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Closed range multipliers and generalized inverses

by

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Abstract. Conditions involving closed range of multipliers on general Banach algebras are studied. Numerous conditions equivalent to a splitting $A = TA \oplus \ker T$ are listed, for a multiplier T defined on the Banach algebra A. For instance, it is shown that $TA \oplus \ker T = A$ if and only if there is a commuting operator S for which $T = TST$ and $S = STS$, that this is the case if and only if such S may be taken to be a multiplier, and that these conditions are also equivalent to the existence of a factorization $T = PB$, where P is an idempotent and B an invertible multiplier. The latter condition establishes a connection to a famous problem of harmonic analysis.

Introduction. In the study of multipliers on, say, commutative semi-simple Banach algebras, in particular in attempting to characterize circumstances under which a multiplier will have closed range, a factorization of the given multiplier as the product of an idempotent and an invertible has kept showing up as a plausible companion—certainly a sufficient, and possibly equivalent, condition for closed range. In some spectacular special cases, namely the group algebras $L^1(G)$ when G is a locally compact abelian group, this equivalence does hold, as was shown by Host and Parreau in 1978. This note takes steps to uncover the precise relationship between the two. We do this by dealing with the issue in somewhat greater generality, and by relating it to the concept of generalized inverse.

Commutativity keeps looming in the present approach, though, and as a consequence the resulting conditions are slightly stronger than those mentioned before. It turns out that for an arbitrary continuous linear operator T on a Banach space X there is a factorization $T = PB$, where P and B commute, and where B is invertible and P idempotent, precisely when $X = TX \oplus \ker T$. Moreover, when X decomposes in this way, TX is necessarily closed. The realization that these conditions also are connected to the existence of a commuting generalized inverse then becomes our starting point.

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When these general equivalences are considered for multipliers, a number of other conditions appear, more of them as we narrow the class of algebras. To single out two sample results, it is shown, in Theorem 10, that a multiplier T on a semisimple Banach algebra A has a decomposition $A = TA \oplus \ker T$ precisely when 0 is isolated from the possibly punctured spectrum $\sigma(T)\setminus\{0\}$; and in Theorem 13 it is shown that if A is a C^*-algebra then the range TA of a multiplier T is closed if and only if $TA \oplus \ker T = A$.

The organization of the material presented here is simple: we move from the general to the particular. Concepts are introduced formally as they are needed.

General linear operators. Let X be a Banach space and let $B(X)$ denote the Banach algebra of all bounded linear operators of X into itself, equipped with the usual operator norm. For $T \in B(X)$, TX and $\ker T$ will denote the range and kernel, respectively, of T.

Much of what follows here is based on the following, essentially a consequence of the open mapping theorem. These facts may well be known to many, but a written source has eluded us, so we include a proof.

LEMMA 1. *Let $T \in B(X)$ and suppose that $TX \cap \ker T = \{0\}$ and that $TX + \ker T$ is closed. Then $T^n X$ is closed for every $n \in \mathbb{N}$.*

Proof. We begin by showing that TX is closed in the given norm. In this norm the space $X_0 := TX \oplus \ker T$ is a Banach space, by assumption. Moreover, it is routine to see that TX is a Banach space, when equipped with the norm

$$|\!|\!| u |\!|\!| := \|u\| + \inf_{v\in X,\, u=Tv} \|v\| .$$

Moreover, since $\|u\| \le |\!|\!| u |\!|\!|$ for any $u \in TX$, the injection $TX \to X_0$ is continuous.

Define $J : TX \times \ker T \to X_0$ by $J(u, v) := u + v$. Then J is a continuous bijection, so by the open mapping theorem, J is bicontinuous, hence $TX = J(TX \times \{0\})$ is closed in X_0, and hence also in X.

Thus T has closed range. Since $\ker T \cap TX = \{0\}$, $\ker T^2 = \ker T$ and thus $\ker T^n = \ker T$ for all $n \in \mathbb{N}$. We can now complete the proof by an inductive argument: if T^n has closed range for some $n \in \mathbb{N}$, then, because $TX \oplus \ker T = TX \oplus \ker T^n$ is closed by assumption, $T^{n+1}X = T^n(TX \oplus \ker T) = T^n(TX \oplus \ker T^n)$ is closed, by [Kato, Lemma 331].

There is an elementary converse: if T^2X is closed then $TX + \ker T$ is closed. Note that no assumption of direct sum is needed. In fact, if T^2X is closed, suppose $Tx_n + z_n \to b$, where z_n is in $\ker T$. Then $T^2x_n \to Tb$, so by assumption there is a $c \in X$ for which $Tb = T^2c$. Since $z := b - Tc \in \ker T$, $b = Tc + z \in TX + \ker T$.

As it happens these observations constitute most of the proof of the following extension to general linear operators of [Aiena–Laursen, Theorem 3.1].

COROLLARY. *Suppose $T \in B(X)$ satisfies $TX \cap \ker T = \{0\}$. Then the following conditions are equivalent.*

(a) *$TX + \ker T$ is norm-closed.*

(b) *$T^n X$ is closed for all $n \in \mathbb{N}$.*

(c) *T^2X is closed.*

(d) *The induced map $T^{\sim} : X/\ker T \to X/\ker T$ has closed range.*

Proof. Only the equivalence (a)⇔(d) remains: as in [Aiena–Laursen], let $Q : X \to X/\ker T$ be the quotient map. Then $T^{\sim}(X/\ker T) = Q(TX + \ker T)$, hence $Q^{-1}(T^{\sim}(X/\ker T)) = TX + \ker T$, so $T^{\sim}(X/\ker T)$ is closed if and only if the same holds for $TX + \ker T$.

We shall say that $T \in B(X)$ has a *generalized inverse*, and write that T has a *g-inverse*, or that T is *g-invertible*, if there is an operator $S \in B(X)$ for which

$$T = TST \quad \text{and} \quad S = STS .$$

Remark 2. (a) There is no gain of generality in requiring only that $T = TST$, because then $S' := STS$ will satisfy $T = TS'T$, as well as $S' = S'TS'$.

(b) If $T = TST$ and $S = STS$ then TS and ST are idempotents for which $TSX = TX$ and $\ker T = \ker ST$.

(c) It is easy to see that a g-inverse rarely is uniquely determined: if T is g-invertible and $T = TST$, then S can be "anything" on $\ker T$. But there is at most one g-inverse which commutes with the given $T \in B(X)$ [Harte–Mbekhta, Theorem 9]. In fact, if S and S' are g-inverses of T, both commuting with T, then $TS' = TSTS' = STS'T = ST$, and hence $S' = S'TS' = S'TS = STS = S$.

THEOREM 3. *If X is a Banach space and $T \in B(X)$, then the following statements are equivalent.*

(a) *T has a commuting g-inverse.*

(b) *$TX \oplus \ker T = X$.*

(c) *$T = PB = BP$, where $B \in B(X)$ is invertible and $P \in B(X)$ is idempotent.*

(d) *$T = TCT$, where $C \in B(X)$ is invertible and $TC = CT$.*

Proof. Suppose (a) holds. Let S be a g-inverse of T for which $TS = ST$. Then the identity $I = TS + (I - TS) = TS + (I - ST)$, where I is the unit of $B(X)$, together with Remark 2(b) show that (b) holds.

Assume (b). By Lemma 1, TX is closed. Moreover, since $T^2X = T(TX) = T(TX \oplus \ker T) = TX$ and $\ker T \cap TX = \{0\}$, it follows that $T|TX$ is invertible. Define $B := (T|TX) \oplus I_{\ker T}$. Clearly B is invertible. If we let P be the projection of X onto TX with $\ker P = \ker T$, then $T = PB = BP$. This proves (c).

If (c) holds then (d) follows with $C := B^{-1}$ by straightforward calculation.

If (d) holds then $S := C^2T$ is a commuting g-inverse of T.

The property "$T = TCT$, where $C \in B(X)$ is invertible" of (d) above is called *decomposably regular* in [Harte–Mbekhta].

Remark 4. The condition (b) of Theorem 3 is equivalent to the condition "$0 \in \mathbb{C}$ is a pole of the resolvent $(T-\lambda)^{-1}$ of order 0 or 1" as well as to the condition "$T^2X = TX$ and $\ker T = \ker T^2$" [Heuser, Propositions 38.4 and 50.2]. This latter condition is also described by the phrase "T has descent and ascent both equal to 1". Recall that T is said to have *descent* n if n is the smallest integer for which $T^nX = T^{n+1}X$ and to have *ascent* n if n is the smallest integer for which $\ker T^n = \ker T^{n+1}$.

Multipliers. We now specialize to multipliers on Banach algebras. We shall let A denote a Banach algebra and assume throughout that A is *without order*. This means that only the zero-element annihilates the whole algebra, i.e. that if $aA = \{0\}$ or $Aa = \{0\}$, then $a = 0$. Just the basics of the theory of multipliers defined on a Banach algebra without order will be needed here, as described in e.g. [Larsen, 1.1]. To recapitulate briefly, a map $T : A \to A$ is called a *multiplier* if $x(Ty) = (Tx)y$ for all $x, y \in A$. Because A is without order, any multiplier turns out to be norm-continuous and linear; the identities $x(Ty) = T(xy)$ and $(Ty)x = T(yx)$ hold for any $x, y \in A$, so that the set $M(A)$ of multipliers may be described as the commutant in $B(A)$ of all operators of multiplication (on the right or on the left) by the elements of the algebra A. The set $M(A)$ is a norm-closed commutative subalgebra of $B(A)$.

Note also that since $x(Ty) = T(xy)$ and $(Ty)x = T(yx)$ for any $x, y \in A$, both the range TA and the kernel $\ker T$ are two-sided ideals of A.

If A is a Banach algebra without order and T is a multiplier on A, then there is at most one g-inverse in $M(A)$ of T; this follows from the commutativity of $M(A)$ and Remark 2(c). We shall see shortly, in Theorem 5, that if $T \in M(A)$ has a commuting g-inverse at all, this will necessarily be a multiplier. This corresponds to the fact that if a multiplier has an inverse (as a linear operator), then this inverse is necessarily a multiplier [Larsen, Theorem 1.1.3].

The next result develops [Harte–Mbekhta, Theorem 9], which proves the implications (i)⇔(v) and (i)⇔(viii). It is also a partial extension of [Aiena–Laursen, Theorem 4.2].

THEOREM 5. *Let A be a Banach algebra without order. Let $T \in M(A)$. Then the following conditions are equivalent.*

(i) *T has a commuting g-inverse.*
(ii) *T has a g-inverse $S \in B(A)$ for which $TS \in M(A)$.*
(iii) *T has a g-inverse $S \in B(A)$ for which TS commutes with T.*
(iv) *T has a g-inverse $S \in M(A)$.*
(v) *$TA \oplus \ker T = A$.*
(vi) *$T^2A = TA$ and $\ker T^2 = \ker T$.*
(vii) *$T = PB = BP$, where $B \in M(A)$ is invertible and $P \in M(A)$ is idempotent.*
(viii) *T is decomposably regular in $M(A)$, i.e. $T = TCT$, where C is an invertible multiplier.*

Proof. (i)⇒(ii). Let S be a commuting g-inverse of T and let $P := TS$. Then, by Remark 2(b), P is an idempotent for which $PA = TA$ and $\ker T = \ker P$, i.e. both kernel and range of P are two-sided ideals. This implies that P is a multiplier, because if $x = Px + (I-P)x$ then $xPy = PxPy + (I-P)xPy$, and since $(I-P)xPy \in \ker P \cap PA = \{0\}$, it follows that $xPy = PxPy$. Similarly, $(Px)y = PxPy$, hence $(Px)y = xPy$ for any $x, y \in A$.

(ii)⇒(iii). Trivial because $M(A)$ is a commutative algebra.

(iii)⇒(v). We know that if $P := TS$ then $PA = TA$. If $x \in A$ then $Tx = Pz$, for suitable $z \in A$, hence if $Px = 0$ then $Tx = P^2z = PTx = TPx = 0$, and so $\ker P \subseteq \ker T$. It follows that $A = TA + \ker T$. If $x \in TA \cap \ker T$ then $x = 0$ provided we show that $xTA = x\ker T = \{0\}$. But $xTA = TxA = \{0\}$, so only $x \ker T = \{0\}$ remains; and if $x = Tz \in TA$, while $y \in \ker T$, then $xy = (Tz)y = z(Ty) = 0$. This yields (v).

(v)⇔(vi). This was noted in Remark 4.

Note that by Theorem 3 we now have the equivalence of (i), (ii), (iii), (v), and (vi).

(v)⇒(vii). If we assume (v) (and hence also (vi)) then the projection $P : A \to A$ with $PA = TA$ and $\ker P = \ker T$ is a multiplier, by (ii). Consequently, $B := T + I - P \in M(A)$. Note that B is the same operator as the operator B described in the proof of Theorem 3, (b)⇒(c), and hence the present B is invertible. Since $T = BP = PB$, (vii) follows.

(vii)⇒(iv). Let $S := PB^{-1}$.

(iv)⇒(v). Since $M(A)$ is commutative, this follows from Theorem 3, (a)⇒(b).

(vii)⇒(viii). If $T = PB = BP$, where $B \in M(A)$ is invertible and $P \in M(A)$ is idempotent, then $TB^{-1}T = PBB^{-1}T = PT = T$.

(viii)⇒(i). If $T = TCT$, where C is an invertible multiplier, then $S := CTC \in M(A)$ is a commuting g-inverse of T.

This completes the proof of Theorem 5.

Semiprime and semisimple algebras. An algebra A is said to be *semiprime* if $\{0\}$ is the only two-sided ideal J for which $J^2 = \{0\}$ [Bonsall–Duncan, Definition IV.30.3]. Alternatively, as is easily seen, A is semiprime if and only if $aAa = \{0\}$ implies that $a = 0$. It is then straightforward to see that a semiprime algebra is without order. We shall need one fact about multipliers on semiprime algebras, related to Remark 4, namely that they have ascent ≤ 1, i.e. that $\ker T^2 = \ker T$: if $T^2x = 0$, then $(Tx)a(Tx) = T(xT(ax)) = T^2(xax) = (T^2x)ax = 0$, for any $a \in A$, hence $Tx = 0$.

THEOREM 6. *Let A be a semiprime Banach algebra. Let $T \in M(A)$. Then the following conditions are equivalent with the conditions mentioned in Theorem* 5.

(ix) $T^2A = TA$, *i.e. T has descent ≤ 1.*

(x) *T has finite descent.*

Proof. Since A is semiprime T has ascent ≤ 1, and so the equivalence of the two conditions listed here is a general fact [Heuser, §38].

(v)⇒(ix). This was noted in proving the implication (b)⇒(c) of Theorem 3.

(ix)⇒(v). This is a consequence of Remark 4.

COROLLARY 7. *Let A be a semiprime Banach algebra. Let $T \in M(A)$. Any of the conditions of Theorem* 5 *entails that* $\operatorname{dist}(0, \sigma(T)\backslash\{0\}) > 0$.

Proof. Obviously only the case $0 \in \sigma(T)$ need concern us. From Theorem 5(v) and Remark 4 we conclude that $0 \in \mathbb{C}$ is a pole of $(T-\lambda)^{-1}$ of order 1. The corollary is immediate from this. We can also argue directly: the operator $T - \lambda$ is invertible if and only if $(T-\lambda)|TA$ and $(T-\lambda)|\ker T$ both are invertible. The corollary then follows since $T|TA$ is invertible, while $\sigma(T|\ker T) = \{0\}$.

COROLLARY 8. *Let A be a semiprime Banach algebra. Let $T \in M(A)$. If $T^2A = TA$ then TA is closed.*

Proof. Combine Theorems 5 and 6 with Lemma 1.

The converse is false (cf. Remark 12). However, there are circumstances in which a converse does hold, e.g. if A is a C^*-algebra, as Theorem 13 below shows. Also, [Host–Parreau, Théorème 1] shows that if A is the usual convolution group algebra $L^1(G)$ of a locally compact abelian group G, then a multiplier with closed range will be the product of an idempotent and an invertible, i.e. satisfy condition (vii) of Theorem 5. Other situations in which a converse holds are described in [Aiena–Laursen, Corollary 4.9].

Recall that a Banach algebra is *semisimple* if the kernels of all irreducible representations have trivial intersection [Bonsall–Duncan, Definition III.24.13]. Any C^*-algebra is semisimple (cf. e.g. [Rickart, Theorem (4.1.19)]). Semisimple Banach algebras are semiprime [Bonsall–Duncan, Proposition IV.30.5], and the two classes are distinct, even within the category of commutative algebras: any commutative algebra which is an integral domain will be semiprime; thus for instance weighted convolution algebras $L^1(\mathbb{R}_+, \omega)$, where the weight ω is chosen so that $\omega(t)^{1/t} \to 0$, as $t \to \infty$, in order to make the algebra radical, will be semiprime, but not semisimple. About multipliers on semisimple Banach algebras we need to note the following.

LEMMA 9. *Let A be a semisimple Banach algebra. If $T \in M(A)$ is nonzero then T is not quasi-nilpotent.*

Proof. Suppose $\|T^n\|^{1/n} \to 0$ as $n \to \infty$. To show that $T = 0$ it suffices to show that $\|(aTx)^n\|^{1/n} = \|(T(ax))^n\|^{1/n} \to 0$ as $n \to \infty$, for every $a, x \in A$, as this would place Tx in the radical of A [Bonsall–Duncan, Proposition III.25.1], hence show that $Tx = 0$, for any $x \in A$. By an easy inductive argument $(Ty)^n = (T^ny)y^{n-1}$ for every $n \in \mathbb{N}$ and every $y \in A$, and hence it follows that

$$\|(aTx)^n\|^{1/n} = \|T^n(ax)(ax)^{n-1}\|^{1/n} \leq \|T^n\|^{1/n}\|ax\| .$$

This proves the lemma.

The next result shows strong analogies to the situation for normal operators on Hilbert space. It is well known that a normal operator N on a Hilbert space H has a nontrivial Riesz decomposition of the form $H = \ker N \oplus NH$ if and only if 0 is an isolated eigenvalue of N.

THEOREM 10. *Let A be a semisimple Banach algebra with multiplier algebra $M(A)$ and let $T \in M(A)$. Then the conditions of Theorems* 5 *and* 6 *are equivalent to*

(xi) $\operatorname{dist}(0, \sigma(T)\backslash\{0\}) > 0$.

Proof. If $\operatorname{dist}(0, \sigma(T)\backslash\{0\})$ is positive, and if $0 \in \sigma(T)$, then by Remark 4 we must show that 0 is a pole of $(T-\lambda)^{-1}$ of order 1, i.e. show that when the Riesz functional calculus provides us with closed T-invariant subspaces M and N for which $A = M \oplus N$ and $0 \notin \sigma(T|M)$, while $\sigma(T|N) = \{0\}$, then $TN = \{0\}$. Clearly $M \subseteq TA$ and $\ker T \subseteq N$. It remains to see that $N \subseteq \ker T$. Since $T|N$ is a quasi-nilpotent multiplier, it follows from Lemma 9 that the proof will be complete once the semisimplicity of N

has been noted; and this is standard (cf. e.g. [Bonsall–Duncan, Corollary III.24.20]).

COROLLARY 11. *Let A be a semisimple Banach algebra. Let $T \in M(A)$. If* $\operatorname{dist}(0, \sigma(T)\setminus\{0\}) > 0$ *then TA is closed.*

Proof. By Theorem 10, $TA = T^2A$. Now apply Corollary 8.

Remark 12. Consider the disc algebra $A := A(\mathbb{D})$ of complex-valued functions, defined and continuous on the closed unit disc $\mathbb{D}$, analytic in its interior, and let T_z be the multiplication operator defined by $(T_z f)(\zeta) := \zeta f(\zeta)$, for every $f \in A(\mathbb{D})$ and every $\zeta \in \mathbb{D}$. Then $T_z \in M(A)$,

$$T_z A = \{f \in A \mid f(0) = 0\}$$

and

$$T_z^2 A = \{f \in A \mid f(0) = f'(0) = 0\}.$$

Both $T_z A$ and $T_z^2 A$ are closed, but obviously distinct. This shows that the converse of Corollary 8 does not hold. Since $0 \in \sigma(T_z)$, but is nonisolated, this example also shows that the converse of Corollary 11 is not true, either.

We can also see that condition (v) of Theorem 5 cannot be relaxed to that of Lemma 1, i.e. to the requirement that $TA \oplus \ker T$ be closed: since $\ker T_z = \{0\}$, $T_z A \oplus \ker T_z$ is closed, but none of the conditions of Theorem 5 holds for T_z. This observation contrasts with [Aiena–Laursen, Theorem 4.2].

There are other important cases in which converses do hold, namely if A is a C^*-algebra. An argument which was also employed in proving [Aiena–Laursen, Proposition 3.3] will establish the following. This result contains an analog of [Harte–Mbekhta, Theorem 8].

THEOREM 13. *Let A be a C^*-algebra and let $T \in M(A)$. Then the following statements are equivalent to* (i)–(x) *of Theorems* 5, 6 *and* 10.

(xii) *TA is closed.*
(xiii) *$TA \oplus \ker T$ is closed.*

Proof. (xiii)⇒(xii). This is Lemma 1.

(xii)⇒(ix). Since TA is a closed two-sided ideal in a C^*-algebra, TA has a bounded approximate identity, hence, by Cohen's factorization theorem, $TA = (TA)^2$. But T is a multiplier, so $TaTb = T^2(ab)$. Consequently, $TA = T^2A$.

Since the implication (v)⇒(xiii) is trivial, the proof is complete.

COROLLARY 14. *A multiplier T with closed range on a C^*-algebra A is injective if and only if it is surjective.*

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