\Lambda_{\varphi_A}}(t, \vec{A}) = \varphi_A(t, \vec{A})$. Denoting by S the unit ball of $M_{\varphi_A^*}(\vec{A}^*)$ we obtain $\sup_{x\in S} tK(1/t, x, \vec{A}^*) = \varphi_A(t)$, i.e. $\sup_{x\in S} K(t, x, \vec{A}^*) = \psi(t)$, where $\psi(t) = t\varphi_A(1/t)$ is also a quasi-power function.

Fix some $\tau > 0$ and a sufficiently small $\varepsilon > 0$. There exists $x_\tau \in S$ such that $K(\tau, x_\tau, \vec{A}^*) \ge (1-\varepsilon)\psi(\tau)$ while $K(t, x_\tau, \vec{A}^*) \le \psi(t)$ for any $t > 0$. It follows from the definition of the K-functional that for each $t > 0$ there

exists $z = z(\tau, t) \in A_0^*$ such that

$$\|z(\tau,t)\|_{A_0^*} + t\|x_\tau - z(\tau,t)\|_{A_1^*} \le (1+\varepsilon)K(t, x_\tau, \vec{A}^*) \le (1+\varepsilon)\psi(t).$$

As was shown in [BK2], since ψ is a quasi-power function, for each $r > 1$ there exists $\lambda > 1$ such that $r\psi(t) \le \psi(\lambda t) \le \lambda\psi(t)/r$, $\forall t > 0$. We fix some $r > 2(1+\varepsilon)/(1-\varepsilon)$ and the corresponding λ. Now we define $u_\tau \in \Delta(\vec{A}^*)$ by

$$u_\tau = z(\tau, \lambda\tau) - z(\tau, \tau/\lambda) = (x_\tau - z(\tau,\tau/\lambda)) - (x_\tau - z(\tau, \lambda\tau)).$$

Then two estimates are valid simultaneously:

$$\begin{aligned} K(s, u_\tau, \vec{A}^*) &\le \|u_\tau\|_{A_0^*} \le \|z(\tau,\lambda\tau)\|_{A_0^*} + \|z(\tau,\tau/\lambda)\|_{A_0^*} \\ &\le (1+\varepsilon)\psi(\lambda\tau) + (1+\varepsilon)\psi(\tau/\lambda), \\ K(s, u_\tau, \vec{A}^*) &\le s\|u_\tau\|_{A_1^*} \le s\|x_\tau - z(\tau,\tau/\lambda)\|_{A_1^*} + s\|x_\tau - z(\tau,\lambda\tau)\|_{A_1^*} \\ &\le (1+\varepsilon)\frac{s\lambda}{\tau}\psi(\tau/\lambda) + (1+\varepsilon)\frac{s}{\lambda\tau}\psi(\lambda\tau), \end{aligned}$$

or by the quasi-concavity of ψ,

$$K(s, u_\tau, \vec{A}^*) \le (1+\varepsilon)(\lambda+1)\psi(\tau), \qquad K(s, u_\tau, \vec{A}^*) \le (1+\varepsilon)(\lambda+1)s\psi(\tau)/\tau.$$

Combining these we obtain

$$K(s, u_\tau, \vec{A}^*) \le (1+\varepsilon)(\lambda+1)\min(1, s/\tau)\psi(\tau). \tag{3}$$

On the other hand, for each t, the K-functional has all the properties of the norm, hence

$$\begin{aligned} &K(\tau, u_\tau, \vec{A}^*) \\ &\quad\ge K(\tau, x_\tau, \vec{A}^*) - K(\tau, z(\tau,\tau/\lambda), \vec{A}^*) - K(\tau, x_\tau - z(\tau,\lambda\tau), \vec{A}^*) \\ &\quad\ge K(\tau, x_\tau, \vec{A}^*) - \|z(\tau,\tau/\lambda)\|_{A_0^*} - \tau\|x_\tau - z(\tau,\lambda\tau)\|_{A_1^*} \\ &\quad\ge (1-\varepsilon)\psi(\tau) - (1+\varepsilon)\psi(\tau/\lambda) - (1+\varepsilon)\psi(\lambda\tau)/\lambda. \end{aligned}$$

But $\psi(\tau/\lambda) \le \psi(\tau)/r$ and $\psi(\lambda\tau) \le \lambda\psi(\tau)/r$, whence

$$K(\tau, u_\tau, \vec{A}^*) \ge (1-\varepsilon)\psi(\tau) - 2(1+\varepsilon)\psi(\tau)/r. \tag{4}$$

By the choice of r,

$$\gamma = (1-\varepsilon) - 2(1+\varepsilon)/r > 0.$$

Setting $v_\tau = u_\tau/(\gamma\psi(\tau))$ we obtain from (3) and (4) the estimate

$$\min(1, s/\tau) \le K(s, v_\tau, \vec{A}^*) \le C\min(1, s/\tau)$$

where $C = (1+\varepsilon)(\lambda+1)/\gamma$.

Thus for each $\tau > 0$ there exists an element in $A_0^* + A_1^*$ whose K-functional is equivalent to $\min(1, s/\tau)$ with equivalence constant independent of τ. Again referring to [BK2] we hence obtain an abundance of K-functionals for the couple $\vec{A}^*$. By Proposition 6 this guarantees the nonequivalence of $M_\varphi(\vec{A}^*)$ and $M_\psi(\vec{A}^*)$ with any nonequivalent φ and ψ. Finally, the duality of generalized Lorentz and Marcinkiewicz spaces gives the analogous nonequivalence for $\Lambda_\varphi(\vec{A})$ and $\Lambda_\psi(\vec{A})$.

It remains to show that for each $\Lambda_\varphi(\vec{A})$ its fundamental function is equivalent to the index. To get a contradiction, suppose that $\psi(t) = \varphi_{\Lambda_\varphi(\vec{A})}(t, \vec{A})$ is not equivalent to $\varphi(t)$; evidently, $\psi(t) \le \varphi(t)$. By definition $\Lambda_\varphi(\vec{A}) \subset A$ for each A with $\varphi_A(t, \vec{A}) \le \varphi(t)$, so $\varphi_A(t) \le \psi(t)$. Hence $\Lambda_\varphi(\vec{A}) = \Lambda_\psi(\vec{A})$, which is impossible. This finishes the proof.

4. The fundamental functions in some special spaces. As a rule, the exact computation of fundamental functions is a difficult problem. It is easier to get estimates of the form $\varphi_A(t, \vec{A}) \le C\varphi(t)$ by proving an inequality

(5) $$\|x\|_A \le C\|x\|_{A_0}\varphi(\|x\|_{A_1}/\|x\|_{A_0})$$

for some quasi-concave function φ and all $x \in \Delta(\vec{A})$. Families of spaces having such a property for a given couple $\vec{A}$ were studied in [P5] under the name of *massives*. A massive is called *homogeneous* if similar inequalities connect any three spaces in the massive. Estimates like (5) occur in almost all abstract constructions of spaces with function indices: the real interpolation method, the constructions of Calderón–Lozanovskiĭ, the functors of Ovchinnikov [O1], Janson [J] etc. An inverse estimate for the fundamental function can be obtained by constructing special elements on which the inequality (5) turns into equality.

EXAMPLE 1. The Hölder spaces H_φ consist of functions x continuous on $[0,1]$ with modulus of continuity $\omega(t,x)$ satisfying

$$\|x\|_{H_\varphi} = \sup_t \omega(t,x)\varphi(t)/t < \infty$$

(two functions differing by a constant are identified). In the case of $\varphi(t) = \varphi_0(t)\theta(\varphi_1(t)/\varphi_0(t))$ where φ_0, φ_1 and θ are quasi-concave, one can easily establish the inequality

(6) $$\|x\|_{H_\varphi} \le \|x\|_{H_{\varphi_0}}\theta(\|x\|_{H_{\varphi_1}}/\|x\|_{H_{\varphi_0}}).$$

If $x(t) = \min(t,\lambda)$ for some $\lambda \in (0,1)$, then $\|x\|_{H_\varphi} = \varphi(\lambda)$ for each quasi-concave φ and equality is achieved in (6), thus

(7) $$\varphi_{H_\varphi}(t, H_{\varphi_0}, H_{\varphi_1}) = \theta(t).$$

Consider now an arbitrary space H_φ intermediate for the couple $(H_{\varphi_0}, H_{\varphi_1})$ (for that, it is sufficient that $\min(\varphi_0,\varphi_1) \le \varphi \le \max(\varphi_0,\varphi_1)$). By the general properties of quasi-concave functions we again obtain (7) with

$$\theta(t) = \sup_s \varphi(s)\min(1/\varphi_0(s), t/\varphi_1(s)).$$

The same "test" functions $x(t) = \min(t,\lambda)$ allow us to establish the inequality (1) with $\vec{A} = (H_0, H_1)$, $\gamma = 1$. Since it is well known that H_φ is an interpolation space for (H_0, H_1) we may apply Propositions 1 and 2, whence

$$\psi_{H_\varphi}(t, H_0, H_1) = \varphi^*_{H_\varphi}(t, H_0, H_1) = t/\varphi(t).$$

EXAMPLE 2. Inequalities of the type (6) are valid for different families of rearrangement invariant spaces if their fundamental functions (in the classical sense) are taken as indices. For instance, Lorentz, Marcinkiewicz and Orlicz spaces are of that kind. At the same time, equality is achieved for each $x(t) = \chi_{[0,\lambda]}(t)$, hence a formula for a "mutually" fundamental function like (7) is also valid for these families. Let us now take three arbitrary rearrangement invariant spaces E, E_0, E_1 such that $E \in \pi(\vec{E})$, $\vec{E} = (E_0, E_1)$ and try to estimate $\varphi_E(t, \vec{E})$ through their classical fundamental functions $\varphi_E(t), \varphi_{E_0}(t), \varphi_{E_1}(t)$. Using characteristic functions one can easily get the inequality

$$\varphi_E(t, \vec{E}) \ge \theta(t) = \sup_s \varphi_E(s)\min(1/\varphi_{E_0}(s), t/\varphi_{E_1}(s)),$$

hence only the opposite inequality is a problem. Sometimes such an inequality can be obtained by using the relation [P1]

$$\Lambda_\varphi(\vec{E}) = \Lambda_\theta(M_{\varphi_0}(\vec{E}), M_{\varphi_1}(\vec{E})), \qquad \varphi = \varphi_0\theta(\varphi_1/\varphi_0),$$

which holds when θ and φ_1/φ_0 are quasi-power functions. Indeed, if $\varphi_E, \varphi_{E_0}, \varphi_{E_1}$ may be taken respectively for $\varphi, \varphi_0, \varphi_1$ in this relation, then

$$\varphi_E(t, \vec{E}) \le \varphi_{\Lambda_\varphi(\vec{E})}(t, M_{\varphi_0}(\vec{E}), M_{\varphi_1}(\vec{E})) \le \theta(t).$$

Notice that in general $\varphi_E(t, \vec{E})$ need not even be equivalent to $\theta(t)$. The corresponding examples are given in [BO], [MM].

EXAMPLE 3. Let E be a Banach lattice on $(0,\infty)$, and w be a measurable almost everywhere positive function. Then $E(w)$ is defined to be the space with the norm $\|x\|_{E(w)} = \|xw\|_E$. We consider the couple $\vec{E}(\vec{w}) = (E(w_0), E(w_1))$ and the intermediate space $E(w)$ with weight $w = w_0\theta(w_1/w_0)$. From convexity of the norm in E it follows easily that

$$\|x\|_{E(w)} \le 2\|x\|_{E(w_0)}\theta(\|x\|_{E(w_1)}/\|x\|_{E(w_0)})$$

for any quasi-concave θ, thus the corresponding estimate for the fundamental function $\varphi_{E(w)}(t, \vec{E}(\vec{w}))$ holds. The inverse estimate depends on the relation between the weight functions w_i, $i = 0,1$, and the measure μ on $(0,\infty)$. Suppose that $\mu\{\alpha < w_1(s)/w_0(s) < \beta\} > 0$ for any α, β from the range of

w_1/w_0. Then

$$\varphi_{E(w)}(t,\vec{E}(\vec{w})) \ge \sup_{\|x\max(w_0,w_1/t)\|_E\le 1} \|xw_0\theta(w_1/w_0)\|_E = \sup_{\|y\|_E\le 1} \left\| \frac{yw_0\theta(w_1/w_0)}{\max(w_0,w_1/t)} \right\| = \operatorname{ess\,sup} \theta(w_1/w_0)\min(1,tw_0/w_1).$$

By the above assumption the "ess sup" may be changed to "sup", hence for all t from the range of w_1/w_0,

$$\varphi_{E(w)}(t,\vec{E}(\vec{w})) \ge \sup\{\theta(w_1/w_0)\min(1,tw_0/w_1)\} = \theta(t),$$

whence $\varphi_{E(w)}(t,\vec{E}(\vec{w})) \sim \theta(t)$.

For weighted sequence spaces the situation changes just because of the violation of the above-mentioned assumption. Both the fundamental function and the K-functional are continuous functions of a continuous argument, while the elements and the weights are discrete; that leads to different variants of nonequivalence and incompleteness of the stores of K-functionals and fundamental functions. Restricting ourselves for simplicity to one weight we consider the couple $\vec{E} = (E, E(w))$, where $w = (w_n)$, $n = 1,2,\ldots$ For an arbitrary quasi-concave θ we then find that $\varphi_{E(\theta(w))}(t,\vec{E}) \ge \theta(t)$ for all $t = w_n$, $n = 1,2,\ldots$ At the remaining points $t > 0$ the fundamental function does not, in fact, depend on $\theta(t)$ but is always less than $2\theta(t)$. Taking into account the quasi-concavity of the fundamental function and the minimal possible $\theta(t)$ we get the estimate

$$\widetilde{\theta}(t)/2 \le \varphi_{E(\theta(w))}(t,\vec{E}) \le 2\widetilde{\theta}(t), \quad \forall t > 0,$$

where $\widetilde{\theta}$ is the piecewise linear function connecting the points $(w_n, \theta(w_n))$. Note that in the case of θ concave the equivalence turns into the exact equality $\varphi_{E(\theta(w))}(t,\vec{E}) = \widetilde{\theta}(t)$.

Thus in the case of a couple of weighted sequence spaces the question about the store of fundamental functions (and also of K-functionals) reduces to the question of equivalence of quasi-concave functions which coincide (are equivalent) on a sequence of values of their argument.

THEOREM 2. *Suppose that two quasi-concave functions φ, ψ take equivalent values on some increasing sequence of positive numbers w_n, $n = 1,2,\ldots$, i.e. $\psi(w_n) \le \varphi(w_n) \le C\psi(w_n)$ for all n. In order that any two such functions be equivalent for all $t > 0$, it is necessary and sufficient that the sequence w_{n+1}/w_n be bounded.*

Proof. Without loss of generality we may assume that $\varphi(w_n) = \psi(w_n) = c_n$ for each n. It suffices to consider the minimal and maximal quasi-concave functions taking the given values at the points $t = w_n$. Let φ, ψ be such functions. Then for $w_n < t < w_{n+1}$,

$$\varphi(t) = \min(c_{n+1}, tc_n/w_n), \quad \psi(t) = \max(c_n, tc_{n+1}/w_{n+1}),$$

therefore

$$1 \le \varphi(t)/\psi(t) \le \max(c_{n+1}/c_n, w_{n+1}/t, t/w_n, c_n w_{n+1}/(c_{n+1}w_n)).$$

By quasi-concavity, $c_n \le c_{n+1}$ and $c_{n+1}/c_n \le w_{n+1}/w_n$, and hence

$$1 \le \varphi(t)/\psi(t) \le w_{n+1}/w_n,$$

which proves the sufficiency.

To prove the necessity we consider the pair φ, ψ as described above for $c_n = \sqrt{w_n}$ and compare their values at $t_n = \sqrt{w_n w_{n+1}}$. In this case $\varphi(t_n) = \sqrt{w_{n+1}}$, $\psi(t_n) = \sqrt{w_n}$ and $\varphi(t_n)/\psi(t_n) = \sqrt{w_{n+1}/w_n}$, while $\varphi(w_n) = \psi(w_n)$ for all n.

The above theorem implies the possibility of existence of coinciding sequence spaces $E(\varphi(w)), E(\psi(w))$ with nonequivalent $\varphi(t), \psi(t)$. This fact, however, is not too informative, until the role of the massive $\{E(\varphi(w))\}$ in the collection of all intermediate spaces of the couple $\vec{E}$ is clarified. This vagueness can be avoided by some concrete choice of the space E (see e.g. [O2]).

PROPOSITION 7. *The spaces $\ell_1(\varphi(w))$ are generalized Lorentz spaces for the couple $\vec{\ell}_1 = (\ell_1, \ell_1(w))$, and the spaces $\ell_\infty(\varphi(w))$ are generalized Marcinkiewicz spaces for the couple $\vec{\ell}_\infty = (\ell_\infty, \ell_\infty(w))$.*

Choosing some sequence of positive numbers $w = (w_n)$ such that $\lim_{n\to\infty} w_{n+1}/w_n = \infty$, we construct a couple $\vec{\ell}_1$ for which the spaces $\Lambda_\varphi(\vec{\ell}_1)$ may coincide for nonequivalent indices. As follows from the proof of Theorem 1, only the minimal such index coincides with the fundamental function of $\Lambda_\varphi(\vec{\ell}_1)$, while the others (nonequivalent) cannot at all be realized as fundamental functions of any intermediate space of the couple $\vec{\ell}_1$. Thus the store of fundamental functions for this couple is incomplete. In particular, for this couple there does not exist any intermediate space with a quasi-power fundamental function.

Passing to dual spaces we get the possibility of coincidence of the spaces $M_\varphi(\vec{\ell}_\infty)$ with nonequivalent indices. By Proposition 6 this means the incompleteness of the store of K-functionals for the couple $\vec{\ell}_\infty$ with weight sequence (w_n) as above. The results of [BK2] then show that there is no $x \in \ell_\infty + \ell_\infty(w)$ which has a quasi-power K-functional $K(t, x, \vec{\ell}_\infty)$.

5. The use of fundamental functions for comparison of interpolation functors. A functor $\mathcal{F}$ acting in the category of Banach couples is called an *interpolation functor* if for each Banach couple $\vec{A}$ it determines the space $\mathcal{F}(\vec{A}) \in \pi(\vec{A})$ such that any two triples $A_0, A_1, \mathcal{F}(\vec{A})$ and $B_0, B_1, \mathcal{F}(\vec{B})$ are interpolation triples for each other. Of course, $\mathcal{F}(\vec{A})$ is always an interpolation space for $\vec{A}$. Applying $\mathcal{F}$ to the couple of one-dimensional spaces $(\mathbb{R}, t\mathbb{R}), t > 0$, we obtain the space $\varphi(t)\mathbb{R}$ with some quasi-concave function φ. This function first appeared in [DKO] and was called the characteristic function of the functor $\mathcal{F}$. The authors have also shown that for any Banach couple $\vec{A}$,

$$\Lambda_\varphi(\vec{A}) \subset \mathcal{F}(\vec{A}) \subset M_\varphi(\vec{A}). \tag{8}$$

Recall that Λ_φ and M_φ are interpolation functors themselves, therefore they are extreme among all having the given characteristic function φ. The space $\mathcal{F}(\vec{A})$ turns out to be intermediate for the couple $(\Lambda_\varphi(\vec{A}), M_\varphi(\vec{A}))$. To describe its position in this couple it is again natural to use fundamental functions. In other words, we have to estimate $\varphi_{\mathcal{F}(\vec{A})}(t, \Lambda_\varphi(\vec{A}), M_\varphi(\vec{A}))$ and $\psi_{\mathcal{F}(\vec{A})}(t, \Lambda_\varphi(\vec{A}), M_\varphi(\vec{A}))$. This will provide the comparison of interpolation functors not only with respect to their imbeddings but also with respect to their proximity to the extreme cases. In general, such an estimate may depend not only on the functors, but also on the couple to which they are applied. However, for real interpolation functors one can give some uniform (nontrivial) estimate independent of the couple $\vec{A}$.

Indeed, in [BK2] it was shown that each real interpolation functor is representable in the form $\vec{A}^K_E$, that is, it yields the space with the norm $\|x\| = \|K(t, x, \vec{A})\|_E$, where E is the value of this functor at the couple $(L_\infty, L_\infty(1/t))$ on the semi-axis $(0, \infty)$. The functors $\Lambda_\varphi, M_\varphi$ are just of this type; let $E_0 = \Lambda_\varphi(L_\infty, L_\infty(1/t))$, $E_1 = M_\varphi(L_\infty, L_\infty(1/t))$. Then

$$\varphi_{\mathcal{F}(\vec{A})}(t, \Lambda_\varphi(\vec{A}), M_\varphi(\vec{A})) = \sup_{\substack{\|K(t,x,\vec{A})\|_{E_0} \le 1 \\ \|K(t,x,\vec{A})\|_{E_1} \le t}} \|K(t, x, \vec{A})\|_E \le \varphi_E(t, E_0, E_1).$$

Note that $\Lambda_\varphi, M_\varphi$ can be replaced by arbitrary real interpolation functors to get an analogous result.

From the results of the previous section it follows that the inequality here may not, in general, be changed to equality, and $\varphi_{\mathcal{F}(\vec{A})}(t, \Lambda_\varphi(\vec{A}), M_\varphi(\vec{A}))$ need not even be equivalent to $\varphi_E(t, \vec{E})$. We do not know whether this is possible for a couple $\vec{A}$ with a complete store of K-functionals.

As an example of a nonreal interpolation functor we consider the extensively studied Calderón–Lozanovskiĭ functor, which generalizes the complex method to the case of a function parameter when $\vec{A}$ is a couple of Banach lattices. For each quasi-concave $\alpha(t)$ it yields the space $\mathcal{L}_\alpha(\vec{A})$ consisting of all $x \in \Sigma(\vec{A})$ representable in the form $|x(s)| = |x_0(s)|\widetilde{\alpha}(|x_1(s)|/|x_0(s)|)$, where $x_i \in A_i$, $i = 0, 1$, and $\widetilde{\alpha}(t) = 1/\alpha(1/t)$. This space is normed by $\|x\| = \inf \max(\|x_0\|_{A_0}, \|x_1\|_{A_1})$, where the infimum is taken over all possible representations of x as above. Applying $\mathcal{L}_\alpha$ to the couple $(\mathbb{R}, t\mathbb{R})$ we obtain its characteristic function equal to $\alpha(t)$ and thus the imbeddings $\Lambda_\alpha(\vec{A}) \subset \mathcal{L}_\alpha(\vec{A}) \subset M_\alpha(\vec{A})$. The first of them means that

$$\varphi_{\mathcal{L}_\alpha(\vec{A})}(t, \vec{A}) \le \alpha(t). \tag{9}$$

Again, as above, we pose the problem of estimating the position of $\mathcal{L}_\alpha(\vec{A})$ in the couple $(\Lambda_\alpha(\vec{A}), M_\alpha(\vec{A}))$. Further results will show the dependence of this estimate on the couple $\vec{A}$ and even the character of this dependence in some cases. We restrict ourselves to two situations, in which the functor $\mathcal{L}_\alpha$ is easily computable in an explicit form. First we study couples of weighted spaces $\vec{E} = (E, E(w))$, for which $\mathcal{L}_\alpha(\vec{E}) = E(\alpha(w))$.

THEOREM 3. *Let $E = L_p$ $(p \ge 1)$ and suppose the weight function w is not equivalent to a constant. Then*

$$\varphi_{\mathcal{L}_\alpha(\vec{L}_p)}(t, \Lambda_\alpha(\vec{L}_p), M_\alpha(\vec{L}_p)) \le Ct^{1/q}, \quad \forall t > 0,\ 1/q = 1 - 1/p, \tag{10}$$

where C does not depend on p and α.

Remark. Just as for sequence spaces, the analog of Proposition 7 is also valid for L_p spaces. Namely, $\mathcal{L}_\alpha(\vec{L}_1) = \Lambda_\alpha(\vec{L}_1)$, which yields $q = \infty$ in (10), while $\mathcal{L}_\alpha(\vec{L}_\infty) = M_\alpha(\vec{L}_\infty)$ yields $q = 1$ in (10). Thus in these extreme cases the inequality (10) is known to hold and even turns into equality by property 4 of fundamental functions. The core of Theorem 3 is to show in what manner the fundamental function varies with p passing from 1 to ∞.

Proof. To get (10) we must take some x satisfying $\|x\|_{\Lambda_\alpha(\vec{L}_p)} \le 1$, $\|x\|_{M_\alpha(\vec{L}_p)} \le t$ and estimate its norm in $\mathcal{L}_\alpha(\vec{L}_p) = L_p(\alpha(w))$. Fix such an x and set $\varrho(s) = K(s, x, \vec{L}_p)$. It was shown in [O2] that by concavity of $\varrho(s)$ there exist two positive numerical sequences $a = (a_n)$, $\sigma = (\sigma_n)$ such that

$$\varrho(s)/2 \le K(s, a, \ell_\infty, \ell_\infty(\sigma)) \le K(s, a, \ell_1, \ell_1(\sigma)) \le 2\varrho(s). \tag{11}$$

It then follows from Proposition 4 that

$$\|a\|_{\Lambda_\alpha(\ell_1, \ell_1(\sigma))} \le 2\|x\|_{\Lambda_\alpha(\vec{L}_p)} \le 2,$$
$$\|a\|_{M_\alpha(\ell_\infty, \ell_\infty(\sigma))} \le 2\|x\|_{M_\alpha(\vec{L}_p)} \le 2t.$$

In view of Proposition 7, this corresponds to the inequalities

$$\|a\|_{\ell_1(\alpha(\sigma))} \le 2, \quad \|a\|_{\ell_\infty(\alpha(\sigma))} \le 2t. \tag{12}$$

Using the Hölder inequality we now obtain

$$\|a\|_{\ell_p(\alpha(\sigma))} \le 2t^{1/q}.$$

Now consider the triples $\vec{L}_p, L_p(\alpha(w))$ and $\vec{\ell}_p, \ell_p(\alpha(\sigma))$. By the Ovchinnikov theorem [O1] they are interpolation triples and from [S] it follows that this interpolation is relatively K-monotone. Hence the elements with equivalent K-functionals (in their own couples) have equivalent norms in the corresponding spaces. It remains to note that, since ℓ_p is intermediate for the couple (ℓ_1, ℓ_∞), the functional $K(s, a, \ell_p, \ell_p(\sigma))$ is also equivalent to $\varrho(s)$ and so

$$\|x\|_{L_p(\alpha(w))} \le Ct^{1/q}.$$

THEOREM 4. *For the fundamental co-function of the space* $\mathcal{L}_\alpha(\vec{L}_p)$ *for the couple* $(\Lambda_\alpha(\vec{L}_p), M_\alpha(\vec{L}_p))$ *the following estimate holds*:

$$\psi_{\mathcal{L}_\alpha(\vec{L}_p)}(t, \Lambda_\alpha(\vec{L}_p), M_\alpha(\vec{L}_p)) \le Ct^{1/p}. \tag{13}$$

Proof. We start with the well-known duality relations

$$\vec{L}_p^* = (L_q(1/w), L_q), \qquad (\mathcal{L}_\alpha(\vec{L}_p))^* = L_q(1/\alpha(w)) = \mathcal{L}_{\alpha^*}(\vec{L}_p^*),$$

where $\alpha^*(t) = t/\alpha(t)$. Moreover, the couple $(\Lambda_{\alpha^*}(\vec{L}_p^*), M_{\alpha^*}(\vec{L}_p^*))$ is a normative dual for $(\Lambda_\alpha(\vec{L}_p), M_\alpha(\vec{L}_p))$. Hence from Theorem 3 it follows that

$$\psi_{\mathcal{L}_\alpha(\vec{L}_p)}(t, \Lambda_\alpha(\vec{L}_p), M_\alpha(\vec{L}_p)) = \varphi_{\mathcal{L}_{\alpha^*}(\vec{L}_p^*)}(t, \Lambda_{\alpha^*}(\vec{L}_p^*), M_{\alpha^*}(\vec{L}_p^*)) \le Ct^{1/p}.$$

COROLLARY. *We have the imbeddings*

$$\Lambda_{t^{1/q}}(\Lambda_\alpha(\vec{L}_p), M_\alpha(\vec{L}_p)) \subset L_p(\alpha(w)) \subset M_{t^{1/q}}(\Lambda_\alpha(\vec{L}_p), M_\alpha(\vec{L}_p)),$$

which means that $L_p(\alpha(w))$ *is an interpolation space with respect to one-dimensional operators for the couple* $(\Lambda_\alpha(\vec{L}_p), M_\alpha(\vec{L}_p))$.

The next result relates to comparison of the functors $\Lambda_\alpha, M_\alpha, \mathcal{L}_\alpha$ applied to the couple $\vec{L} = (L_\infty, L_1)$, for which they yield, respectively, the Lorentz, Marcinkiewicz and Orlicz spaces with fundamental function α. We also use the function

$$m(t) = \sup_s \alpha(ts)/\alpha(s)$$

which is widely used in interpolation theory.

THEOREM 5. *The fundamental function and co-function of the space* $\mathcal{L}_\alpha(\vec{L})$ *for the couple* $(\Lambda_\alpha(\vec{L}), M_\alpha(\vec{L}))$ *satisfy the inequalities*

$$\varphi_{\mathcal{L}_\alpha(\vec{L})}(t, \Lambda_\alpha(\vec{L}), M_\alpha(\vec{L})) \le Ctm(1/t), \tag{14}$$

$$\psi_{\mathcal{L}_\alpha(\vec{L})}(t, \Lambda_\alpha(\vec{L}), M_\alpha(\vec{L})) \le Cm(t). \tag{15}$$

Proof. As in the proof of Theorem 3 we fix x such that $\|x\|_{\Lambda_\alpha(\vec{L})} \le 1$, $\|x\|_{M_\alpha(\vec{L})} \le t$ and define $\varrho(s) = K(s, x, \vec{L})$. We also take sequences a, σ satisfying (11), which leads to inequalities (12). From this point on, the proof becomes different because instead of $\ell_p(\alpha(\sigma))$ we need $\ell_m(\alpha(\sigma)) = \mathcal{L}_m(\ell_\infty(\alpha(\sigma)), \ell_1(\alpha(\sigma)))$. It then follows from (9) and from property 6 of fundamental functions that $\|a\|_{\ell_m(\alpha(\sigma))} \le 2tm(1/t)$.

The next step consists in establishing the imbedding $\ell_m(\alpha(\sigma)) \subset \mathcal{L}_\alpha(\ell_\infty, \ell_1(\sigma))$. Let $a \in \ell_m(\alpha(\sigma))$, and let $a_0 \in \ell_\infty$, $a_1 \in \ell_1$ be taken in such a way that $a = a_0\widetilde{m}(a_1/a_0)$ and

$$\|a\|_{\ell_m(\alpha(\sigma))} \ge \max(\|a_0\alpha(\sigma)\|_{\ell_\infty}, \|a_1\alpha(\sigma)\|_{\ell_1}) - \varepsilon$$

for some $\varepsilon > 0$. Suppose that $u_0 = a_0\alpha(\sigma)$, $u_1 = a_1\alpha(\sigma)/\sigma$, $u = u_0\widetilde{\alpha}(u_1/u_0)$. Then

$$\begin{aligned}\|u\|_{\mathcal{L}_\alpha(\ell_\infty, \ell_1(\sigma))} &\le \max(\|u_0\|_{\ell_\infty}, \|u_1\sigma\|_{\ell_1}) \\ &= \max(\|a_0\alpha(\sigma)\|_{\ell_\infty}, \|a_1\alpha(\sigma)\|_{\ell_1}) \le \|a\|_{\ell_m(\alpha(\sigma))} + \varepsilon.\end{aligned}$$

On the other hand,

$$\begin{aligned}u = u_0\widetilde{\alpha}(u_1/u_0) &= a_0\alpha(\sigma)\widetilde{\alpha}(a_1/(\sigma a_0)) \\ &= a_0\alpha(\sigma)/\alpha(\sigma a_0/a_1) \ge a_0\widetilde{m}(a_1/a_0) = a,\end{aligned}$$

whence

$$\|a\|_{\mathcal{L}_\alpha(\ell_\infty, \ell_1(\sigma))} \le \|a\|_{\ell_m(\alpha(\sigma))} + \varepsilon.$$

It remains to let $\varepsilon \to 0$.

Now note that, in view of (11), $K(s, a, \ell_\infty, \ell_1(\sigma)) \sim K(s, x, \vec{L})$ and that interpolation from $(\ell_\infty, \ell_1(\sigma), \mathcal{L}_\alpha(\ell_\infty, \ell_1(\sigma)))$ to $(\vec{L}, \mathcal{L}_\alpha(\vec{L}))$ is relatively K-monotone, which implies that $\|x\|_{\mathcal{L}_\alpha(\vec{L})} \le Ctm(1/t)$ and (14) is proved. To prove (15) we pass to dual spaces taking into account that

$$\vec{L}' = \vec{L}, \quad (\mathcal{L}_\alpha(\vec{L}))' = \mathcal{L}_{\alpha^*}(\vec{L}), \quad (\Lambda_\alpha(\vec{L}))' = M_{\alpha^*}(\vec{L}), \quad (M_\alpha(\vec{L}))' = \Lambda_{\alpha^*}(\vec{L})$$

and

$$\sup_s \alpha^*(ts)/\alpha^*(s) = tm(1/t).$$

COROLLARY. *The following imbeddings hold*:

$$\Lambda_{\widetilde{m}^*}(\Lambda_\alpha(\vec{L}), M_\alpha(\vec{L})) \subset \mathcal{L}_\alpha(\vec{L}) \subset M_{m^*}(\Lambda_\alpha(\vec{L}), M_\alpha(\vec{L})).$$

Unlike the corollary to Theorem 4, we obtain here generalized Lorentz and Marcinkiewicz spaces with different indices, which does not allow us to claim that $\mathcal{L}_\alpha(\vec{L})$ is an interpolation space with respect to one-dimensional operators acting in the indicated spaces. The coincidence of indices and the interpolation property can only be obtained for $\alpha(t) = t^{1/p}$ with any $p \ge 1$. Note one more peculiarity of all results in this section. It follows directly from the definitions that for any couple $\vec{A}$ we have $\Lambda_\alpha(\vec{A}) \subset M_\alpha(\vec{A})$ with imbedding constant 1, and thus for each space A lying between $\Lambda_\alpha(\vec{A})$ and $M_\alpha(\vec{A})$ both $\varphi_A(t, \Lambda_\alpha(\vec{A}), M_\alpha(\vec{A}))$ and $\psi_A(t, \Lambda_\alpha(\vec{A}), M_\alpha(\vec{A}))$ are constant for $t \ge 1$. Hence the estimates of fundamental functions and co-functions obtained in this section provide some useful information only for $0 < t < 1$, characterizing the decay of these functions as $t \to 0$.

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