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An application of two-parameter martingales in harmonic analysis

by

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Abstract. Some duality results and some inequalities are proved for two-parameter Vilenkin martingales, for Fourier backwards martingales and for Vilenkin and Fourier coefficients.

1. Introduction. Gundy and Varopoulos [11] have proved some inequalities for one-parameter backwards martingales generated by the Fourier series of a function.

Those results are here generalized to two-parameter martingales and to backwards martingales generated by Vilenkin systems and the Fourier system, respectively. First of all the usual martingale inequalities are proved for such systems. Martingale Hardy spaces generated by the L_p norm of the maximal function or of the quadratic variation are equivalent to the L_p space when $1 < p < \infty$. The Hardy space H_p generated by the L_p norm of the conditional quadratic variation is, in general, different from the ones above. However, all Hardy spaces considered in this paper are equivalent for "bounded" Vilenkin martingales and for "bounded" Fourier backwards martingales.

The dual space of H_p $(0 < p < \infty)$ is found. It is $\Lambda_2(\alpha)$ for $0 < p < 1$ $(\alpha = 1/p - 1)$, BMO_2 for $p = 1$, and H_q for $1 < p < \infty$ $(1/p + 1/q = 1)$. Finally, it is proved that the l_2 norm of the "defective" Vilenkin and Fourier coefficients can be estimated by the H_1 or L_p norm of the function.

2. Vilenkin martingales. In this paper $\Omega = [0,1) \times [0,1)$, $\mathcal{A}$ is the σ-algebra of Borel sets and P is Lebesgue measure. Let $(p_n, n \in \mathbb{N})$ and $(q_n, n \in \mathbb{N})$ be two sequences of natural numbers ≥ 2. Set $P_0 = Q_0 = 1$

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and

$$P_{n+1} := \prod_{k=0}^{n} p_k, \quad Q_{n+1} := \prod_{k=0}^{n} q_k \quad (n \in \mathbb{N}).$$

Every $x \in [0,1)$ can be uniquely written in the following way:

$$x = \sum_{k=0}^{\infty} \frac{x_k}{P_{k+1}}, \quad 0 \le x_k < p_k, \ x_k \in \mathbb{N}.$$

If there are two different forms, choose the one for which $\lim_{k\to\infty} x_k = 0$. The functions

$$r_n(x) := \exp \frac{2\pi \imath x_n}{p_n}, \quad r'_n(y) := \exp \frac{2\pi \imath y_n}{q_n}$$

are called *generalized Rademacher functions*, where $\imath := \sqrt{-1}$.

Let $\mathcal{A}_n$ and $\mathcal{A}'_m$ be the σ-algebras generated by $\{r_0, \ldots, r_{n-1}\}$ and by $\{r'_0, \ldots, r'_{m-1}\}$, respectively, and let $\mathcal{F}_{n,m}$ be the σ-algebra generated by $\mathcal{A}_n \times \mathcal{A}'_m$, i.e. $\mathcal{F}_{n,m} = \sigma(\mathcal{A}_n \times \mathcal{A}'_m)$, $\mathcal{F}_{n,\infty} := \sigma(\bigcup_{k=0}^{\infty} \mathcal{F}_{n,k})$ and $\mathcal{F}_{\infty,m} := \sigma(\bigcup_{k=0}^{\infty} \mathcal{F}_{k,m})$. It is easy to see that $(\mathcal{F}_{n,m})$ is nondecreasing and

$$\mathcal{F}_{n,m} = \sigma\{[kP_n^{-1}, (k+1)P_n^{-1}) \times [lQ_m^{-1}, (l+1)Q_m^{-1}) : \\ 0 \le k < P_n, \ 0 \le l < Q_m\}.$$

The Kronecker product system of two one-parameter Vilenkin systems is called a *two-parameter Vilenkin system* $(w_{n,m}; n, m \in \mathbb{N})$, i.e.

$$w_{n,m}(x,y) := w_n(x) w'_m(y) := \prod_{k=0}^{\infty} r_k(x)^{n_k} \prod_{l=0}^{\infty} r'_l(y)^{m_l}$$

where $n = \sum_{k=0}^{\infty} n_k P_k$, $m = \sum_{l=0}^{\infty} m_l Q_l$, $0 \le n_k < P_k$, $0 \le m_l < Q_l$ and $n_k, m_l \in \mathbb{N}$.

The conditional expectation operator with respect to $\mathcal{F}_{n,m}$ will be denoted by $E_{n,m}$ $(n, m \in \mathbb{N} \cup \{\infty\})$. For the (complex) space $L_p(\Omega, \mathcal{A}, P)$ and for its norm we use the shorter notation L_p and $\|\cdot\|_p$.

The two-parameter Vilenkin–Fourier series and the Vilenkin–Fourier coefficients of an integrable function f are given by

$$f(x) \sim \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} c_{k,l} w_{k,l}(x,y) \quad \text{and} \quad c_{k,l} := \widehat{f}(k,l) := E(f \overline{w}_{k,l}),$$

respectively. For simplicity we always suppose that $\widehat{f}(k,0) = \widehat{f}(0,k) = 0$ $(k \in \mathbb{N})$.

Let $f_{n,m}$ be the (P_n, Q_m)th partial sum of the Vilenkin–Fourier series of f. It is easy to see ([16]) that

$$f_{n,m}(x,y) = \sum_{k=0}^{P_n-1} \sum_{l=0}^{Q_m-1} c_{k,l} w_{k,l}(x,y) \\ = P_n Q_m \int_{I_{n,m}(x,y)} f \, dP = E_{n,m} f(x,y)$$

where $I_{n,m}(x,y)$ denotes the atom of $\mathcal{F}_{n,m}$ containing (x,y) $(n, m \in \mathbb{N}$, $(x,y) \in \Omega)$, that is to say, $(f_{n,m}; n, m \in \mathbb{N})$ is the martingale relative to $(\mathcal{F}_{n,m})$ obtained from f.

The martingale difference sequence is given by

$$d_{n+1,m+1} f := f_{n+1,m+1} - f_{n+1,m} - f_{n,m+1} + f_{n,m} \\ = \sum_{k=P_n}^{P_{n+1}-1} \sum_{l=Q_m}^{Q_{m+1}-1} c_{k,l} w_{k,l}, \\ d_{0,k} f := d_{k,0} := 0 \quad (k \in \mathbb{N}).$$

This can be rewritten as

$$d_{n+1,m+1} f = \sum_{i=1}^{p_n-1} \sum_{j=1}^{q_m-1} v_{n,m}^{(i,j)} r_n^i r_m'^j \tag{1}$$

where every $v_{n,m}^{(i,j)}$ is $\mathcal{F}_{n,m}$-measurable.

The following notations will be used for a function $f \in L_1$:

$$f^* := \sup_{n,m} |f_{n,m}|, \quad S(f) := \Big(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |d_{n,m} f|^2\Big)^{1/2},$$

$$s(f) := \Big(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} E_{n,m} |d_{n+1,m+1} f|^2\Big)^{1/2}.$$

Since for $i, l = 1, \ldots, p_n - 1$ we have

$$E_{n,\infty}(r_n^i) = 0, \quad E_{n,\infty}(r_n^i \overline{r_n^l}) = \delta(i-l), \quad |r_n^i| = 1, \tag{2}$$

we obtain

$$s(f) = \Big(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{i=1}^{p_n-1} \sum_{j=1}^{q_m-1} |v_{n,m}^{(i,j)}|^2\Big)^{1/2}. \tag{3}$$

Denote by H_p the space of functions for which

$$\|f\|_{H_p} := \|s(f)\|_p < \infty.$$

In martingale theory it is well known that if $f \in H_p$ then $f_{n,m}$ converges a.e. and also in L^p-norm as $\min(n,m) \to \infty$ $(p \ge 1$, see [14]).

Let us introduce the concept of a stopping time. A mapping ν which maps Ω into the subsets of $\mathbb{N}^2 \cup \{\infty\}$ is said to be a *stopping time* relative to $(\mathcal{F}_{n,m})$ if the elements of $\nu(\omega)$ are incomparable (i.e. if $(k,l),(n,m) \in \nu(\omega)$ then neither $(k,l) \le (n,m)$ nor $(n,m) \le (k,l)$; of course, $(k,l) < \infty$ for all $k,l \in \mathbb{N}$) and if for all $(n,m) \in \mathbb{N}^2$,

$$\{\omega \in \Omega : (n,m) \in \nu(\omega)\} =: \{(n,m) \in \nu\} \in \mathcal{F}_{n,m}.$$

We use the notation $(k,l) \ll (n,m)$ if $k < n$ and $l < m$ hold at the same time. For $H \subset \mathbb{N}^2$ we write $H \ll (n,m)$ if there exists a pair $(k,l) \in H$ such that $(k,l) \ll (n,m)$. So we immediately see ([20]) that ν is a stopping time if and only if

$$\{\nu \not\ll (n,m)\} \in \mathcal{F}_{n-1,m-1} \quad (n,m \in \mathbb{N}).$$

As in the one-parameter case, we can define a *stopped martingale* $(f^\nu_{n,m})$ for an arbitrary martingale f relative to a stopping time ν:

$$f^\nu_{n,m} := \sum_{i\le n}\sum_{j\le m} \chi(\{\nu \not\ll (i,j)\}) d_{i,j} f$$

where $\chi(A)$ denotes the characteristic function of a set A. Since $\{\nu \not\ll (i,j)\} \in \mathcal{F}_{i-1,j-1}$ it follows that $(f^\nu_{n,m}; n,m \in \mathbb{N})$ is a martingale (see [22]).

Using the stopped martingale we can define the BMO_2 and the $\Lambda_2(\alpha)$ spaces as follows. $\Lambda_2(\alpha)$ $(\alpha \ge 0)$ denotes the space of functions $f \in L_2$ for which

$$\|f\|_{\Lambda_2(\alpha)} := \sup_\nu \{P(\nu \ne \infty)^{-1/2-\alpha} \|f - f^\nu\|_2\} < \infty$$

where the supremum is taken over all stopping times. The $\Lambda_2(0)$ norm and space will be denoted by BMO_2.

3. Fourier backwards martingales. We denote by $+$ the group operation of the group $[0,1)$, namely, addition modulo 1. For a positive integer $r \ge 1$ denote by $\mathcal{G}_r$ the σ-algebra of all r^{-1}-periodic Borel subsets of $[0,1)$. Set $\mathcal{G}_{r,s} := \sigma(\mathcal{G}_r \times \mathcal{G}_s)$. The conditional expectation of a function $f \in L_1$ with respect to $\mathcal{G}_{r,s}$ is given by

$$E(f \mid \mathcal{G}_{r,s})(x,y) = r^{-1}s^{-1} \sum_{i=0}^{r-1}\sum_{j=0}^{s-1} f\left(x + \frac{i}{r}, y + \frac{j}{s}\right) \quad ((x,y) \in \Omega).$$

If f is expanded in a Fourier series

$$f(x,y) \sim \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \widehat{f}(k,l) \exp(\imath 2\pi kx) \exp(\imath 2\pi ly)$$

then (cf. [11])

$$E(f \mid \mathcal{G}_{r,s})(x,y) \sim \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \widehat{f}(kr, ls) \exp(\imath 2\pi krx) \exp(\imath 2\pi lsy)$$

where the Fourier coefficients of f are defined by

$$\widehat{f}(k,l) := \int_\Omega f(x,y) \exp(-\imath 2\pi kx) \exp(-\imath 2\pi ly)\, dx\, dy.$$

Consider the decreasing sequence of σ-algebras $\mathcal{F}_{n,m} := \mathcal{G}_{P_n,Q_m}$ $(n,m \in \mathbb{N})$. Then, for $(x,y) \in \Omega$,

$$f_{n,m}(x,y) := E_{n,m} f(x,y) = P_n^{-1} Q_m^{-1} \sum_{i=0}^{P_n-1}\sum_{j=0}^{Q_m-1} f\left(x + \frac{i}{P_n}, y + \frac{j}{Q_m}\right)$$

is a backwards martingale with respect to $(\mathcal{F}_{n,m})$. Let $\mathcal{F}_{n,\infty} := \bigcap_{k=0}^\infty \mathcal{F}_{n,k}$, $\mathcal{F}_{\infty,m} := \bigcap_{k=0}^\infty \mathcal{F}_{k,m}$ and assume that $\widehat{f}(k,0) = \widehat{f}(0,k) = 0$ $(k \in \mathbb{Z})$. It is known (see [12]) that, in this case, $f_{n,m} \to f_{\infty,m} = 0$ $(n \to \infty)$, $f_{n,m} \to f_{n,\infty} = 0$ $(m \to \infty)$ and $f_{n,m} \to 0$ $(n,m \to \infty)$, all this in L_1 norm.

Now the martingale difference sequence is given by

$$d_{n,m} f := f_{n,m} - f_{n+1,m} - f_{n,m+1} + f_{n+1,m+1} \quad (n,m \in \mathbb{N}).$$

The martingale maximal and square functions are defined as in Section 2. It is easy to see that if f is a trigonometric polynomial then

$$S(f)(x,y) = \Big(\sum_{n=0}^\infty\sum_{m=0}^\infty \Big|\sum_{p_n \nmid k}\sum_{q_m \nmid l} \widehat{f}(kP_n, lQ_m) \exp(\imath 2\pi kP_n x) \exp(\imath 2\pi lQ_m y)\Big|^2\Big)^{1/2}.$$

It can be seen ([11], [15]) that in this case, similarly to (1), $d_{n,m} f$ can also be rewritten as

$$d_{n,m} f = \sum_{i=1}^{p_n-1}\sum_{j=1}^{q_m-1} u^{(i,j)}_{n,m} \varrho^i_n \varrho'^j_m$$

where every $u^{(i,j)}_{n,m}$ is $\mathcal{F}_{n+1,m+1}$-measurable and $\varrho_n(x) := \exp(\imath 2\pi P_n x)$, $\varrho'_m(y) := \exp(\imath 2\pi Q_m y)$.

The functions ϱ^i_n satisfy (2), more exactly, for $i,l = 1,\ldots,p_n - 1$ we have

$$E_{n+1,0}(\varrho^i_n) = 0, \quad E_{n+1,0}(\varrho^i_n \overline{\varrho^l_n}) = \delta(i-l), \quad |\varrho^i_n| = 1.$$

Consequently, for the conditioned square function we obtain

(4)
$$s(f) := \Big(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} E_{n+1,m+1}|d_{n,m}f|^2\Big)^{1/2} = \Big(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=1}^{p_n-1}\sum_{j=1}^{q_m-1} |u_{n,m}^{(i,j)}|^2\Big)^{1/2}.$$

The H_p space as well as the stopping time are defined as in Section 2. It can be shown similarly to [22] that ν is a stopping time if and only if

$$\{\nu \not\gg (n,m)\} \in \mathcal{F}_{n+1,m+1} \quad (n,m \in \mathbb{N}).$$

Now the *stopped martingale* $(f_{n,m}^{\nu})$ for a martingale f is given by

$$f_{n,m}^{\nu} := \sum_{i\ge n}\sum_{j\ge m} \chi(\{\nu \not\gg (i,j)\})d_{i,j}f.$$

In this case $\Lambda_2(\alpha)$ $(\alpha \ge 0)$ denotes the space of functions $f \in L_2$ for which

$$\|f\|_{\Lambda_2(\alpha)} := \sup_{\nu}\{P(\nu \ne (0,0))^{-1/2-\alpha}\|f - f^{\nu}\|_2\} < \infty$$

where the supremum is taken over all stopping times. The $\Lambda_2(0)$ norm and space will again be denoted by BMO_2.

4. Results. The theorems of this section hold both for Vilenkin martingales and for Fourier backwards martingales.

THEOREM 1.

(i) $$\|f\|_p \le \|f^*\|_p \le \Big(\frac{p}{p-1}\Big)^2\|f\|_p \quad (1 < p < \infty),$$

(ii) $$c_p\|S(f)\|_p \le \|f^*\|_p \le C_p\|S(f)\|_p \quad (1 < p < \infty),$$

(iii) $$\|f^*\|_p \le C_p\|s(f)\|_p, \quad \|S(f)\|_p \le C_p\|s(f)\|_p \quad (0 < p \le 2),$$

(iv) $$\|s(f)\|_p \le C_p\|f^*\|_p \quad (2 \le p < \infty),$$

(v) *if the sequences* (p_n) *and* (q_n) *are bounded then*

$$\|s(f)\|_p \sim \|S(f)\|_p \sim \|f^*\|_p \quad (0 < p < \infty).$$

THEOREM 2. *The dual space of* H_p *where* $0 < p \le 1$ *is* $\Lambda_2(\alpha)$ *with* $\alpha = 1/p - 1$.

The theorems for martingales can be found in [3], [4], [6], [10], [13] and [22]. The proofs for backwards martingales are similar. In general, $\|s(f)\|_p$ is not equivalent to $\|S(f)\|_p$. The H_1 space is equivalent neither to the Hardy space considered by Gundy and Varopoulos in [11] nor to the classical Hardy space, even in the one-parameter case. However, in the one-parameter case, if $p_n = r$ (examined in [11]) then our backwards H_1 space is equivalent to the Hardy space studied in [11].

To characterize the dual of H_p $(1 < p < \infty)$ we need the following inequality due to Stein ([17], p. 103) in the one-parameter case.

THEOREM 3. *Let* $X = (X_{n,m})$ *be a not necessarily adapted function sequence and* (k_n), (l_m) *be two sequences of natural numbers. Then*

$$\Big\|\Big(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|E_{k_n,l_m}X_{n,m}|^2\Big)^{1/2}\Big\|_p \le C_p\Big\|\Big(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|X_{n,m}|^2\Big)^{1/2}\Big\|_p \quad (1 < p < \infty)$$

where C_p *depends only on* p.

Proof. The proof is similar to Stein's original proof. Let us introduce another well known definition. $L_p(l_r)$ $(1 \le p, r \le \infty)$ denotes the space of function sequences $\xi = (\xi_n, n \in \mathbb{N}^2)$ for which

$$\|\xi\|_{L_p(l_r)} := \Big\|\Big(\sum_{n\in\mathbb{N}^2}|\xi_n|^r\Big)^{1/r}\Big\|_p < \infty.$$

The following lemma can be proved similarly to the Riesz representation theorem.

LEMMA. *The dual of* $L_p(l_r)$ *is* $L_q(l_s)$ *whenever* $1 \le p, r < \infty$, $1/p + 1/q = 1$ *and* $1/r + 1/s = 1$.

We shall use the following generalization of the Riesz convexity theorem: Let T be a linear operator which maps function sequences to function sequences. Suppose that T is a bounded operator from $L_{p_0}(l_{q_0})$ to itself and from $L_{p_1}(l_{q_1})$ to itself. Then T is also bounded from $L_{p_t}(l_{q_t})$ to itself where

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1} \quad (0 \le t \le 1).$$

The proof is very similar to that of the original Riesz convexity theorem (see e.g. [1]). Now consider the operator T defined by

$$T : (X_{n,m}; n,m \in \mathbb{N}) \to (E_{k_n,l_m}X_{n,m}; n,m \in \mathbb{N}).$$

T is bounded on $L_p(l_p)$ $(1 < p < \infty)$ since

$$E\Big(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|E_{k_n,l_m}X_{n,m}|^p\Big) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}E|E_{k_n,l_m}X_{n,m}|^p \le \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}E|X_{n,m}|^p.$$

On the other hand, by the maximal inequality (see Theorem 1(i)) we obtain

$$E(\sup_{n,m}|E_{k_n,l_m}X_{n,m}|)^p \le E(\sup_{n,m}\sup_{k,l}E_{k,l}|X_{n,m}|)^p \le E(|Z^*|^p) \le C_p E(\sup_{n,m}|X_{n,m}|)^p$$

where $Z := \sup_{n,m}|X_{n,m}|$, which shows the boundedness of T on $L_p(l_\infty)$ $(1 < p \le \infty)$. Applying now the generalized Riesz convexity theorem we conclude that T is bounded on $L_p(l_q)$ if $1 < p \le q \le \infty$. In particular, if $1 < p \le 2$ then T is bounded on $L_p(l_2)$.

For $2 < p < \infty$ we prove the theorem with a duality argument. By the Lemma we have

$$\Big\|\Big(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|E_{k_n,l_m}X_{n,m}|^2\Big)^{1/2}\Big\|_p = \sup_{\|Y\|_{L_q(l_2)}\le 1} E\Big[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(E_{k_n,l_m}X_{n,m})Y_{n,m}\Big]$$

where $1/p+1/q=1$. Moreover,

$$E\Big[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(E_{k_n,l_m}X_{n,m})Y_{n,m}\Big] = E\Big[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}X_{n,m}(E_{k_n,l_m}Y_{n,m})\Big] \le \|(X_{n,m})\|_{L_p(l_2)}\|(E_{k_n,l_m}Y_{n,m})\|_{L_q(l_2)} \le C_q\|(X_{n,m})\|_{L_p(l_2)},$$

which completes the proof of the theorem. ■

If (p_n) or (q_n) is unbounded then H_p is not equivalent to L_p. So the following theorem contains a new result.

THEOREM 4. *The dual space of H_p is H_q where $1 < p < \infty$ and $1/p + 1/q = 1$.*

P r o o f. We give the proof for Vilenkin martingales. The proof for Fourier backwards martingales is similar. Set $\mathcal{F}_{n,m}^{(i,j)} := \mathcal{F}_{n,m}$ $(i = 1,\dots,p_n-1;\ j=1,\dots,q_m-1)$. Now we consider the function sequences of the form

$$X = (X_{n,m}^{(i,j)};\, i=1,\dots,p_n-1;\, j=1,\dots,q_m-1;\, n,m\in\mathbb{N}).$$

The inequality in Theorem 3 can be written as

$$\Big\|\Big(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=1}^{p_n-1}\sum_{j=1}^{q_m-1}|E_{k_n,l_m}X_{n,m}^{(i,j)}|^2\Big)^{1/2}\Big\|_p \le C_p\Big\|\Big(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=1}^{p_n-1}\sum_{j=1}^{q_m-1}|X_{n,m}^{(i,j)}|^2\Big)^{1/2}\Big\|_p \tag{5}$$

if $1 < p < \infty$. Denote by ${}^{a}L_p(l_2)$ the subspace of $L_p(l_2)$ of adapted sequences X relative to $\mathcal{F}_{n,m}^{(i,j)}$.

We show that the dual of ${}^{a}L_p(l_2)$ is ${}^{a}L_q(l_2)$ whenever $1 < p < \infty$ and $1/p+1/q=1$. Note that this result does not hold even in the one-parameter case for $p=1$. Indeed, if for every $Y \in {}^{a}L_q(l_2)$,

$$l_Y(X) := E\Big(\sum_{n,m,i,j}X_{n,m}^{(i,j)}Y_{n,m}^{(i,j)}\Big) \quad (X\in {}^{a}L_p(l_2))$$

then l_Y is in the dual of ${}^{a}L_p(l_2)$ and

$$\|l_Y\| \le \|Y\|_{L_q(l_2)}.$$

Conversely, if l is in the dual of ${}^{a}L_p(l_2)$ then, by the Banach–Hahn theorem, it can be extended with the same norm to the whole $L_p(l_2)$ space. Hence there exists a Y from the dual space of $L_p(l_2)$, i.e. $Y \in L_q(l_2)$, such that $l = l_Y$ and

$$\|Y\|_{L_q(l_2)} \le \|l\|.$$

As

$$l(X) = E\Big(\sum_{n,m,i,j}X_{n,m}^{(i,j)}E_{n,m}Y_{n,m}^{(i,j)}\Big),$$

by (5) with $k_n = n$ and $l_m = m$ we get

$$\|(E_{n,m}Y_{n,m}^{(i,j)})\|_{L_q(l_2)} \le \|Y\|_{L_q(l_2)} \le \|l\|.$$

Using (3) (or (4)) the theorem follows immediately from the isometry between H_p and ${}^{a}L_p(l_2)$. ■

The dual of H_∞ is considered in [19] for Walsh martingales.

Now we generalize Theorem 3 of Gundy and Varopoulos [11].

THEOREM 5. *If $f\in L_1$ then*

$$\Big(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=1}^{p_n-1}\sum_{j=1}^{q_m-1}|\widehat{f}(iP_n,jQ_m)|^2\Big)^{1/2} \le C\|f\|_{H_1} \tag{6}$$

where $\widehat{f}(\cdot,\cdot)$ denotes the Vilenkin–Fourier or the Fourier coefficients of f.

P r o o f. We prove this theorem for Vilenkin martingales only. The proof for Fourier backwards martingales is similar. First of all we show that if

$$g := \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=1}^{p_n-1}\sum_{j=1}^{q_m-1}\widehat{g}(iP_n,jQ_m)w_{iP_n,jQ_m} \in L_2 \tag{7}$$

then

$$\|g\|_{\mathrm{BMO}_2} \le \|g\|_2 = \Big(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=1}^{p_n-1}\sum_{j=1}^{q_m-1}|\widehat{g}(iP_n,jQ_m)|^2\Big)^{1/2}. \tag{8}$$

Using the definition of the stopped martingale we have

$$\|g\|_{\mathrm{BMO}_2} = \sup_\nu P(\nu \neq \infty)^{-1/2}\|g - g^\nu\|_2$$
$$= \sup_\nu P(\nu \neq \infty)^{-1/2} \times \Big(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} E[\chi(\nu \ll (n+1,m+1))|d_{n+1,m+1}g|^2]\Big)^{1/2}$$
$$= \sup_\nu P(\nu \neq \infty)^{-1/2} \times \Big(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} E[\chi(\nu \ll (n+1,m+1))E_{n,m}|d_{n+1,m+1}g|^2]\Big)^{1/2}.$$

Since

$$d_{n+1,m+1}g = \sum_{i=1}^{p_n-1}\sum_{j=1}^{q_m-1} \widehat{g}(iP_n, jQ_m)w_{iP_n,jQ_m}$$

we get, by (2),

$$E_{n,m}|d_{n+1,m+1}g|^2 = \sum_{i=1}^{p_n-1}\sum_{j=1}^{q_m-1} |\widehat{g}(iP_n, jQ_m)|^2.$$

Hence

$$\|g\|_{\mathrm{BMO}_2} \le \sup_\nu P(\nu \neq \infty)^{-1/2} \times \Big(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} E\Big[\chi(\nu \neq \infty)\sum_{i=1}^{p_n-1}\sum_{j=1}^{q_m-1} |\widehat{g}(iP_n, jQ_m)|^2\Big]\Big)^{1/2}$$
$$= \Big(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=1}^{p_n-1}\sum_{j=1}^{q_m-1} |\widehat{g}(iP_n, jQ_m)|^2\Big)^{1/2},$$

which proves (8).

If $f \in L_2$ then by Riesz representation theorem

$$\Big(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=1}^{p_n-1}\sum_{j=1}^{q_m-1} |\widehat{f}(iP_n, jQ_m)|^2\Big)^{1/2} = \sup_g E(fg)$$

where the supremum is taken over all g of the form (7) with $\|g\|_2 = 1$. Using (8) and Theorem 2 we obtain

$$|E(fg)| \le C\|f\|_{H_1}\|g\|_{\mathrm{BMO}_2} \le C\|f\|_{H_1},$$

which, on the one hand, proves (6) for $f \in L_2$. On the other hand, we have shown in [22] that L_2 is dense in H_1. Theorem 5 follows easily from this. ∎

It is not known whether (6) holds if we write on the right hand side of the inequality $C_p\|f\|_p$ instead of $C\|f\|_{H_1}$ whenever $f \in L_p$ and (p_n) or (q_n) are unbounded.

The following Corollary for Fourier backwards martingales can be proved in the same way as Gundy and Varopoulos have proved Corollary 1 in [11] for one parameter. Let p, r, q, s be primes and

$$\mathcal{F}_{n_1,n_2,n_3,n_4} := \mathcal{G}_{p^{n_1},q^{n_3}} \cap \mathcal{G}_{r^{n_2},s^{n_4}} = \mathcal{G}_{p^{n_1}r^{n_2},q^{n_3}s^{n_4}}.$$

Since Theorems 1 and 2 are also valid for four parameters, similarly to Theorem 5 we can prove that for $1 < p < \infty$,

$$\Big(\sum_{n_1,n_2,n_3,n_4} |\widehat{f}(p^{n_1}r^{n_2}, q^{n_3}s^{n_4})|^2\Big)^{1/2} \le C_p\|f\|_p.$$

Of course, there also exists a Hardy space for which the previous inequality holds. However, this Hardy space is different from the ones studied above. Applying this method for several parameters we get

COROLLARY. *Let $f \in L_p$ $(1 < p < \infty)$ and $a_1, \ldots, a_{m_1}$, $b_1, \ldots, b_{m_2}$ be arbitrary integers greater than* 1. *Then, for the Fourier coefficients, we obtain*

$$\Big(\sum_{n_1,\ldots,n_{m_1}}\sum_{k_1,\ldots,k_{m_2}} |\widehat{f}(a_1^{n_1}\ldots a_{m_1}^{n_{m_1}}, b_1^{k_1}\ldots b_{m_2}^{k_{m_2}})|^2\Big)^{1/2} \le C_p\|f\|_p.$$

For one parameter and for bounded (p_n) the characterization of H_1 by means of conjugate functions can be found in [11] and [18]. It is an open question whether this characterization can be extended to the two-parameter or to the unbounded case.

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Closed range multipliers and generalized inverses

by

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Abstract. Conditions involving closed range of multipliers on general Banach algebras are studied. Numerous conditions equivalent to a splitting $A = TA \oplus \ker T$ are listed, for a multiplier T defined on the Banach algebra A. For instance, it is shown that $TA \oplus \ker T = A$ if and only if there is a commuting operator S for which $T = TST$ and $S = STS$, that this is the case if and only if such S may be taken to be a multiplier, and that these conditions are also equivalent to the existence of a factorization $T = PB$, where P is an idempotent and B an invertible multiplier. The latter condition establishes a connection to a famous problem of harmonic analysis.

Introduction. In the study of multipliers on, say, commutative semisimple Banach algebras, in particular in attempting to characterize circumstances under which a multiplier will have closed range, a factorization of the given multiplier as the product of an idempotent and an invertible has kept showing up as a plausible companion—certainly a sufficient, and possibly equivalent, condition for closed range. In some spectacular special cases, namely the group algebras $L^1(G)$ when G is a locally compact abelian group, this equivalence does hold, as was shown by Host and Parreau in 1978. This note takes steps to uncover the precise relationship between the two. We do this by dealing with the issue in somewhat greater generality, and by relating it to the concept of generalized inverse.

Commutativity keeps looming in the present approach, though, and as a consequence the resulting conditions are slightly stronger than those mentioned before. It turns out that for an arbitrary continuous linear operator T on a Banach space X there is a factorization $T = PB$, where P and B commute, and where B is invertible and P idempotent, precisely when $X = TX \oplus \ker T$. Moreover, when X decomposes in this way, TX is necessarily closed. The realization that these conditions also are connected to the existence of a commuting generalized inverse then becomes our starting point.

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