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Factorization through Hilbert space and the dilation of $L(X,Y)$-valued measures

by

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Abstract. We present a general necessary and sufficient algebraic condition for the spectral dilation of a finitely additive $L(X,Y)$-valued measure of finite semivariation when X and Y are Banach spaces. Using our condition we derive the main results of Rosenberg, Makagon and Salehi, and Miamee without the assumption that X and/or Y are Hilbert spaces. In addition we relate the dilation problem to the problem of factoring a family of operators through a single Hilbert space.

0. Introduction. Let X, Y be two Banach spaces and T be an $L(X,Y)$-valued (f.a.) measure on an algebra Σ of subsets of a space Ω. Following the original work of Masani [4] on c.a.o.s. dilation ($X = \mathbb{C}$ and Y a Hilbert space), the problem of the spectral dilation of T was introduced by Rosenberg [9] and solved under the condition of 2-majorizability for the case in which X is finite-dimensional and Y is any Hilbert space. This was extended to X, Y Hilbert spaces in [3]. An alternate general condition was proposed by Makagon and Salehi [3] and several examples were derived. In [5], the work was extended to the case when X is a Banach space and Y is a Hilbert space. In Richard [8], the general problem for X, Y Banach spaces was solved using 2-majorizability and was related to the work of Lindenstrauss and Pełczyński [2] on Hilbertian operators in Banach spaces.

In our work, following recent ideas of Pisier [6], we establish Lemma 1 which is exploited to derive Corollary 1 from which the main results of [3], [5] and [9] are easily derived. It should be noted that the proof in [3] reduces their condition to 2-majorizability and fails in case Y is a Banach space. In addition to obtaining a general condition for dilation which includes ([3], [5], [9]), we are able to relate the dilation problem to the problem of factoring a family of operators through a single Hilbert space. It should be noted that

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our solution is algebraic and free from any topological considerations as in [3] and [5]. In addition, the sufficient conditions of [3] and [5] are not satisfied by spectral measures. Some simple consequences of Corollary 1 also include a generalization of the main result of [4] on c.a.o.s. measures.

Throughout this work Ω is an arbitrary set, Σ is an algebra of subsets of Ω, X and Y are (complex) Banach spaces, and $L(X,Y)$ is the space of all bounded linear operators from X to Y with the usual operator norm. As usual, we write $L(X)=L(X,X)$, $X^*=L(X,\mathbb{C})$ where $\mathbb{C}$ is the complex plane, $\mathbb{R}^n$ is n-dimensional Euclidean space, and $\mathbb{N}$ is the set of positive integers.

A set function $T:\Sigma\to L(X,Y)$ is *finitely additive* (f.a.) provided $T(\Delta_1\cup\Delta_2)=T(\Delta_1)+T(\Delta_2)$ for all disjoint Δ_1,Δ_2 in Σ and has *finite semivariation* provided $\|T\|_{\mathrm{sv}}=\sup\{\|\sum_{k=1}^n t_kT(\Delta_k)\|\}$ is finite, where the supremum is taken over all finite Σ-partitions (Δ_k) of Ω and scalars (t_k) with $|t_k|\le 1$.

1. Spectral dilation of operator-valued set functions. In [8], a connection was established between the 2-majorizability of $L(X,Y)$-valued measures and a result of Lindenstrauss–Pełczyński ([7], Theorem 2.4) concerning Hilbertian operators which gives an explicit form of the factoring Hilbert space. The technique used in [8] was an equivalent form of the relation $(y_j)<(x_i)$ in ([7], page 22). For this the following definition of the spectral dilation of an $L(X,Y)$-valued measure was introduced in [8].

DEFINITION 1. A set function $T:\Sigma\to L(X,Y)$ is said to have a *finitely additive* (f.a.) *spectral dilation* if there exists a Hilbert space H, a f.a. spectral measure E on (Ω,Σ) in H, and bounded linear operators $V_1:X\to H$ and $V_2:H\to Y$ such that for each Δ in Σ, $T(\Delta)=V_2E(\Delta)V_1$.

Recent techniques introduced by Pisier [6] and our Lemma 1 motivate the following definition which is a generalization of the condition for factorization of an operator given by Lindenstrauss–Pełczyński.

DEFINITION 2. A f.a. set function $T:\Sigma\to L(X,Y)$ of finite semivariation has *Property* (L–P) provided there is some constant C such that for all $n\in\mathbb{N}$, disjoint $\Delta_1,\dots,\Delta_n$ in Σ, $x_1,\dots,x_n$ in X and complex matrices $(a_{ij}^{(k)})$, $i,j,k=1,\dots,n$, satisfying

$$(1)\qquad \sum_{i=1}^n\Big|\sum_{j=1}^n a_{ij}^{(k)}\beta_j\Big|^2\le\sum_{j=1}^n|\beta_j|^2,\qquad \forall(\beta_j)\in\mathbb{R}^n,\ k=1,\dots,n,$$

we have

$$\sum_{i=1}^n\Big\|\sum_{j=1}^n\sum_{k=1}^n a_{ij}^{(k)}T(\Delta_k)x_j\Big\|^2\le C\sum_{j=1}^n\|x_j\|^2.$$

In order to prove Theorem 1 we get, in our special case, a strengthened form of the relation $(z_i)<(x_i)$ introduced by Pisier [6]. This form allows us to produce matrices $(a_{ij}^{(k)})$ which, uniformly in k, satisfy the relation $(y_j)<(x_i)$ in [7]. Stated simply, our main result is as follows:

THEOREM 1. *If a set function $T:\Sigma\to L(X,Y)$ has Property* (L–P) *then T has a finitely additive spectral dilation.*

We use the following notation. Let μ be a c.a. measure on (Ω,Σ) with $\mu(\Omega)=1$ and $L^2(\mu)$ denote the Hilbert space of square integrable functions on Ω (up to μ-equivalence) with the usual inner product. We let S be the collection of (complex-valued) Σ-simple functions on Ω with the supremum norm and define $L_T:S\to L(X,Y)$ by

$$L_T(f)=\sum_{k=1}^n\alpha_kT(\Delta_k),\qquad f=\sum_{k=1}^n\alpha_k1_{\Delta_k}\in S.$$

We let $S\otimes X$ be the algebraic tensor product of S and X and denote by $\widehat{L}_T:S\otimes X\to Y$ the linear map defined by

$$\widehat{L}_T(z)=\sum_{i=1}^n L_T(f_i)x_i\qquad\text{for } z=\sum_{i=1}^n f_i\otimes x_i\in S\otimes X.$$

Let $I=L(X,L^2(\mu))$. For $\xi\in I$ and $z=\sum_{i=1}^n f_i\otimes x_i\in S\otimes X$ we write

$$\xi\cdot z=\sum_{i=1}^n f_i\xi(x_i).$$

Finally, for $z_1,\dots,z_n\in S\otimes X$ and $x_1,\dots,x_n\in X$ we write $(z_i)<(x_i)$ provided for every $\xi\in I$ we have

$$\sum_{i=1}^n\|\xi\cdot z_i\|_2^2\le\sum_{i=1}^n\|\xi(x_i)\|_2^2.$$

LEMMA 1. *If $(z_i)<(x_i)$ then there exist disjoint $\Delta_1,\dots,\Delta_n$ in Σ and $n\times n$ matrices $(a_{ij}^{(1)}),\dots,(a_{ij}^{(n)})$ satisfying* (1) *such that for each $i=1,\dots,n$,*

$$z_i=\sum_{j=1}^n\Big(\sum_{k=1}^n a_{ij}^{(k)}1_{\Delta_k}\Big)\otimes x_j.$$

Proof. Suppose $(z_i)<(x_i)$. Then there are functions f_{ij} in S such that for each i, $z_i=\sum_{j=1}^n f_{ij}\otimes x_j$ (see [6], Remark 2.3).

Let $f_{ij}=\sum_{k=1}^n a_{ij}^{(k)}1_{\Delta_k}\in S$. Fix k' such that $\mu(\Delta_{k'})>0$ and $x^*\in X^*$.

Consider $\xi = 1_{\Delta_{k'}} \otimes x^* \in I$. Then for each $i = 1, \dots, n$,

$$\xi \cdot z_i = \sum_{j=1}^n \Big(\sum_{k=1}^n a_{ij}^{(k)} 1_{\Delta_k}\Big) 1_{\Delta_{k'}} x^*(x_j) = \sum_{j=1}^n a_{ij}^{(k')} 1_{\Delta_{k'}} x^*(x_j) = 1_{\Delta_{k'}} x^*\Big(\sum_{j=1}^n a_{ij}^{(k')} x_j\Big).$$

So since $(z_i) < (x_i)$,

$$\sum_{i=1}^n \|\xi \cdot z_i\|_2^2 = \sum_{i=1}^n \mu(\Delta_{k'}) \Big| x^*\Big(\sum_{j=1}^n a_{ij}^{(k')} x_j\Big)\Big|^2 \le \sum_{i=1}^n \|\xi(x_i)\|_2^2 = \sum_{i=1}^n \mu(\Delta_{k'}) |x^*(x_i)|^2.$$

Since $\mu(\Delta_{k'}) > 0$, we have

$$\sum_{i=1}^n \Big| x^*\Big(\sum_{j=1}^n a_{ij}^{(k')} x_j\Big)\Big|^2 \le \sum_{i=1}^n |x^*(x_i)|^2 \quad \text{for every } x^* \text{ in } X^*,$$

which by ([7], Proposition 1) implies the matrices $(a_{ij}^{(k')})$ satisfy (1). ■

We now come to the proof of Theorem 1 which is an adaptation of that of Theorem 1.1 of [6] by using Property (L–P).

Proof of Theorem 1. Suppose $T : \Sigma \to L(X, Y)$ has Property (L–P). If $z_1, \dots, z_n \in S \otimes X$ and $x_1, \dots, x_n \in X$ with $(z_i) < (x_i)$ then by Lemma 1, $z_i = \sum_{j=1}^n (\sum_{k=1}^n a_{ij}^{(k)} 1_{\Delta_k}) \otimes x_j$ where the matrices $(a_{ij}^{(k)})$ satisfy (1) so since T has Property (L–P),

$$(2) \qquad \sum_{i=1}^n \|\widehat{L}_T(z_i)\|^2 = \sum_{i=1}^n \Big\| \sum_{j=1}^n L_T\Big(\sum_{k=1}^n a_{ij}^{(k)} 1_{\Delta_k}\Big) x_j \Big\|^2 = \sum_{i=1}^n \Big\| \sum_{j=1}^n \sum_{k=1}^n a_{ij}^{(k)} T(\Delta_k) x_j \Big\|^2 \le C \sum_{j=1}^n \|x_j\|^2.$$

Recall $I = L(X, L^2(\mu))$. Let Λ be the space of all functions $\varphi : I \to \mathbb{R}$ such that for some finite sequence $x_1, \dots, x_n$ in X,

$$|\varphi(\xi)| \le \sum_{i=1}^n \|\xi(x_i)\|_2^2 \quad \text{for all } \xi \text{ in } I,$$

and let Λ_+ be the set of all positive functions in Λ.

For φ in Λ define $p(\varphi) = \inf\{C \sum_{i=1}^n \|x_i\|^2\}$ where the infimum is taken

over all finite sets (x_i) in X such that

$$\varphi(\xi) \le \sum_{i=1}^n \|\xi(x_i)\|_2^2 \quad \text{for all } \xi \text{ in } I.$$

For φ in Λ_+ define $q(\varphi) = \sup\{\sum_{i=1}^n \|\widehat{L}_T(z_i)\|^2\}$ where the supremum is taken over all finite sets (z_i) in $S \otimes X$ such that

$$\sum_{i=1}^n \|\xi \cdot z_i\|_2^2 \le \varphi(\xi) \quad \text{for all } \xi \text{ in } I.$$

Then p is subadditive on Λ_+ and both are positive homogeneous. Also by (2), $q(\varphi) \le p(\varphi)$ for all φ in Λ_+. By the Hahn–Banach Theorem ([6], Theorem 0.3) there is a linear form $f : \Lambda \to \mathbb{R}$ such that

$$(3) \qquad q(\varphi) \le f(\varphi) \quad \text{for all } \varphi \text{ in } \Lambda_+, \text{ and}$$

$$(4) \qquad f(\varphi) \le p(\varphi) \quad \text{for all } \varphi \text{ in } \Lambda.$$

Let $\Lambda + i\Lambda$ be the complexification of Λ; extend f linearly to a linear form on $\Lambda + i\Lambda$ and continue to call this extension f.

Next define the space H of all functions $g : I \to L^2(\mu)$ such that the function $\xi \to \|g(\xi)\|_2^2$ is in Λ. If g, g' are in H, the function $\xi \to \langle g(\xi), g'(\xi)\rangle_2$ is in $\Lambda + i\Lambda$ since

$$|\langle g(\xi), g'(\xi)\rangle_2| \le \|g(\xi)\|_2 \|g'(\xi)\|_2 \le \Big(\sum \|\xi(x_i)\|_2^2\Big)^{1/2} \Big(\sum \|\xi(y_i)\|_2^2\Big)^{1/2} \le \sum \|\xi(x_i)\|_2^2 + \sum \|\xi(y_i)\|_2^2$$

for some finite sets (x_i) and (y_i) in X.

An inner product can thus be defined on H by $\langle g, g'\rangle = f(\langle g(\cdot), g'(\cdot)\rangle_2)$. After passing to the quotient of the kernel of the associated seminorms we obtain a Hilbert space $\widehat{H}$.

For x in X let $\widehat{x} : I \to \widehat{H}$ be defined by $\widehat{x}(\xi) = \xi(x)$. Consider $\varphi(\xi) = \langle \widehat{x}(\xi), \widehat{x}(\xi)\rangle \in \Lambda$. Since $\varphi(\xi) = \|\xi(x)\|_2^2$ for all ξ in I, $p(\varphi) \le C\|x\|^2$. Also, by definition, $\langle \widehat{x}, \widehat{x}\rangle = f(\langle \widehat{x}(\cdot), \widehat{x}(\cdot)\rangle)$, giving $\langle \widehat{x}, \widehat{x}\rangle = f(\varphi) \le p(\varphi) \le C\|x\|^2$. There is, therefore, a linear operator $V_1 : X \to \widehat{H}$ with $\|V_1\| \le C^{1/2}$ such that $V_1 x$ is the equivalence class of $\widehat{x}$ in $\widehat{H}$.

For $h_1, \dots, h_n$ in S and $x_1, \dots, x_n$ in X consider for $\xi \in I$,

$$\varphi(\xi) = \Big\| \sum_{i=1}^n h_i \widehat{x}_i(\xi) \Big\|^2.$$

If $z = \sum h_i \otimes x_i$ then $\|\xi \cdot z\|_2^2 = \varphi(\xi)$ so by (3),

$$\|\widehat{L}_T(z)\|^2 = \Big\| \sum_{i=1}^n L_T(h_i) x_i \Big\|^2 \le q(\varphi) \le f(\varphi).$$

Moreover, $\varphi \geq 0$ and for ξ in I, and

$$\varphi(\xi) = \Big\| \sum_{i=1}^n h_i \xi(x_i) \Big\|^2 \leq \Big(\sum_{i=1}^n \|h_i\| \|\xi(x_i)\|_2 \Big)^2$$
$$\leq \Big(\sum_{i=1}^n \|h_i\|^2 \Big) \Big(\sum_{i=1}^n \|\xi(x_i)\|_2^2 \Big) \equiv \varrho_n^2 \sum_{i=1}^n \|\xi(x_i)\|_2^2 .$$

Letting $y_i = \varrho_n x_i$, $i = 1, \ldots, n$, and $\xi \in I$, we have $\varphi \in \Lambda$, since

$$\varphi(\xi) \leq \sum_{i=1}^n \|\xi(y_i)\|_2^2 .$$

The operator $\Pi : L(L^2(\mu)) \to L(\widehat{H})$ defined by $\Pi(h)g(\xi) = hg(\xi)$ is a representation (up to equivalence classes).

For $h_1, \ldots, h_n$ in S and $x_1, \ldots, x_n$ in X,

$$\Big\| \sum_{i=1}^n \Pi(h_i) V_1 x_i \Big\|_H = \Big\| \sum_{i=1}^n \Pi(h_i) \widehat{x}_i \Big\|_H = \Big\| \sum_{i=1}^n h_i \widehat{x}_i \Big\|_H$$
$$= \Big\langle \sum_{i=1}^n h_i \widehat{x}_i, \sum_{i=1}^n h_i \widehat{x}_i \Big\rangle$$
$$= f\Big(\Big\langle \sum_{i=1}^n h_i \widehat{x}_i(\cdot), \sum_{i=1}^n h_i \widehat{x}_i(\cdot) \Big\rangle \Big)$$
$$\equiv f(\varphi) \geq \Big\| \sum_{i=1}^n L_T(h_i) x_i \Big\|^2 .$$

Here we have identified S as a subspace of $L(L^2(\mu))$. Thus one can define a linear map $V_2 : \overline{\mathrm{span}}(\Pi(S)V_1 X) \to Y$ with $\|V_2\| \leq 1$ such that $V_2(\sum_{i=1}^n \Pi(h_i) V_1 x_i) = \sum_{i=1}^n L_T(h_i) x_i$. Extend this operator V_2 to a continuous linear operator on $\widehat{H}$ to Y with $\|V_2\| \leq 1$ and continue to call this extension V_2.

For $\Delta \in \Sigma$ consider $E(\Delta) = \Pi(1_\Delta)$. Then $E : \Sigma \to L(\widehat{H})$ and since Π is a representation,

$$[E(\Delta)]^2 = [\Pi(1_\Delta)]^2 = \Pi(1_\Delta^2) = \Pi(1_\Delta) = E(\Delta),$$
$$[E(\Delta)]^* = [\Pi(1_\Delta)]^* = \Pi(\overline{1}_\Delta) = E(\Delta),$$
$$E(\Delta_1 \cap \Delta_2) = \Pi(1_{\Delta_1} \cap 1_{\Delta_2}) = \Pi(1_{\Delta_1} 1_{\Delta_2}) = \Pi(1_{\Delta_1}) \Pi(1_{\Delta_2})$$
$$= E(\Delta_1) E(\Delta_2),$$

and $E(\Omega) = I$ so E is a spectral measure in $\widehat{H}$. Moreover, $V_2 E(\Delta) V_1 x = V_2 \Pi(1_\Delta) V_1 x = L_T(1_\Delta) x = T(\Delta) x$ for x in X, which completes the proof. ∎

The following corollary is an analogue of ([7], Corollary 2.5).

COROLLARY 1. *Let $T : \Sigma \to L(X,Y)$ be finitely additive of finite semivariation. The following are equivalent:*

(a) *T has a finitely additive spectral dilation.*

(b) *For all n and for all n-tuples of $n \times n$ unitary matrices $((a_{ij}^{(1)}), \ldots, (a_{ij}^{(n)}))$ we have for all $x_1, \ldots, x_n$ in X and $\Delta_1, \ldots, \Delta_n$ disjoint in Σ that there is a constant C with*

$$\sum_{i=1}^n \Big\| \sum_{j=1}^n \sum_{k=1}^n a_{ij}^{(k)} T(\Delta_k) x_j \Big\|^2 \leq C \sum_{j=1}^n \|x_j\|^2 .$$

Proof. Suppose (a) holds. Let $T(\Delta) = V_2 E(\Delta) V_1$ where E is a spectral measure in some Hilbert space H. For $(a_{ij}^{(1)}), \ldots, (a_{ij}^{(n)})$ unitary, $x_1, \ldots, x_n$ in X and $\Delta_1, \ldots, \Delta_n$ disjoint in Σ we have

$$\sum_i \Big\| \sum_j \sum_k a_{ij}^{(k)} T(\Delta_k) x_j \Big\|_Y^2 = \sum_i \Big\| \sum_j \sum_k a_{ij}^{(k)} V_2 E(\Delta_k) V_1 x_j \Big\|_Y^2$$
$$\leq \|V_2\|^2 \sum_i \sum_j \sum_{j'} \sum_{k'} a_{ij}^{(k)} \overline{a_{ij'}^{(k')}} \langle E(\Delta_{k'}) E(\Delta_k) V_1 x_j,\ V_1 x_{j'} \rangle .$$

Since $\Delta_1, \ldots, \Delta_n$ are disjoint this equals

$$\|V_2\|^2 \sum_k \sum_j \sum_{j'} \Big(\sum_i a_{ij}^{(k)} \overline{a_{ij'}^{(k)}} \Big) \langle E(\Delta_k) V_1 x_j, V_1 x_{j'} \rangle .$$

Using this and the fact that for each k, $(a_{ij}^{(k)})$ is unitary we obtain

$$\sum_i \Big\| \sum_j \sum_k a_{ij}^{(k)} T(\Delta_k) x_j \Big\|_Y^2 \leq \|V_2\|^2 \sum_k \sum_j \langle E(\Delta_k) V_1 x_j,\ V_1 x_j \rangle$$
$$= \|V_2\|^2 \sum_j \Big\langle E\Big(\bigcup_k \Delta_k \Big) V_1 x_j, V_1 x_j \Big\rangle$$
$$\leq \|V_2\|^2 \|V_1\|^2 \sum_j \|x_j\|^2 .$$

Suppose (b) holds. By Theorem 1 it suffices to show that T has Property (L–P). Fix $x_1, \ldots, x_n$ in X and $\Delta_1, \ldots, \Delta_n$ disjoint in Σ and consider $\sum_{i=1}^n \| \sum_{j=1}^n \sum_{k=1}^n a_{ij}^{(k)} T(\Delta_k) x_j \|^2$ where the matrices $(a_{ij}^{(k)})$, $k = 1, \ldots, n$, each satisfy (1). Let B_n be the collection of all n-tuples of $n \times n$ matrices satisfying (1). Then B_n is convex and the extreme points of B_n consist of the set of all n-tuples of $n \times n$ unitary matrices. Thus for these fixed $x_1, \ldots, x_n$ and $\Delta_1, \ldots, \Delta_n$, the supremum of $\sum_{i=1}^n \| \sum_{j=1}^n \sum_{k=1}^n a_{ij}^{(k)} T(\Delta_k) x_j \|^2$ over B_n is attained for an n-tuple of unitary matrices $((a_{ij}^{(1)}), \ldots, (a_{ij}^{(n)}))$. It follows that (b) implies T has Property (L–P). ∎

2. Some special cases. In this section we derive the extensions of the main results in [3], [5], [8] and [9] concerning the spectral dilation for $L(X,Y)$-valued measures. We use the following notation. For a Banach space X, we let $\overline{X}^*$ denote the set of all continuous conjugate linear functionals on X. For x^* in $\overline{X}^*$ or x^* in X^* we let $\langle x^*, x\rangle$ denote the evaluation of the functional x^* at the point x in X. By a *positive operator* F in $L(X,\overline{X}^*)$ we mean an operator satisfying $\langle Fx_1, x_2\rangle = \overline{\langle Fx_2, x_1\rangle}$ and $\langle Fx, x\rangle \ge 0$ for all x, x_1, x_2 in X. The collection of all positive operators in $L(X,\overline{X}^*)$ will be denoted by $L^+(X,\overline{X}^*)$. Additionally, for Banach spaces X, Y and $S \in L(X,Y)$ we define the conjugate adjoint $\overline{S}^* \in L(\overline{Y}^*, \overline{X}^*)$ of S by $\overline{S}^*(y) = y^* \circ S$, $y^* \in Y^*$.

DEFINITION 3. A f.a. set function $T : \Sigma \to L(X,Y)$ is said to have a *finitely additive* (f.a.) *two-majorant* F if $F : \Sigma \to L^+(X,\overline{X}^*)$ is finitely additive and there is a constant C such that for each positive integer n, $x_1,\ldots,x_n$ in X and disjoint $\Delta_1,\ldots,\Delta_n$ in Σ,

$$\Big\|\sum_{j=1}^n T(\Delta_j)x_j\Big\|^2 \le C\sum_{j=1}^n \langle F(\Delta_j)x_j, x_j\rangle .$$

PROPOSITION 1. *A f.a. set function $T : \Sigma \to L(X,Y)$ has a f.a. spectral dilation if and only if T has a f.a. 2-majorant $F : \Sigma \to L^+(X,\overline{X}^*)$.*

Proof. Taking $F(\Delta) = \overline{V}_1^* E(\Delta)V_1$, the necessity is obvious. To prove the sufficiency we use Corollary 1. Let n be a positive integer, $x_1,\ldots,x_n$ be in X, $\Delta_1,\ldots,\Delta_n$ be disjoint in Σ and $((a_{ij}^{(1)}),\ldots,(a_{ij}^{(n)}))$ be an n-tuple of $n\times n$ unitary matrices. Then

$$\sum_{i=1}^n\Big\|\sum_{j=1}^n\sum_{k=1}^n a_{ij}^{(k)}T(\Delta_k)x_j\Big\|^2 = \sum_{i=1}^n\Big\|\sum_{k=1}^n T(\Delta_k)\Big(\sum_{j=1}^n a_{ij}^{(k)}x_j\Big)\Big\|^2$$
$$\le \sum_{i=1}^n\Big\|\sum_{k=1}^n\Big\langle F(\Delta_k)\Big(\sum_{j=1}^n a_{ij}^{(k)}x_j\Big), \sum_{j=1}^n a_{ij}^{(k)}x_j\Big\rangle .$$

This equals $\sum_k\sum_j\sum_{j'}(\sum_i a_{ij}^{(k)}\overline{a_{ij'}^{(k)}})\langle F(\Delta_k)x_j, x_{j'}\rangle$, which since each $(a_{ij}^{(k)})$, $k=1,\ldots,n$, is unitary equals

$$\sum_{k=1}^n\sum_{j=1}^n\Big\langle F(\Delta_k)x_j, x_j\Big\rangle = \sum_{j=1}^n\Big\langle F\Big(\bigcup_k \Delta_k\Big)x_j, x_j\Big\rangle \le \|F\|_{\rm sv}\sum_{j=1}^n\|x_j\|^2$$

where $\|F\|_{\rm sv}$ is the semivariation of F. ■

PROPOSITION 2. *Let $T : \Sigma \to L(X,Y)$ be finitely additive. The following are equivalent.*

(a) *There exists a constant C such that for any $x_1^l, x_2^l,\ldots,x_n^l$ in X, $l=1,\ldots,N$, and for any disjoint $\Delta_1,\ldots,\Delta_n$ in Σ,*

$$\sum_{l=1}^N\Big\|\sum_{i=1}^n T(\Delta_i)x_i^l\Big\|^2 \le C\sup\Big\{\Big|\sum_{l=1}^N\sum_{i=1}^n\langle F(\Delta_i)x_i^l, x_i^l\rangle\Big|\Big\}$$

where the supremum is taken over all f.a. $L(X,\overline{X}^*)$-valued set functions with $\|F\|_{\rm sv}\le 1$.

(b) *T has a f.a. spectral dilation.*

Proof. Implication (b)⇒(a) is obvious from the necessity in Proposition 1. We use Corollary 1 to show (a) implies (b). Let $x_1,\ldots,x_n$ be in X, $\Delta_1,\ldots,\Delta_n$ be disjoint in Σ and $((a_{ij}^{(1)}),\ldots,(a_{ij}^{(n)}))$ be an n-tuple of unitary matrices. Then since each $(a_{ij}^{(k)})$, $k=1,\ldots,n$, is unitary we have, with the supremum taken as above,

$$\sum_{i=1}^n\Big\|\sum_{j=1}^n\sum_{k=1}^n a_{ij}^{(k)}T(\Delta_k)x_j\Big\|^2 = \sum_{i=1}^n\Big\|\sum_{k=1}^n T(\Delta_k)\Big(\sum_{j=1}^n a_{ij}^{(k)}x_j\Big)\Big\|^2$$
$$\le C\sup\Big\{\Big|\sum_{i=1}^n\sum_{k=1}^n\Big\langle F(\Delta_k)\Big(\sum_{j=1}^n a_{ij}^{(k)}x_j\Big), \sum_{j=1}^n a_{ij}^{(k)}x_j\Big\rangle\Big|\Big\}$$
$$= C\sup\Big\{\Big|\sum_{k=1}^n\sum_{j=1}^n\sum_{j'=1}^n\sum_{i=1}^n a_{ij}^{(k)}\overline{a_{ij'}^{(k)}}\langle F(\Delta_k)x_j, x_{j'}\rangle\Big|\Big\}$$
$$= C\sup\Big\{\Big|\sum_{k=1}^n\sum_{j=1}^n\langle F(\Delta_k)x_j, x_j\rangle\Big|\Big\}$$
$$\le C\sup\Big\{\Big|\sum_{j=1}^n\Big\langle F\Big(\bigcup_{k=1}^n\Delta_k\Big)x_j, x_j\Big\rangle\Big|\Big\}$$
$$\le C\sup\Big\{\|F\|_{\rm sv}\sum_{j=1}^n\|x_j\|^2\Big\} = C\sum_{j=1}^n\|x_j\|^2 . \ ■$$

This gives the main result of [5], which includes that of [3]. We now give some sufficient conditions by verifying Property (L–P) which depend on the geometry of X and/or Y. For this we need the definition of type 2 and cotype 2 spaces.

Let $D = \{-1,+1\}^\infty$ and μ be the infinite product of $\frac12[\delta_{-1}+\delta_{+1}]$ on D where δ_t is the Dirac measure. We denote by $\varepsilon_n : D \to \{-1,+1\}$ the nth coordinate of D. A Banach space X is *of type* 2 if there is a constant C such that for any finite sequence $x_1,\ldots,x_n$ in X, $\|\sum_{j=1}^n\varepsilon_j x_j\|_2^2 \le C\sum_{j=1}^n\|x_j\|^2$. A Banach space Y is *of cotype* 2 if for any sequence $y_1, y_2,\ldots$ in Y, $\sum\varepsilon_i y_i$ converging μ-almost everywhere implies $\sum\|y_i\|^2 < \infty$.

An operator $U : X \to Y$ is 2-*summing* if there is a constant C such that for any finite sequence $x_1, \ldots, x_n$ in X we have $\sum_{i=1}^{n} \|Ux_i\|^2 \le C \sup\{\sum_{i=1}^{n} |\varphi(x_i)|^2 : \varphi \in X^*, \|\varphi\| \le 1\}$.

We next establish an extension of the main result of [1].

LEMMA 2. *Let Y be a Banach space of cotype 2 and T be a f.a. Y-valued set function of finite semivariation on (Ω, Σ). Then there exists a Hilbert space H, a f.a. spectral measure E in H, a bounded linear operator $V : H \to Y$ and a unique x_0 in H such that for each Δ in Σ, $T(\Delta) = VE(\Delta)x_0$. In addition, $\|V\| \le 1$.*

Proof. Let K be the Stone space associated with (Ω, Σ), $C(K)$ be the space of continuous functions on the compact space K equipped with the supremum norm, and $B(\Sigma)$ be the uniform closure of the set of all complex-valued Σ-simple functions on Ω. Then ([1]) there is an isometric isomorphism $\tau : C(K) \to B(\Sigma)$. Define $L_T : B(\Sigma) \to Y$ to be the continuous linear extension of the operator

$$L_T(f) = \sum_{k=1}^{n} T(\Delta_k)\alpha_k, \qquad f = \sum_{k=1}^{n} 1_{\Delta_k}\alpha_k,$$
$$\Delta_1, \ldots, \Delta_n \text{ in } \Sigma,\ \alpha_1, \ldots, \alpha_n \text{ in } \mathbb{C}.$$

Since Y is of cotype 2, $L_T \circ \tau : C(K) \to Y$ is 2-summing ([7], p. 62). By the Pietsch Factorization Theorem, there is a probability measure λ' on K and a constant C such that

$$\|L_T(\tau f)\|^2 \le C \int_K |\tau f(k)|^2 \, \lambda'(dk) \qquad \text{for all } f \in C(K).$$

Since τ, being an isomorphism between the complex algebras $C(K)$ and $B(\Sigma)$, satisfies $|\tau^{-1}g|^2 = \tau^{-1}|g|^2$ for all g in $B(\Sigma)$, it follows that

$$\|\Phi_T(g)\|^2 \le C \int_\Omega |g(\omega)|^2 \, \lambda(d\omega)$$

where $\lambda(A) = \int_K (\tau^{-1} \circ 1_A)(k)\, \lambda'(dk)$ is finitely additive and positive. Taking $g = \sum_{k=1}^{n} 1_{\Delta_k}\alpha_k$ in $B(\Sigma)$ yields

$$\Big\| \sum_{k=1}^{n} T(\Delta_k)\alpha_k \Big\|^2 \le C \sum_{k=1}^{n} |\alpha_k|^2 \lambda(\Delta_k)$$

so λ is a finitely additive 2-majorant of T. Here we regard T as being $L(\mathbb{C}, Y)$-valued. By Proposition 1 there is a Hilbert space H, a spectral measure E in H and bounded linear operators $V_1 : \mathbb{C} \to H$ and $V_2 : H \to Y$ such that for Δ in Σ,

$$T(\Delta) = V_2 E(\Delta) V_1 = V_2 E(\Delta) x_0$$

for some unique x_0 in H. Taking $V = V_2$ completes the proof since the fact that $\|V\| \le 1$ follows from the proof of Theorem 1. ∎

For a Banach space X and an algebra Σ of subsets of a set Ω we denote by $M(X)$ the space of all finitely additive set functions on (Ω, Σ) of bounded semivariation and by $S(X)$ the space of all X-valued Σ-simple functions on Ω. For f in $S(X)$ of the form $f = \sum_{k=1}^{n} 1_{\Delta_k} x_k$, $\Delta_1, \ldots, \Delta_n$ in Σ, $x_1, \ldots, x_n$ in X, we let

$$\|f\|_\infty = \sup\Big\{ \Big| \sum_{k=1}^{n} \langle m(\Delta_k), x_k \rangle \Big| : m \in M(X^*), \|m\|_{\mathrm{sv}} \le 1 \Big\}.$$

Then $(S(X), \|\cdot\|_\infty)$ is a normed linear space. Moreover, the map $T_m : (M(X^*), \|\cdot\|_{\mathrm{sv}}) \to (S(X), \|\cdot\|_\infty)^*$ defined by

$$T_m(f) = \sum_{k=1}^{n} \langle m(\Delta_k), x_k \rangle, \qquad f = \sum_{k=1}^{n} 1_{\Delta_k} x_k,$$

is an antilinear isometry ([3], Lemma 2). We denote the closure of $S(X)$ with respect to the norm $\|\cdot\|_\infty$ simply by $\overline{(S(X), \|\cdot\|_\infty)}$.

For any $f = \sum_{k=1}^{n} 1_{\Delta_k} x_k \in S(X)$ and $T : \Sigma \to L(X, Y)$, f.a. and of finite semivariation, we define $\Phi_T \in L((S(X), \|\cdot\|_\infty), Y)$ by $\Phi_T(f) = \sum_{k=1}^{n} T(\Delta_k) x_k$ and refer to Φ_T as the operator associated with T. We extend Φ_T linearly without increasing norm to $\overline{(S(X), \|\cdot\|_\infty)}$ and continue to call this extension Φ_T.

PROPOSITION 3. *Let X be a Banach space of type 2. If $\Phi_T : \overline{(S(X), \|\cdot\|_\infty)} \to Y$ is 2-summing then $T : \Sigma \to L(X, Y)$ has a f.a. spectral dilation.*

Proof. Let $((a_{ij}^{(1)}), \ldots, (a_{ij}^{(n)}))$ be an n-tuple of unitary matrices, $x_1, \ldots, x_n$ be in X and $\Delta_1, \ldots, \Delta_n$ be disjoint in Σ. Suppose $\Phi_T : \overline{(S(X), \|\cdot\|_\infty)} \to Y$ is 2-summing. Then letting

$$f_i = \sum_{k=1}^{n} 1_{\Delta_k} \Big(\sum_{j=1}^{n} a_{ij}^{(k)} x_j \Big) \in S(X)$$

we obtain

$$\sum_{i=1}^{n} \Big\| \sum_{j=1}^{n} \sum_{k=1}^{n} a_{ij}^{(k)} T(\Delta_k) x_j \Big\|^2 = \sum_{i=1}^{n} \Big\| \Phi_T \Big(\sum_{k=1}^{n} 1_{\Delta_k} \Big(\sum_{j=1}^{n} a_{ij}^{(k)} x_j \Big) \Big) \Big\|^2$$
$$= \sum_{i=1}^{n} \|\Phi_T(f_i)\|^2 \le C \sup\Big\{ \sum_{i=1}^{n} |\varphi(f_i)|^2 : \varphi \in (S(X), \|\cdot\|_\infty)^*, \|\varphi\| \le 1 \Big\}$$
$$= C \sup\Big\{ \sum_{i=1}^{n} \Big| \sum_{j=1}^{n} \sum_{k=1}^{n} \langle m(\Delta_k), a_{ij}^{(k)} x_j \rangle \Big|^2 : m \in M(X^*), \|m\|_{\mathrm{sv}} \le 1 \Big\}.$$

Fix $m \in M(X^*)$, with $\|m\|_{\mathrm{sv}} \le 1$. Since X is of type 2, X^* is of cotype 2 so by Lemma 2, there is a Hilbert space H, a finitely additive spectral measure E in H, a bounded operator $V_m : H \to Y$ and a unique x_0 in H such that $m(\Delta) = V_m E(\Delta) x_0$ with $\|V_m\| \le 1$. So,

$$\begin{aligned}\sum_{j=1}^{n}\Big|\sum_{k=1}^{n}\langle m(\Delta_k), a_{ij}^{(k)}x_j\rangle\Big|^2 &= \sum_{j=1}^{n}\Big|\sum_{k=1}^{n}\langle V_m E(\Delta_k)x_0, a_{ij}^{(k)}x_j\rangle\Big|^2\\ &\le \|x_0\|^2\Big\|\sum_{j=1}^{n}\sum_{k=1}^{n}E(\Delta_k)\overline{V}_m^* a_{ij}^{(k)}x_j\Big\|^2\\ &= \|x_0\|^2\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{j'=1}^{n}\sum_{k'=1}^{n}\langle V_m E(\Delta_{k'})E(\Delta_k)\overline{V}_m^* a_{ij}^{(k)}x_j, a_{ij'}^{(k')}x_{j'}\rangle\\ &= \|x_0\|^2\sum_{j=1}^{n}\sum_{j'=1}^{n}\sum_{k=1}^{n}\langle V_m E(\Delta_k)\overline{V}_m^* a_{ij}^{(k)}x_j, a_{ij}^{(k)}x_{j'}\rangle\\ &= \|x_0\|^2\sum_{j=1}^{n}\sum_{j'=1}^{n}\sum_{k=1}^{n}a_{ij}^{(k)}\overline{a_{ij'}^{(k)}}\langle V_m E(\Delta_k)\overline{V}_m^* x_j, x_{j'}\rangle\,.\end{aligned}$$

Summing over i yields

$$\begin{aligned}&\sum_{i=1}^{n}\Big\|\sum_{j=1}^{n}\sum_{k=1}^{n}a_{ij}^{(k)}T(\Delta_k)x_j\Big\|^2\\ &\quad\le C\sup\Big\{\sum_{i=1}^{n}\Big|\sum_{j=1}^{n}\sum_{k=1}^{n}\langle m(\Delta_k), a_{ij}^{(k)}x_j\rangle\Big|^2 : m\in M(X^*),\ \|m\|_{\mathrm{sv}}\le 1\Big\}\\ &\quad\le C\|x_0\|^2\sup\Big\{\sum_{j=1}^{n}\langle V_m E(\Delta)\overline{V}_m^* x_j, x_j\rangle : \|m\|_{\mathrm{sv}}\le 1\Big\}\\ &\quad\le C\|x_0\|^2\|V_m\|^2\sum_{j=1}^{n}\|x_j\|^2,\end{aligned}$$

which by Corollary 1 completes the proof as $\|V_m\|^2 \le 1$ for all m. ■

References

[1] S. D. Chatterji, *Orthogonally scattered dilation of Hilbert space valued set functions*, in: Measure Theory, Oberwolfach 1981, D. Kölzow and D. Maharam-Stone (eds.), Lecture Notes in Math. 945, Springer, New York, 1982, 269–281.

[2] J. Lindenstrauss and A. Pełczyński, *Absolutely summing operators in $\mathcal{L}_p$-spaces and their applications*, Studia Math. 29 (1968), 275–326.

[3] A. Makagon and H. Salehi, *Spectral dilation of operator-valued measures and its application to infinite-dimensional harmonizable processes*, ibid. 85 (1987), 257–297.

[4] P. Masani, *Quasi-isometric measures and their applications*, Bull. Amer. Math. Soc. 76 (1970), 427–528.

[5] A. G. Miamee, *Spectral dilation of $L(B,H)$-valued measures and its application to stationary dilation for Banach space valued processes*, Indiana Univ. Math. J. 38 (1989), 841–860.

[6] G. Pisier, *Completely bounded maps between sets of Banach space operators*, ibid. 39 (1990), 249–277.

[7] —, *Factorization of Linear Operators and Geometry of Banach Spaces*, CBMS Regional Conf. Ser. in Math. 60, Amer. Math. Soc., Providence, R.I., 1986.

[8] P. Richard, *Harmonizability, V-boundedness, and stationary dilation of Banach-valued processes*, in: Probability in Banach Spaces, 8, Proc. Eighth Internat. Conf., R. Dudley, M. Hahn and J. Kuelbs (eds.), Birkhäuser, Boston, 1992, 189–205.

[9] M. Rosenberg, *Quasi-isometric dilations of operator-valued measures and Grothendieck's inequality*, Pacific J. Math. 103 (1982), 135–161.

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