[16] G. Köthe, *Topological Vector Spaces*, Springer, Berlin, 1969.
[17] V. B. Moscatelli, *Fréchet spaces without continuous norm and without a basis*, Bull. London Math. Soc. 12 (1980), 63–66.
[18] J. Mujica, *A completeness criterion for inductive limits of Banach spaces*, in: Functional Analysis, Holomorphy and Approximation Theory II, J. A. Barroso (ed.), North-Holland, Amsterdam, 1984, 319–329.
[19] H. Pfister, *Bemerkungen zum Satz über die Separabilität der Fréchet-Montel Räume*, Arch. Math. (Basel) 27 (1976), 86–92.
[20] J. Schmets, *Spaces of Vector-Valued Continuous Functions*, Lecture Notes in Math. 1003, Springer, Berlin, 1983.
[21] M. Valdivia, *Semi-Suslin and dual metric spaces*, in: Functional Analysis, Holomorphy and Approximation Theory, J. A. Barroso (ed.), North-Holland, Amsterdam, 1982, 445–459.

Added in proof:

[22] S. Dierolf and P. Domański, *Compact subsets of coechelon spaces*, to appear.
[23] J. Bonet, P. Domański and J. Mujica, *Complete spaces of vector-valued holomorphic germs*, to appear.

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Received August 21, 1992 (2988)

STUDIA MATHEMATICA 107 (1) (1993)

Integrability theorems for trigonometric series

by

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Abstract. We show that, if the coefficients (a_n) in a series $a_0/2 + \sum_{n=1}^{\infty} a_n \cos(nt)$ tend to 0 as $n \to \infty$ and satisfy the regularity condition that

$$\sum_{m=0}^{\infty} \left\{ \sum_{j=1}^{\infty} \left[\sum_{n=j2^m}^{(j+1)2^m-1} |a_n - a_{n+1}| \right]^2 \right\}^{1/2} < \infty,$$

then the cosine series represents an integrable function on the interval $[-\pi, \pi]$. We also show that, if the coefficients (b_n) in a series $\sum_{n=1}^{\infty} b_n \sin(nt)$ tend to 0 and satisfy the corresponding regularity condition, then the sine series represents an integrable function on $[-\pi, \pi]$ if and only if $\sum_{n=1}^{\infty} |b_n|/n < \infty$. These conclusions were previously known to hold under stronger restrictions on the sizes of the differences $\Delta a_n = a_n - a_{n+1}$ and $\Delta b_n = b_n - b_{n+1}$. We were led to the mixed-norm conditions that we use here by our recent discovery that the same combination of conditions implies the integrability of Walsh series with coefficients (a_n) tending to 0.

We also show here that this condition on the differences implies that the cosine series converges in L^1-norm if and only if $a_n \log n \to 0$ as $n \to \infty$. The corresponding statement also holds for sine series for which $\sum_{n=1}^{\infty} |b_n|/n < \infty$. If either type of series is assumed *a priori* to represent an integrable function, then weaker regularity conditions suffice for the validity of this criterion for norm convergence.

1. Introduction. We outline one proof of the integrability results in this section, and comment further on that proof in Section 2. We present another proof of the integrability results in Sections 5 and 6. We also state two theorems about L^1-norm convergence in Section 1, and show in Section 3 how these statements follow from the integrability results. We begin this section by recalling some earlier work on these questions, and we say more in Sections 4 and 5 about how our results compare with other work.

About eighty years ago, W. H. Young [36] related integrability of series to properties of differences of coefficients by showing that if the coefficients in a cosine series tend to 0 and form a convex sequence, then the series

1991 *Mathematics Subject Classification*: Primary 42A32, 42A16; Secondary 46E39, 46E40, 46E30, 46E35.

Research of the second author partially supported by NSERC grant number 4822.

converges to a nonnegative integrable function on the interval $(0,\pi]$, and the cosine series is the Fourier series of that function.

About ten years later, Kolmogorov [20] showed that the integrability conclusion still holds for cosine series with coefficients that tend to 0 and satisfy the condition that

$$\sum_{j=0}^{\infty}(j+1)|\Delta^2 a_j| = \sum_{k=0}^{\infty}\sum_{j=k}^{\infty}|\Delta^2 a_j| < \infty .$$

Such sequences (a_n) are called *quasiconvex*. Convex sequences that tend to 0 are quasiconvex, because their differences Δa_n and $\Delta^2 a_n = \Delta a_n - \Delta a_{n+1}$ are nonnegative, and the series $\sum_{k=n}^{\infty}\Delta a_k$ and $\sum_{j=k}^{\infty}\Delta^2 a_j$ converge to a_n and Δa_k respectively. Hence $\sum_{k=0}^{\infty}\sum_{j=k}^{\infty}\Delta^2 a_j = a_0$ for such sequences. Another way to state the condition of quasiconvexity is that there is a convex sequence (c_n) that tends to 0 so that $|\Delta^2 a_n| = \Delta^2 c_n$ for all n. Kolmogorov also showed that such cosine series converge in L^1-norm if and only if $a_n \log n \to 0$ as $n \to \infty$. Accounts of this work can be found in books [4, 37].

After an interval of about fifteen years, S. Sidon [27] showed that integrability of cosine series follows from a seemingly complicated condition that is weaker than quasiconvexity. About thirty-five years after that, S. A. Telyakovskiĭ [35] gave a new proof of Sidon's theorem, and pointed out that Sidon's condition is equivalent to the requirement that $a_n \to 0$ as $n \to \infty$ and that there be a nonincreasing sequence (A_n) with the properties that $\sum_{n=0}^{\infty} A_n < \infty$ and $|\Delta a_n| \le A_n$ for all n. Another way to state this condition is that the sizes of the first differences Δa_n are majorized by the first differences of a convex sequence that tends to 0.

As we will explain in Section 4, this restriction on the size of the differences is equivalent to the requirement that

$$\sum_{m=0}^{\infty} 2^m \max_{2^{m-1}<n\le 2^m} |\Delta a_n| < \infty . \tag{1.1}$$

In the late 1970's, G. A. Fomin [13] introduced the weaker condition that

$$\sum_{m=1}^{\infty} 2^m \left(\frac{1}{2^{m-1}} \sum_{2^{m-1}<n\le 2^m} |\Delta a_n|^p\right)^{1/p} < \infty \tag{1.2}$$

for finite positive values of p, and showed that if (a_n) tends to 0 and satisfies condition (1.2) for some $p > 1$, then the cosine series is integrable. People have also considered integrability of sine series, and shown [35], [13] that the corresponding restrictions on the coefficients (b_n) imply integrability if and only if $\sum_{n=1}^{\infty}|b_n|/n < \infty$.

Several other proofs of Fomin's theorem have been found [29], [6], [14], [21], and it has been rediscovered at least once [7]. Except as noted below, the condition that we use here is strictly weaker than all conditions on the sizes of individual differences that have been previously shown to imply integrability.

The present paper mainly reports on work that we did in the summer of 1991. Six months later, we learned that N. Tanović-Miller and various co-authors [24, 8, 9] had, in papers that had not yet appeared in mid-1991, also proved integrability theorems with the same kinds of conditions on the sizes of differences. In view of this, we have slightly modified the presentation of our results to stress the parts that differ from theirs.

We use some notation and some very basic ideas from the theory [15] of amalgams of L^p and ℓ^q spaces. This theory is usually presented on noncompact nondiscrete groups like the real line $\mathbb{R}$ but the notions also make sense on the discrete group $\mathbb{Z}$ of all integers and on the unit circle $\mathbb{T}$. Given a positive integer M cover $\mathbb{Z}$ with translates of the set $\{0,\ldots,M-1\}$ by multiples of M. Given a function d on $\mathbb{Z}$ and given indices p and q in the interval $[0,\infty]$, let $\|d\|_{p,q,M}$ denote the quantity obtained by first computing the ℓ^p-norms of the restriction of d to each set in the cover and then computing the ℓ^q-norm of the resulting sequence. For instance,

$$\|d\|_{1,2,M} = \left\{\sum_{j=-\infty}^{\infty}\Big[\sum_{n=jM}^{(j+1)M-1}|d(n)|\Big]^2\right\}^{1/2} .$$

We use $\|d\|'_{1,2,M}$ to denote the quantity obtained by proceeding as above but omitting the two middle intervals $[-M,0)$ and $[0,M)$. Thus

$$\|d\|'_{1,2,M} = \left\{\sum_{j=-\infty}^{-2} + \sum_{j=1}^{\infty}\Big[\sum_{n=jM}^{(j+1)M-1}|d(n)|\Big]^2\right\}^{1/2} .$$

We will follow [6] in mostly working with the complex form

$$\sum_{n=-\infty}^{\infty} c(n)e^{int}$$

of trigonometric series. We use function notation for the coefficients to save on subscripts. We will say that a series with coefficients $(c(n))$ *represents an integrable function*, or simply that the series is *integrable* if there is a function F in $L^1(-\pi,\pi)$ so that

$$c(n) = \widehat{F}(n) = \int_{-\pi}^{\pi} F(t)e^{-int}\,\frac{dt}{2\pi}$$

for all integers n. By the Riemann–Lebesgue lemma, a necessary condition for integrability is that $c(n) \to 0$ as $n \to \pm\infty$, but it is well known [37, §5.1] that this is not sufficient.

We combine the assertions made in the abstract as follows. Call a sequence $(c(n))$ *regular* if it tends to 0 at $\pm\infty$ and its differences $\Delta c(n) = c(n) - c(n+1)$ satisfy the condition that

$$\sum_{m=0}^{\infty} \|\Delta c\|'_{1,2,2^m} < \infty. \tag{1.3}$$

Call the sequence *sufficiently symmetric* if

$$\sum_{n=1}^{\infty} \frac{|c(n) - c(-n)|}{n} < \infty. \tag{1.4}$$

THEOREM 1. *If the coefficients in a trigonometric series form a regular sequence, then the series represents an integrable function if and only if the coefficient sequence is sufficiently symmetric.*

Before outlining a proof, we comment on the conditions appearing above. Let $\|c\|_\Delta$ denote the sum of the series appearing on the left in condition (1.3) and $\|c\|_\Sigma$ denote the sum on the left in (1.4). Then the sum of the two functionals $\|\cdot\|_\Delta$ and $\|\cdot\|_\Sigma$ is a norm on the space of regular sufficiently symmetric sequences. This space of sequences is complete with respect to this norm. It follows from the closed-graph theorem that the mapping that sends each coefficient sequence to the corresponding L^1-function is a bounded operator from the space of sequences to L^1. In fact, our methods show directly that there are absolute constants A and B for which the L^1-norm of the function represented by the trigonometric series with coefficients $(c(n))$ is bounded above by $A\|c\|_\Delta + B\|c\|_\Sigma$.

For cosine series, the coefficients in the complex form of the series are even, and condition (1.4) is automatically satisfied. Condition (1.3) reduces to the one specified in the abstract. For sine series with coefficients $(b(n))$, condition (1.4) reduces to the statement that $\sum_{n=1}^{\infty} |b(n)|/n < \infty$.

Condition (1.3) is equivalent to the regularity condition obtained by replacing the indices (2^m) by any sequence (k_m) for which the ratios k_{m+1}/k_m all lie between constants a and b with $1 < a < b < \infty$. It is also equivalent to the requirement that

$$\sum_{M=1}^{\infty} \frac{\|\Delta c\|'_{1,2,M}}{M} < \infty.$$

It follows from regularity that the sequence $(\Delta c(n))$ belongs to ℓ^1, that is, that the original sequence $(c(n))$ has bounded variation. A sequence $(c(n))$ with bounded variation is sufficiently symmetric if and only if

$$\sum_{m=0}^{\infty} |c(2^m) - c(-2^m)| < \infty. \tag{1.5}$$

Again, the indices (2^m) can be replaced here by any sequence (k_m) that grows rapidly enough but not too rapidly.

Now fix a regular sequence $(c(n))$. Given two positive numbers a and b, denote the union of the intervals $[a, b]$ and $[-b, -a]$ by $\pm[a, b]$. It is well known that the sequence of Dirichlet kernels $D_n(t) = \sum_{k=-n}^{n} e^{int}$ is uniformly bounded on each set $\pm[\delta, \pi]$ with $\delta > 0$. Since the sequence $(c(n))$ tends to 0 at $\pm\infty$ and has bounded variation, it follows by summation by parts that the series $\sum_{n=-\infty}^{\infty} c(n)e^{int}$ converges uniformly on each such set $\pm[\delta, \pi]$. Denote the sum of the series by $F(t)$. Matters essentially reduce to determining whether $F(t)$ has a bad enough singularity at $t = 0$ to prevent F from being integrable.

Let $f(t) = (1 - e^{-it})F(t)$. The series $\sum_{n=-\infty}^{\infty} \Delta c(n)e^{int}$ converges absolutely to $f(t)$, and hence $\widehat{f}(n) = \Delta c(n)$ for all n. The function F is integrable if and only if

$$\int_{-\pi}^{\pi} |f(t)| \frac{dt}{|t|} < \infty.$$

For each natural number m let E_m be the union $\pm(\pi/2^{m+1}, \pi/2^m]$. Then F is integrable if and only if

$$\sum_{m=0}^{\infty} \int_{E_m} |f(t)| \frac{dt}{|t|} < \infty.$$

Now in each set E_m the quantity $|t|$ is nearly constant, and the measure of E_m is about equal to any value of $|t|$ in E_m. So the sum above is finite if and only if the sum, on m, of the average values of $|f(t)|$ in the various sets E_m is finite.

We estimate these average values by splitting the series for f into the partial sum $s_m(t) = \sum_{|n|\le 2^m} \Delta c(n)e^{int}$ of order 2^m and the tail $T_m(t) = f(t) - s_m(t)$. Then $\widehat{T}_m(n) = \Delta c(n)$ if $|n| > 2^m$ and $\widehat{T}_m(n) = 0$ otherwise. In particular, $\|\widehat{T}_m\|_{1,2,2^m} \le \|\Delta c\|'_{1,2,2^m}$. This is exactly what we need to control the size of $\int_{E_m} |T_m(t)|\,dt$.

Denote the measure of any subset E of $[-\pi, \pi]$ by $|E|$. When $|E| > 0$ and g is an integrable function on E, denote the average $(1/|E|)\int_E g$ by g_E.

LEMMA 2. *There is a constant C so that if g is an integrable function on $[-\pi, \pi]$, and if I is an interval of length $\pi/2^{m+1}$ in $[-\pi, \pi]$, then*

$$|g|_I \le C\|\widehat{g}\|_{1,2,2^m}. \tag{1.6}$$

We will comment on various proofs of the lemma in the next section. It follows from Lemma 2 that

$$\sum_{m=0}^{\infty} |T_m|_{E_m} \le C \sum_{m=0}^{\infty} \|\Delta c\|'_{1,2,2^m} = C\|c\|_\Delta . \tag{1.7}$$

Given any integrable function f on $[-\pi,\pi)$, let $s_m(t) = \sum_{n=-2^m}^{2^m} \widehat{f}(n)e^{int}$, with the convention that $s_{-1} \equiv 0$. Also let $f_m = s_m - s_{m-1}$ when $m \ge 0$. Variants of the following statement have been used in earlier work [6, 14] on this topic. We will prove it at the beginning of the next section.

LEMMA 3. *The sums s_m have the property that*

$$\frac{1}{|E_m|}\int_{E_m} |s_m(t) - s_m(0)|\,dt \le C2^{-m}\sum_{j=0}^{m} 2^j \|\widehat{f_j}\|_1 . \tag{1.8}$$

For the function f obtained from F as above, adding these estimates as m runs from 0 to ∞, and reversing the order of summation on the right yields that

$$\sum_{m=0}^{\infty} |s_m - s_m(0)|_{E_m} \le 2C\sum_{j=0}^{\infty} \|\widehat{f_j}\|_1 = 2C\|\widehat{f}\|_1 = 2C\|\Delta c\|_1 . \tag{1.9}$$

Inequalities (1.7) and (1.9) combine to imply that, for a series with regular coefficients, the function F is integrable if and only if the sum on m of the numbers $|s_m(0)|$ is finite.

Now $s_m(0) = c(-2^m) - c(2^m+1)$. The two series $\sum_{m=0}^{\infty} |c(-2^m) - c(2^m+1)|$ and $\sum_{m=0}^{\infty} |c(2^m) - c(-2^m)|$ differ from each other by no more than $\sum_{m=0}^{\infty} |\Delta c(2^m)|$. This sum is finite because regularity implies bounded variation. So the sum on m of the numbers $|s_m(0)|$ is finite if and only if the series in condition (1.5) converges.

Thus F is integrable if and only if condition (1.5) holds. This condition is equivalent, for sequences with bounded variation, to condition (1.4). When F is integrable, its Fourier coefficients and those of f must satisfy the relation $\widehat{f}(n) = \Delta\widehat{F}(n)$ for all n because $f(t) = (1-e^{-it})F(t)$. But also $\widehat{f}(n) = \Delta c(n)$ for all n. Therefore the sequences $\widehat{F}$ and c can differ only by a constant, which must be 0 because both sequences vanish at infinity.

To summarize, regularity implies that the series converges except at 0 to $F(t)$. Adding sufficient symmetry leads to the conclusion that F is integrable, and that the given trigonometric series represents F. On the other hand, if a series with regular coefficients $c(n)$ represents an integrable function, G say, then the function $f : t \mapsto (1-e^{-it})G(t)$ will have two properties. First, $\widehat{f}(n) = \Delta c(n)$ for all n; second, the function $t \mapsto f(t)/t$ is integrable. Applying Lemmas 2 and 3 to f then yields that the sequence $(c(n))$ must be sufficiently symmetric. This completes the proof of Theorem 1.

We conclude this section by stating our results on norm convergence. We will prove them in Section 3.

THEOREM 4. *If the coefficients $c(n)$ in a trigonometric series form a regular and sufficiently symmetric sequence, then the series converges in L^1-norm if and only if*

$$c(n)\log|n| \to 0 \quad \text{as } n \to \pm\infty . \tag{1.10}$$

The norm-convergence criterion (1.10) also follows from weaker restrictions on the sizes of the differences of the coefficients, provided that the series is assumed *a priori* to represent an integrable function. Denote the indicator function of any set E by 1_E, even though this is inconsistent with the convention that g_E denotes the average value of the function g in the set E. Also denote the sum $\sum_{m=0}^{\infty} \|d\|'_{1,2,2^m}$ by $\|d\|'$. Call a sequence $(c(n))$ *asymptotically regular* if

$$\lim_{\lambda\to 1+}\limsup_{M\to\infty} \|(\Delta c)\cdot 1_{\pm[M,\lambda M]}\|' = 0 . \tag{1.11}$$

We will show that if the Fourier coefficients of an integrable function, F say, form an asymptotically regular sequence, then the Fourier series of F converges in L^1-norm if and only if

$$\widehat{F}(n)\log|n| \to 0 \quad \text{as } n \to \pm\infty . \tag{1.12}$$

An even weaker regularity condition suffices here. Given an integer K and a sequence $(d(n))_{n=-\infty}^{\infty}$, let

$$\|d\|^*_K = \sum_{m=0}^{K} \|d\|'_{1,2,2^m} . \tag{1.13}$$

Note that when $2^m > \lambda M$, the middle intervals that are deleted in computing $\|\cdot\|'_{1,2,2^m}$ include the intervals $\pm[M,\lambda M]$. So $\|(\Delta c)\cdot 1_{\pm[M,\lambda M]}\|'_{1,2,2^m} = 0$ for these values of m, and

$$\|(\Delta c)\cdot 1_{\pm[M,\lambda M]}\|' = \|(\Delta c)\cdot 1_{\pm[M,\lambda M]}\|^*_J \quad \text{when } 2^J \ge \lambda M .$$

Given numbers $\lambda > 1$ and $M > 0$, let $K(\lambda, M)$ be the smallest integer with the property that $2^{K(\lambda,M)} \ge (\lambda-1)M$. Say that the sequence c is *asymptotically locally regular* if

$$\lim_{\lambda\to 1+}\limsup_{M\to\infty} \|(\Delta c)\cdot 1_{\pm[M,\lambda M]}\|^*_{K(\lambda,M)} = 0 . \tag{1.14}$$

It is easy to verify that regularity implies asymptotic regularity, and that this implies asymptotic local regularity; moreover, one can devise examples to show that the converse implications are false.

THEOREM 5. *If the Fourier coefficients of an integrable function F form an asymptotically locally regular sequence, then the Fourier series of F converges in L^1-norm if and only if $\widehat{F}(n)\log|n| \to 0$ as $n \to \pm\infty$.*

2. Proofs of the lemmas. We begin with Lemma 3. Clearly,

$$|s_m - s_m(0)|_{E_m} \le \sum_{n=-2^m}^{2^m} |\widehat{f}(n)| \sup\{|e^{int} - 1| : t \in E_m\}. \tag{2.1}$$

Now $|e^{int} - e^{in0}| \le |nt|$, since $|(d/dt)e^{int}| = |n|$. Also, $|t| \le \pi/2^m$ for all t in E_m. So the supremum in the sum above is at most $|n|\pi/2^m$. Insert this upper bound, and then split the sum into pieces corresponding to the functions f_j. This yields inequality (1.8) with $C = \pi$, thus completing the proof of Lemma 3.

In the context of Lemma 2, we can use L^∞-norms on $[-\pi, \pi]$ and ℓ^1-norms on the integers to get the elementary estimate $|g|_{E_m} \le \|g\|_\infty \le \|\widehat{g}\|_1$. We can rewrite this ℓ^1-norm as $\|\widehat{g}\|_{1,1,2^m}$. The point of Lemma 2 is that in bounding $|g|_{E_m}$ we can replace the middle index 1 in $\|\widehat{g}\|_{1,1,2^m}$ by a 2, and use a multiple of $\|\widehat{g}\|_{1,2,2^m}$, which can be much smaller than $\|\widehat{g}\|_{1,1,2^m}$.

In proving Lemma 2, we will work mainly with L^2-norms and ℓ^2-norms. Given a function g and an interval I satisfying the hypotheses in the lemma, translate I and g so that I is centred at 0. This has no effect on $|g|_I$, or $|\widehat{g}|$, or $\|\widehat{g}\|_{1,2,2^m}$.

Fix $m \ge 0$, and let $h_m(t) = 2^{-m}\sum_{n=0}^{2^m-1} e^{int}$. Then the real part of h_m is bounded below by $1/\sqrt{2}$ on the interval $I = [-\pi/2^{m+2}, \pi/2^{m+2}]$. Again use the notation 1_I for the indicator function of I. Then

$$\begin{aligned}\int_I |g| &\le \int \sqrt{2}|h_m g| \cdot 1_I \le \sqrt{2}\|1_I\|_2\|h_m g\|_2 \\ &= \sqrt{2}\sqrt{\pi/2^{m+1}}\,\|\widehat{h_m g}\|_2 .\end{aligned} \tag{2.2}$$

We show below that $\|\widehat{h_m g}\|_2 \le 2\sqrt{2^{-m}}\|\widehat{g}\|_{1,2,2^m}$. Dividing the inequality above by $|I|$ yields inequality (1.6) with $C = 4/\sqrt{\pi}$.

Of course, $\widehat{h_m g}$ is the convolution of $\widehat{h}_m$ and $\widehat{g}$. Now $\widehat{h}_m$ vanishes outside the interval $[0, 2^m)$ and $\|\widehat{h}_m\|_2 = 2^{-m/2}$. Let $\widehat{g}_j$ be the restriction of $\widehat{g}$ to the interval $[j2^m, (j+1)2^m)$, and suppose initially that the pieces $\widehat{g}_j$ vanish when j is odd. The various convolution products $\widehat{h}_m * \widehat{g}_j$ with j even have disjoint supports. So, in this case, $(\|\widehat{h}_m * \widehat{g}\|_2)^2$ is equal to the sum over even j's of the terms $(\|\widehat{h}_m * \widehat{g}_j\|_2)^2$. By Young's inequality for convolution, each of these terms is bounded above by $(\|\widehat{h}_m\|_2\|\widehat{g}_j\|_1)^2 = 2^{-m}(\|\widehat{g}_j\|_1)^2$. Adding over even j's yields that

$$(\|\widehat{h}_m * \widehat{g}\|_2)^2 \le 2^{-m}(\|\widehat{g}\|_{1,2,2^m})^2 \tag{2.3}$$

provided that $\widehat{g}_j$ vanishes for all odd values of j. The same estimate holds when $\widehat{g}_j$ vanishes for all even values of j. In general, simply split $\widehat{g}$ into two parts vanishing in alternate intervals of length 2^m, and apply the estimate (2.3) separately to each part to complete the proof of the lemma.

There are many precedents for Lemma 2. It is equivalent by a duality argument to the statement that if k is a function that vanishes outside some interval, I say, of length $\pi/2^{m+1}$ and if $\|k\|_\infty \le 2^{m+1}/\pi$, then $\|\widehat{k}\|_{\infty,2,2^m} \le C$. If the function k also has the property that its average value is 0, then it is an *atom* as in [11], and the conclusion about the Fourier coefficients of k can be proved as in [11, p. 574]. If the average value, c say, of k is not 0, then split k as $c1_I + a$, where a is an atom supported by the interval I. It is easy to verify by direct calculation that the coefficients of $c1_I$ also have the required property.

The corresponding statement about Fourier transforms of atoms on the real line has been rediscovered repeatedly [2, p. 71]. Again the extension to other bounded functions supported by intervals is easy, and the inequality on the circle follows from the one on the line. Alternatively, on the line, one can use a change of variable to reduce matters to proving the estimate

$$\|\widehat{f}\|_{\infty,2,1} \le C\|f\|_\infty$$

for functions f that vanish outside some interval of length 1. This estimate, with the L^∞-norm on the right replaced by an L^2-norm, which is smaller for such functions f, can be found in work of Plancherel and Pólya [25].

Another application of duality leads to the estimate

$$\|g\|_{2,\infty,1} \le C\|\widehat{g}\|_{1,2,1}, \tag{2.4}$$

which is an endpoint for F. Holland's extension [18] of the Hausdorff–Young theorem to amalgams. This estimate and versions of it that follow by rescaling the variables are the analogues of Lemma 2 on the real line. Our method of proof transfers easily to that setting, and has the advantage of avoiding the use of duality. Conversely, it is easy to deduce Lemma 2 from inequality (2.4).

There is a striking similarity between the regularity condition (1.3) and a condition that arose in unpublished work by C. Fefferman. He showed that a sequence (d_n) has the property that $d \cdot \widehat{G} \in \ell^1$ for all G in classical H^1 if and only if $\sup_M \|d\|'_{1,2,M} < \infty$. It suffices here to take the supremum over values of M that are powers of 2, and then the condition is that

$$\sup_{m\ge 0}\left\{\sum_{j=1}^{\infty}\Big[\sum_{n=j2^m}^{(j+1)2^m-1}|d_n|\Big]^2\right\}^{1/2} < \infty. \tag{2.5}$$

Various proofs of the multiplier theorem were published in [28], [31], [5], and [19]. A key step in all of these proofs is to show that condition (2.5) implies that the series

$$\sum_{n=0}^{\infty} d_n e^{int} \tag{2.6}$$

represents a function with bounded mean oscillation.

In [28] and [31] this is accomplished by first proving a dual statement about coefficients of H^1-functions. Except for the use of duality, the methods used in [28] and [31] are similar to those used in the present paper. These methods can be used to give a direct proof that the series (2.6) must belong to BMO if its coefficients satisfy condition (2.5). Note first that the condition implies that $d \in \ell^2$, so that the series (2.6) does represent a function, f say, in $L^2[-\pi,\pi]$. Let I be an interval with length lying between $\pi/2^{m+2}$ and $\pi/2^{m+1}$, and split the series for f into pieces s_m and T_m as in the proof of Theorem 1. The proof of Lemma 2 also applies to intervals like I and yields that

$$|T_m|_I \le 2C\|\widehat{T}_m\|_{1,2,2^m}\,.$$

The norm above is majorized by the supremum, K say, in (2.5). It follows that

$$|T_m - (T_m)_I|_I \le 4CK\,.$$

Similarly, the proof of Lemma 3 applies with the set E_m replaced by the interval I, and yields that

$$|s_m - (s_m)_I|_I \le 2C2^{-m}\sum_{j=0}^{m} 2^j\|\widehat{f}_j\|_1\,. \tag{2.7}$$

Inserting the upper bound $\|\widehat{f}_j\|_1 \le K$ in (2.7) completes the proof that $f \in BMO$.

Given an integrable function f and a positive integer m, let

$$OSC_m(f) = \sup\{|f - f_I|_I : \pi/2^{m+1} < |I| \le \pi/2^m\}\,.$$

Then $f \in BMO$ if and only if the sequence $(OSC_m(f))_{m=0}^{\infty}$ is bounded. The proof of Theorem 1 shows that if

$$\sum_{m=0}^{\infty} \|\widehat{f}\|'_{1,2,2^m} < \infty\,, \tag{2.8}$$

then the sequence $(OSC_m(f))$ belongs to ℓ^1. This smoothness property of f is the endpoint case $MO^0_{\infty,1}$ of a class of mean-oscillation conditions introduced by Ricci and Taibleson [26]. The spaces $MO^{\alpha}_{s,r}$ with $\alpha > 0$ and $1 \le s, r \le \infty$ coincide with certain Besov spaces [26, 12]. The proof of integrability in [14] used properties of Besov spaces.

To see that the conclusion of Theorem 1 holds with $\|F\|_1$ bounded above by $A\|c\|_\Delta + B\|c\|_\Sigma$, first note that $|F(t)| \le (\pi/2)|f(t)|/|t|$. Then use the splitting

$$f(t) = s_m(0) + [s_m(t) - s_m(0)] + T_m(t) \tag{2.9}$$

when $t \in E_m$. By inequality (1.7),

$$\sum_{m=0}^{\infty} \int_{E_m} |T_m(t)|\frac{dt}{|t|} \le 2C\|c\|_\Delta$$

for all sequences c that tend to 0 at $\pm\infty$. By inequality (1.9),

$$\sum_{m=0}^{\infty} \int_{E_m} |s_m(t) - s_m(0)|\frac{dt}{|t|} \le 4C\|\Delta c\|_1\,.$$

Moreover, there are constants C'' and B so that

$$\sum_{m=0}^{\infty} \int_{E_m} |s_m(0)|\frac{dt}{|t|} \le \sum_{m=0}^{\infty} 2|s_m(0)| \le C''\|\Delta c\|_1 + B\|c\|_\Sigma\,.$$

Finally, $\|\Delta c\|_1 \le 2\|c\|_\Delta$.

3. Proofs of the norm-convergence theorems. By Theorem 1, the hypotheses in Theorem 4 imply that the series represents a function, F say, in $L^1(T)$. Since regularity implies asymptotic local regularity, Theorem 4 follows from Theorem 5.

It is still worth while giving a separate proof of Theorem 4, to clarify the pattern in the more complicated proof of Theorem 5. For each positive integer N, let V_N be the trigonometric polynomial with coefficients that are equal to 1 on the interval $[-N, N]$, that vanish outside the interval $(-2N, 2N)$, and that are linear on each of the intervals $[N, 2N]$ and $[-2N, -N]$. As is well known, these (de la Vallée-Poussin) kernels can be written in the form $2K_{2N} - K_N$ for suitable Fejér kernels K_N, and the convolutions $F * V_N$ converge in L^1-norm to F.

If the partial sums S_N of the series converge in L^1-norm, then they must also converge to F. So the series converges in L^1-norm if and only if $\|F*V_N - S_N\|_1 \to 0$ as $N \to \infty$. The coefficients of the difference $F*V_N - S_N$ vanish outside the intervals $\pm(N, 2N)$. Split this difference into pieces L_N and R_N with coefficients vanishing outside the intervals $(-2N, -N)$ and $(N, 2N)$ respectively. It is clear that if $\|L_N\|_1 \to 0$ and $\|R_N\|_1 \to 0$ as $N \to \infty$, then $\|F * V_N - S_N\|_1 \to 0$ as $N \to \infty$. The converse implication holds too, because R_N is equal to the convolution of $F * V_N - S_N$ with the polynomial $t \mapsto e^{iNt}V_N(t)$, which has L^1-norm no larger than 3.

In fact, $\|R_N\|_1 \to 0$ as $N \to \infty$ if and only if $c(N)\log N \to 0$ as $N \to \infty$, and $\|L_N\|_1 \to 0$ as $N \to \infty$ if and only if $c(-N)\log N \to 0$. To verify this for R_N let $\widetilde{R}_N(t) = R_N(-t)$ for all t, and consider the sum $\widetilde{R}_N + R_N$. Then

$\|\widetilde{R}_N\|_1 = \|R_N\|_1$ and the frequencies of $\widetilde{R}_N$ all lie in the interval $(-2N, -N)$. Applying the observations in the previous paragraph to $\widetilde{R}_N + R_N$ shows that its L^1-norm tends to 0 as $N \to \infty$ if and only if this is true for the norm of R_N alone.

We fill the gap in the spectrum by letting $G_N = \widetilde{R}_N + c(N)D_N + R_N$. We will see below that the hypotheses of Theorem 4 imply that $\|G_N\|_1 \to 0$ as $N \to \infty$. Assuming this, we have $\|R_N\|_1 \to 0$ as $N \to \infty$ if and only if $\|c(N)D_N\|_1 \to 0$ as $N \to \infty$. Now the latter condition holds if and only if $c(N)\log N \to 0$ as $N \to \infty$.

To estimate $\|G_N\|_1$, observe that its Fourier coefficients are constant except in the two intervals $[-2N, -N]$ and $[N, 2N]$, where $\widehat{G}_N$ coincides with $\widehat{V}_N \cdot \widehat{F}$. For values of n in these intervals, split $\Delta\widehat{G}_N(n)$ as

$$\widehat{V}_N(n)\Delta c(n) + c(n+1)\Delta\widehat{V}_N(n) = d_N(n) + e_N(n), \quad \text{say}.$$

Set $d_N(n)$ and $e_N(n)$ equal to 0 for all other values of n. The hypothesis that $\|c\|_\Delta < \infty$ and the fact that $|\widehat{V}_N(n)| \le 1$ for all n imply for each value of m that $\|d_N\|_{1,2,2^m} \to 0$ as $N \to \infty$. Also

$$\|d_N\|' = \sum_{m=0}^{\infty} \|d_N\|'_{1,2,2^m} \to 0 \quad \text{as } N \to \infty.$$

It is easy to verify that $\|V_N\|_\Delta \le 20$ for all N. This fact and the hypothesis that $c(n) \to 0$ as $n \to \pm\infty$ imply that $\|e_N\|' \to 0$ as $N \to \infty$. So $\|\widehat{G}_N\|_\Delta \to 0$ as $N \to \infty$; moreover, $\|\widehat{G}_N\|_\Sigma = 0$, because $\widehat{G}_N$ is even. Finally, $\|G_N\|_1 \le A\|\widehat{G}_N\|_\Delta + B\|\widehat{G}_N\|_\Sigma$, so that $\|G_N\|_1$ does indeed tend to 0 as $N \to \infty$.

To prove Theorem 5, we use condition (1.13) with the parameter λ running through the range of the sequence $(1+1/k)_{k=1}^{\infty}$. Given $\varepsilon > 0$, we choose integers $k > 1$ and H so that

$$\|\widehat{F} \cdot 1_{\pm[M, M(1+1/k)]}\|^*_{K(\lambda,M)} < \varepsilon \tag{3.1}$$

when $\lambda = 1+1/k$ and $M > H$. We will not change k in the rest of the proof, but, if necessary, we will replace H by larger integers to ensure that other conditions also hold.

We use the modified (de la Vallée-Poussin) kernels $V_{k,N}$ with coefficients that vanish outside the interval $(-(k+1)N, (k+1)N)$, are equal to 1 on the interval $[-kN, kN]$, and are linear on the intervals $\pm[kN, (k+1)N]$. Since $V_{k,N} = [(k+1)K_{(k+1)N} - K_{kN}]/k$, we have $\|F * V_{k,N} - F\|_1 \to 0$ as $N \to \infty$. We alter the choice of H so that

$$\|F * V_{k,N} - F\|_1 < \varepsilon \tag{3.2}$$

whenever $kN > H$.

Given $M > H$, let N be the smallest integer for which $kN \ge M$. Matters reduce to controlling the size of the L^1-norm of the difference $F * V_{k,N} - S_M$. Again, split this difference into pieces L_M and R_M. Then $\widehat{R}_M$ vanishes off $[M, (k+1)N]$. Since $k(N-1) < M$, it follows that $N < M/k + 1$ and that $(k+1)N \le \lambda M + k$. So $[M, (k+1)N] \subset [M, \lambda M + k]$. We require that $H \ge k^2$ in order to ensure that $k < M/k = (\lambda - 1)M$. Then the support of $\widehat{R}_M$ is included in $[M, M + 2(\lambda-1)M]$, and this is included in $[M, 3M]$. It follows as before that $\|R_M\|_1$ and $\|L_M\|_1$ are both bounded above by $3\|L_M + R_M\|_1$. The idea again is to show that $\|R_M\|_1$ is small if and only if $c(M)\log|M|$ is small.

It follows from inequality (3.1) and the inclusion of the support of $\widehat{R}_M$ in $[M, M+2(\lambda-1)M]$ that $\|R_M\|^*_{K(\lambda,M)} \le 2\varepsilon$. On the other hand, $\|R_M\|'$ can be much larger that this, because it is computed by summing all terms $\|\widehat{R}_M\|'_{1,2,2^m}$ with $2^m \le \lambda M$, rather than restricting the summation to terms with $2^m \le (\lambda - 1)M$.

We work instead with the function Q_M given by $t \mapsto e^{i(N-M)t}R_M(t)$. This polynomial has the same L^1-norm as R_M; also $\|Q_M\|^*_{K(\lambda,M)} \approx \|R_M\|^*_{K(\lambda,M)}$. The frequencies of Q_M all lie in $[N, N + (k+1)N - M]$. Rewrite the right endpoint of this interval as $(kN-M)+2N$. Then $kN - M < k$ because $k(N-1) < M$; also, our requirement that $H \ge k^2$ implies that $k < M/k \le (kN)/k = N$. So $(kN - M) < N$, and $\widehat{Q}_M$ vanishes off $[N, 3N)$. Hence,

$$\|Q_M\|' \le 2\|Q_M\|^*_{K(\lambda,M)},$$

which is bounded above by 4ε. Let $\widetilde{Q}_M(t) = Q_M(-t)$ for all t, and deal with $\|\widetilde{Q}_M + Q_M\|_1$ by letting $G_M = \widetilde{Q}_M + \widehat{F}(M)D_N + Q_M$.

The differences of the coefficients of G_M vanish outside the set $\pm[N, 3N]$. On this set, split these differences into terms $d_M(n)$ and $e_M(n)$ as in the proof of Theorem 4, but replacing $c(n)$ by $\widehat{F}(n)$, and V_N by the polynomial $P_{M,N}$ with the following properties. The coefficients of $P_{M,N}$ vanish outside the interval $[-(k+2)N + M, (k+2)N - M]$, they are equal to 1 on the interval $[-(k+1)N + M, (k+1)N - M]$, and they are linear on each of the intervals $\pm[(k+1)N - M, (k+2)N - M]$.

Much as before, $\|d_M\|' < 4\varepsilon$ whenever $M > H$. Now use the fact that $\widehat{F}$ vanishes at $\pm\infty$ to further alter the choice of H so that $|\widehat{F}(n)| < \varepsilon$ whenever $|n| > H$. Then $\|e_M\|' < 40\varepsilon$ for all $M > H$. It follows that $\|G_M\|_1 < 44\varepsilon$ for all such values of M.

We now suppose that the Fourier series of F converges in L^1-norm, and we further alter the choice of H so that we also have $\|F - S_M\|_1 < \varepsilon$ for all $M > H$. It follows that

$$\|L_M + R_M\|_1 = \|F * V_{k,N} - S_M\|_1 < 2\varepsilon \tag{3.3}$$

for all such values of M. This implies that $\|R_M\|_1 < 6\varepsilon$, and hence that

$$\|G_M - \widehat{F}(M)D_N\|_1 = \|\widetilde{Q}_M + Q_M\|_1 < 12\varepsilon$$

for all $M > H$. But also, $\|G_M\|_1 < 44\varepsilon$, so that $\|\widehat{F}(M)D_N\|_1 < 56\varepsilon$ for all such values of M.

In any case, there is a positive constant C so that $\|D_N\|_1 \geq C\log N$ for all $N \geq 1$. Here, $N \geq M/k$, and $M \geq H$. As noted earlier, our requirement that $H \geq k^2$ implies that $k < N$; hence $N^2 > kN \geq M$. Therefore,

$$|\widehat{F}(M)\log M| \leq 2|\widehat{F}(M)\log N| \leq (2/C)\|\widehat{F}(M)D_N\|_1 \leq (112/C)\varepsilon\,.$$

So hypothesis (1.14) and L^1-norm convergence of the series imply that $\widehat{F}(M)\log M \to 0$ as $M \to \infty$. By symmetry, these assumptions also imply that $\widehat{F}(-M)\log M \to 0$ as $M \to \infty$.

Finally, drop the assumption of L^1-norm convergence, but assume condition (1.12). Then choose $H \geq k^2$ so that conditions (3.1) and (3.2) both hold when $M > H$, and so that $\|\widehat{F}(-M)D_N\|_1$ and $\|\widehat{F}(M)D_N\|_1$ are both bounded above by ε for all such values of M. With this choice of H,

$$\|\widetilde{Q}_N + Q_N\|_1 = \|G_M - \widehat{F}(M)D_N\|_1 < 45\varepsilon\,.$$

Then $\|R_M\|_1 = \|Q_N\|_1 \leq 225\varepsilon$, and this is also an upper bound for $\|L_M\|_1$. So

$$\|F * V_{k,N} - S_M\|_1 = \|L_M + R_M\|_1 \leq 450\varepsilon,$$

and hence $\|S_M - F\|_1 \leq 451\varepsilon$ for all such values of M. So conditions (1.14) and (1.12) do indeed imply L^1-convergence of the Fourier series of F.

4. Conditions on individual differences. We begin with the equivalence between condition (1.1) and the Sidon–Telyakovskiĭ condition, as an example of a type of argument that we will use again. Suppose that the Sidon–Telyakovskiĭ condition holds. Then the smallest sequence (A_n) with the properties specified is the one given by $A_n = \sup\{|\Delta a_k| : k \geq n\}$. Since this sequence (A_n) is nonnegative and nonincreasing, the condition that $\sum_{n=0}^{\infty} A_n < \infty$ is equivalent to the requirement that

$$\sum_{m=1}^{\infty} 2^m \sup_{n\geq 2^{m-1}} |\Delta a_n| = \sum_{m=1}^{\infty} 2^m A_{2^m} < \infty\,. \tag{4.1}$$

It follows that condition (1.1) holds. On the other hand,

$$\sup_{n\geq 2^{m-1}} |\Delta a_n| \leq \sum_{j=m-1}^{\infty} \max_{2^{j-1}<n\leq 2^j} |\Delta a_n|\,.$$

Multiplying by 2^m, adding on m, and reversing the order of summation on

the right shows that

$$\sum_{m=1}^{\infty} 2^m \sup_{n\geq 2^{m-1}} |\Delta a_n| \leq 4\sum_{j=0}^{\infty} 2^j \max_{2^{j-1}<n\leq 2^j} |\Delta a_n|\,.$$

So condition (1.1) implies condition (4.1).

Fomin's condition (1.2) can be rewritten in the form

$$2\sum_{m=1}^{\infty} 2^{(m-1)/p'}\Big(\sum_{2^{m-1}<n\leq 2^m} |\Delta a_n|^p\Big)^{1/p} < \infty\,, \tag{4.2}$$

where p' denotes the conjugate index given by $1/p + 1/p' = 1$. Then p' is finite when $p > 1$. In these cases the factors $2^{(m-1)/p'}$ grow geometrically as $m \to \infty$, and the same kind of argument as above shows that condition (4.2) implies the seemingly stronger condition that

$$\sum_{m=1}^{\infty} 2^{(m-1)/p'}\Big(\sum_{n=2^{m-1}}^{\infty} |\Delta a_n|^p\Big)^{1/p} < \infty\,. \tag{4.3}$$

When $p = 1$, condition (4.2) becomes the requirement that $(\Delta a_n) \in \ell^1$, while condition (4.3) states that

$$\sum_{m=1}^{\infty}\Big(\sum_{n=2^{m-1}}^{\infty} |\Delta a_n|\Big) < \infty\,. \tag{4.4}$$

As in [13], reversing the order of summation in condition (4.4) shows that it is equivalent to the condition that

$$\sum_{n=0}^{\infty} |\Delta a_n|\log(n+1) < \infty\,, \tag{4.5}$$

which is strictly stronger than merely requiring that $(\Delta a_n) \in \ell^1$.

In fact, it is known [37, Chapter 5] that, for sequences (a_n) tending to 0, condition (4.5) implies integrability of cosine series with coefficients (a_n), but monotonicity does not, and hence neither does requiring that $(\Delta a_n) \in \ell^1$. To relate condition (4.5) to regularity, proceed as in the discussion just before the proof of Lemma 2. Rewrite (4.4) in the form $\sum_{m=0}^{\infty} \|\Delta a\|'_{1,1,2^m} < \infty$, and replace the middle index 1 in $\|\Delta a\|'_{1,1,2^m}$ by a 2 to get the condition that

$$\|\Delta a\|' = \sum_{m=0}^{\infty} \|\Delta a\|'_{1,2,2^m} < \infty\,. \tag{4.6}$$

This step is valid, since ℓ^1-norms are stronger than ℓ^2-norms.

We can pass from Fomin's condition to condition (4.6) in a similar way. By Hölder's inequality, if condition (1.2) is satisfied for some value of p, then it is satisfied for all smaller values of p. So we may assume that $1 < p \le 2$. Then we rewrite (4.3) in the form

$$\sum_{m=0}^{\infty} 2^{m/p'} \|\Delta a\|'_{p,p,2^m} < \infty . \tag{4.7}$$

Consider $\|\Delta a\|'$, and apply Hölder's inequality to the inner sums to get

$$\sum_{n=j2^m}^{(j+1)2^m-1} |\Delta a_n| \le 2^{m/p'} \Big[\sum_{n=j2^m}^{(j+1)2^m-1} |\Delta a_n|^p \Big]^{1/p} . \tag{4.8}$$

Fix m and regard the right side of (4.8) as a sequence indexed by $j \ge 1$. Because $p \le 2$, the ℓ^2-norm of this sequence is majorized by its ℓ^p-norm, which equals the summand on the left in (4.7). Hence Fomin's condition implies that $\|a\|_\Delta < \infty$.

In Tanović-Miller's extension [33] of Fomin's condition the indices 2^m are replaced by indices k_m with the property that $k_{m+1}/k_m > a$ for all m and some constant $a > 1$, and condition (4.2) is replaced by the requirement that

$$\sum_{m=0}^{\infty} k_m^{1/p'} \Big(\sum_{k_m \le n < k_{m+1}} |\Delta a_n|^p \Big)^{1/p} < \infty . \tag{4.9}$$

It is then also required that

$$\sum_{m=0}^{\infty} \sum_{n=k_m}^{k_{m+1}-1} |\Delta a_n| \log\Big(\frac{k_{m+1}}{k_m}\Big) < \infty . \tag{4.10}$$

It is still assumed that $p > 1$, and it still turns out that if this combination of conditions holds for one value of p in the interval $(1,\infty]$, then it also holds for all smaller values. When the ratios k_{m+1}/k_m form a bounded sequence, condition (4.9) is equivalent to Fomin's condition (4.2), and the second requirement (4.10) reduces to the condition that $(\Delta a_n) \in \ell^1$, which follows from (4.2). When there is no upper bound on the sequence (k_{m+1}/k_m), however, the combination of conditions (4.9) and (4.10) does not [33, p. 508] imply Fomin's condition.

In relating condition (4.6) to this combination, we may assume that $1 < p \le 2$. We first consider a sequence (d_n) that vanishes outside one of the intervals $[k_M, k_{M+1})$, and we seek an estimate for $\|d\|'$ in terms of

$$k_M^{1/p'} \Big(\sum_{k_M \le n < k_{M+1}} |d_n| \Big)^{1/p} \quad \text{and} \quad \sum_{k_M \le n < k_{M+1}} |d_n| \log(k_{M+1}/k_M) .$$

Any sequence can be split into such special sequences (d_n), and we can sum such estimates over all values of M to deduce regularity from conditions (4.9) and (4.10).

In any case, $\|d\|'_{1,2,2^m} \le \|d\|'_{1,p,2^m}$ because ℓ^p-norms are stronger than ℓ^2-norms here. Cover the support of d with intervals of the form $[j2^m, (j+1)2^m)$, and apply Hölder's inequality in each interval in the cover to get

$$\|d\|'_{1,p,2^m} \le 2^{m/p'} \Big(\sum_{k_M \le n < k_{M+1}} |\Delta d_n|^p \Big)^{1/p} .$$

Then add these upper bounds for all values of m for which $2^m \le k_M$, and get at most

$$C_p k_M^{1/p'} \Big(\sum_{k_M \le n < k_{M+1}} |\Delta d_n|^p \Big)^{1/p} .$$

In this step, it is necessary that $p > 1$.

The term $\|d\|'_{1,2,2^m}$ is also bounded above by $\sum_{k_M < n \le k_{M+1}} |d_n|$. Adding this fixed upper bound over values of m with $k_M < 2^m < k_{M+1}$ yields at most

$$2 \log\Big(\frac{k_{M+1}}{k_M}\Big) \sum_{n=k_M}^{k_{M+1}-1} |d_n| .$$

Finally, $\|d\|'_{1,2,2^m} = 0$ for all values of m for which $2^m \ge k_{M+1}$. So Tanović-Miller's condition implies condition (4.6).

To get examples where condition (4.6) holds but the ones used earlier do not, consider series in which the coefficients are constant in long intervals. Suppose for instance that $\Delta a_n = 0$ unless n is a power of 2. Then regularity becomes the requirement that

$$\sum_{m=0}^{\infty} \sum_{n=2^m}^{\infty} |\Delta a_n|^2 < \infty . \tag{4.11}$$

Let $a_n = 1/(m+1)$ for all integers n in each interval $(2^m, 2^{m+1}]$. Then

$$\Delta a_{2^m} = 1/(m+1)(m+2) \qquad \text{for all } m ,$$

and $\Delta a_n = 0$ for all other values of n. It is easy to check that condition (4.11) holds, but that conditions (4.1), (4.2), and (4.5) do not. Similarly, if condition (4.9) held for this sequence (a_n), then it would follow that $\sum_{m=1}^{\infty} k_m^{1/p'}/\log^2(k_m) < \infty$, but in fact this series must diverge when $p > 1$.

It is known [3] that cosine series where the coefficients tend to 0 and have differences that are 0 except on a lacunary set are integrable if and only if condition (4.11) holds.

QUESTION 1. *Suppose for a sequence (d_n) that the conditions that $a_n \to 0$ as $n \to \infty$ and that $|\Delta a_n| \le |d_n|$ for all n imply integrability of the cosine* *series with coefficients* (a_n). *Does it follow that*

$$\|d\|' = \sum_{m=0}^{\infty} \|d\|'_{1,2,2^m} < \infty?$$

As noted above, the answer is yes if $d_n = 0$ off a lacunary set of n's, but the question seems open in general. It is known [32] that on the real line the condition that $\|f\|_{1,2} < \infty$ characterizes the measurable functions f with the property that every measurable function with smaller absolute value than f has a Fourier transform that is a function. Question 1 is complicated by the fact that regularity involves a combination of overlapping amalgam norms.

Fefferman's theorem about condition (2.5) can be rephrased as follows. A sequence $(d_n)_{n=1}^{\infty}$ has the property that $\sup_{m\ge 0} \|d\|'_{1,2,2^m} < \infty$ if and only if every series $\sum_{n=1}^{\infty} e_n e^{int}$ with $|e_n| \le |d_n|$ for all n belongs to BMO.

QUESTION 2. *Does the condition that* $\|d\|' < \infty$ *hold if and only if every series with coefficients majorized by* $|d|$ *belongs to the space* $MO^0_{\infty,1}$?

In [10] and [33] there are discussions of various conditions that were known at the time to imply that property (1.12) characterizes L^1-norm convergence of Fourier series. The weakest of these conditions that involves only the sizes of individual difference of coefficients is that there be some index $p > 1$ for which

$$\lim_{\lambda\to 1+} \limsup_{M\to\infty} M^{1/p'} \|(\Delta\widehat{F}) \cdot 1_{\pm[M,\lambda M]}\|_p = 0\,. \tag{4.12}$$

Subsequently, it was shown [30] that property (1.12) characterizes L^1-norm convergence under the seemingly weaker condition that there be some value of $\lambda > 1$ and some index $p > 1$ for which

$$\sup_M M^{1/p'} \|(\Delta\widehat{F}) \cdot 1_{\pm[M,\lambda M]}\|_p < \infty\,. \tag{4.13}$$

This condition is discussed further in [16] and in [22, p. 205], where it is pointed out that if the condition holds for one value of $\lambda > 1$ then it holds for all $\lambda > 1$.

As in the comparison of integrability theorems, condition (4.12) implies asymptotic regularity. On the other hand, examples with lacunary differences of coefficients show that there are regular sequences that do not satisfy condition (4.12). These examples also show that condition (4.13) can fail for such sequences.

Another way to explain this is to observe that if condition (4.13) holds for some index $p > 1$, then condition (4.12) holds for all smaller values of p. To verify this, we suppose that (4.13) holds when $p = q$; as noted above, we may take $\lambda = 2$. We then fix an index $p < q$, and let $1/r = 1/p - 1/q$. We

consider values of λ in the interval $(1, 2]$, and apply Hölder's inequality to get

$$\|(\Delta\widehat{F}) \cdot 1_{\pm[M,\lambda M]}\|_p \le \|(\Delta\widehat{F}) \cdot 1_{\pm[M,2M]}\|_q \cdot \|1_{\pm[M,\lambda M]}\|_r\,.$$

By inequality (4.13), with $p = q$, there is a constant, C say, so that the ℓ^q-norm above is no larger than $CM^{-1/q'}$. Provided that $(\lambda - 1)M \ge 1$, there are no more than $4(\lambda-1)M$ integers in the set $\pm[M, \lambda M]$. Then the ℓ^r-norm above is no larger than $[4(\lambda-1)M]^{1/r}$. Combining these estimates yields that

$$M^{1/p'} \|(\Delta\widehat{F}) \cdot 1_{\pm[M,\lambda M]}\|_p \le M^{1/p'} CM^{-1/q'} [4(\lambda-1)M]^{1/r}$$

for all $M \ge 1/(\lambda - 1)$. In particular, this provides an upper bound for the limit supremum of the left side above. The right side simplifies to $C[4(\lambda-1)]^{1/r}$, and this tends to 0 as $\lambda \to 1+$, because $1/r > 0$ here.

5. Other conditions on differences. So far, we have only considered conditions on the sizes of individual differences. Fomin proved his integrability theorem for cosine series by showing that conditions (1.1) and (1.2) imply that

$$\sum_{m=2}^{\infty} \Bigg| \sum_{k=1}^{[m/2]} \frac{\Delta a_{m-k} - \Delta a_{m+k}}{k} \Bigg| < \infty\,. \tag{5.1}$$

Telyakovskiĭ had shown earlier [34] that this condition on the sizes of combinations of differences implies integrability of cosine series when the coefficients also tend to 0 and have bounded variation. The example with $a_n = 1/(m+1)$ for all integers n in each interval $(2^m, 2^{m+1}]$ shows that (4.6) does not imply (5.1). We do not know whether Telyakovskiĭ's conditions for integrability imply regularity.

Another sufficient condition for integrability involves differences with various step lengths. To state it, we revert to using the complex form of trigonometric series. Given a sequence $(c(n))_{n=-\infty}^{\infty}$ of coefficients and an integer k, let $\Delta_k c(n) = c(n) - c(n+k)$. We will explain below how standard methods show that if $c(n) \to 0$ as $n \to \pm\infty$, and if

$$\sum_{m=0}^{\infty} 2^{-m/2} \|\Delta_{2^m} c\|_2 < \infty\,, \tag{5.2}$$

then the trigonometric series with coefficients $(c(n))$ is integrable.

Later in this section and in the next section, we explain why, for sequences tending to 0, conditions (1.3) and (1.4) imply condition (5.2), thus providing an alternate proof of part of Theorem 1. In general, condition (5.2) does not imply condition (1.3), because it does not imply that the sequence (c_n) has bounded variation. In the next section, we show that condition (5.2) does imply conditions (1.3) and (1.4) for sequences that tend monotonically to 0 at $\pm\infty$ in the following sense. In addition to requiring that the sequence tend to 0 at $\pm\infty$, we also require that there be a positive integer N so that the sign of Δc is constant in the interval $[N, \infty)$ and also constant in the interval $(-\infty, N]$. If a cosine or sine series has real-valued coefficients that tend monotonically to 0 in the usual sense, then the corresponding complex form of the series has coefficients that satisfy the monotonicity condition above.

We deal first with the fact that condition (5.2) implies integrability of series with coefficients that tend to 0. The corresponding implication on the real line, with the summation on m running from $-\infty$ to ∞ was regarded as known twenty-five years ago [17]. We include a proof of the version that we need here to clarify its relation to Theorem 1.

If condition (5.2) holds then, in particular, $\|\Delta c\|_2 < \infty$, so that the series with coefficients $(\Delta c(n))$ represents a function, f say, in L^2. As in the proof of Theorem 1, matters essentially reduce to showing that the function $F : t \mapsto f(t)/(1 - e^{-it})$ is integrable. For each nonnegative integer m, let $F_m(t) = F(t)(1 - e^{-i2^m t})$ for all t. Then

$$F_m(t) = F(t)(1 - e^{-it}) \sum_{k=0}^{2^m-1} e^{-ikt} = f(t) \sum_{k=0}^{2^m-1} e^{-ikt}, \tag{5.3}$$

for all t. Because of this, the coefficient $\widehat{F}_m(n)$ is the sum of the coefficients $\Delta c(j)$ of f as j runs through the interval $[n, n + 2^m)$, and is therefore equal to $\Delta_{2^m} c(n)$ for all n.

On the other hand, the real part of $(1 - e^{-i2^m t})$ is at least 1 for all points t in the set E_m that was used in the proof of Theorem 1. So,

$$\int_{E_m} |F(t)| \frac{dt}{2\pi} \le \int |1_{E_m} F_m| \le \|1_{E_m}\|_2 \|F_m\|_2$$
$$= 2^{-(m+1)/2} \|\widehat{F}_m\|_2 = 2^{-(m+1)/2} \|\Delta_{2^m} c\|_2 .$$

Adding these inequalities as m runs from 0 to ∞ yields that $F \in L^1$ with

$$\|F\|_1 \le \frac{1}{\sqrt{2}} \sum_{m=0}^{\infty} 2^{-m/2} \|\Delta_{2^m} c\|_2 .$$

Note that the factor $\sum_{k=0}^{2^m-1} e^{-ikt}$ that appears in line (5.3) is just 2^m times the conjugate of the factor $h_m(t)$ that was used in the proof of Lemma 2. The same factor could have been used in both proofs, but the choices made seemed more natural in each context.

As mentioned earlier, conditions (1.3) and (1.4) imply condition (5.2) for all sequences that tend to 0 at $\pm\infty$. In the next section, we will prove this directly, using many of the same ideas as our first proof of Theorem 1. We can prove this implication much more quickly, however, by observing that in proving Theorem 1, we actually showed more, namely that

$$\sum_{m=0}^{\infty} 2^{-m/2} \|F \cdot 1_{E_m}\|_2 < \infty . \tag{5.4}$$

We also deduced this from the assumption that the coefficients tend to 0 and satisfy condition (5.2). Denote the left side of inequality (5.2) by $\|c\|_{B(1/2,2,1)}$. Then condition (5.4) implies that $\|\widehat{F}\|_{B(1/2,2,2)} < \infty$. This corresponds to well-known [23] characterizations of more general Besov spaces on $\mathbb{R}$.

The special case that we need here can be proved very simply. It suffices to estimate $2^{-m/2}\|(F \cdot 1_{E_m})^\wedge\|_{B(1/2,2,1)}$ in terms of $2^{-m/2}\|F \cdot 1_{E_m}\|_2$. To simplify the notation suppose that the function F vanishes off the set E_m. The sequence $(\Delta_{2^k} \widehat{F}(n))_{n=-\infty}^{\infty}$ consists of the coefficients of the function $F_k : t \mapsto (1 - e^{-i2^k t})F(t)$; so the norm of this sequence is equal to $\|F_k\|_2$. In any case, this norm is bounded above by $2\|F\|_2$; in the present case, the norm is also bounded above by $2^{k-m}\pi\|F\|_2$, because of the restriction on the support of F_k. Use the first upper bound when $k \ge m$, and use the second one otherwise. The outcome is that $\|\widehat{F}\|_{B(1/2,2,1)} \le C\|F\|_2$ if F vanishes off E_m.

In [8] Buntinas and Tanović-Miller denote the condition that we call regularity by dv^2; they show that dv^2 implies another condition that they call cv^2, and they show that this second condition implies integrability of cosine series. Condition cv^2 does not imply condition dv^2 because it does not imply bounded variation. Their proof of integrability actually shows, however, that cv^2 implies condition (5.4), and hence implies condition (5.2). The conditions in [34], namely that the coefficients (a_n) in a cosine series tend to 0 have bounded variation and satisfy (5.1), also imply (5.4), but our proof of this is too complicated to include here.

Condition (5.2) is the dual version of the smoothness condition that arises in Bernstein's theorem on absolute convergence of Fourier series. It is easy to formulate dual versions of our conditions that also imply absolute convergence. Suppose for instance that f is continuous on the interval $[-\pi, \pi]$, with $f(-\pi) = f(\pi)$, and that f is differentiable in the interior of this interval except possibly at 0. The analogue of condition (1.3) in this setting is the requirement that

$$\sum_{m=0}^{\infty} \|f'\|'_{1,2,\pi/2^m} < \infty . \tag{5.5}$$

Here, the functional $\|\cdot\|'_{1,2,\pi/2^m}$ is defined by using the interval $[0, \pi/2^m)$ and disjoint translates of it to cover $[-\pi, \pi)$, and then computing ℓ^2 combinations of L^1-norms over the intervals in the cover, but omitting the two middle intervals. Our method shows that if f satisfies the conditions specified above then

$$\sum_{n=-\infty}^{\infty} |\widehat{f}(n)| < \infty \quad \text{if and only if} \quad \int_0^{\pi} \frac{|f(t)-f(-t)|}{t}\,dt < \infty . \tag{5.6}$$

As above, however, the method actually proves the stronger conclusion tha

$$\sum_{m=0}^{\infty} 2^{m/2}\Big[\sum_{2^m\le|n|<2^{m+1}} |\widehat{f}(n)|^2\Big]^{1/2} < \infty , \tag{5.7}$$

and this is equivalent to the requirement that $f \in B(1/2,2,1)$. Half o this equivalence is the well-known fact that Bernstein's method shows tha absolute convergence follows from the assumption that $f \in B(1/2,2,1)$.

6. Another proof of integrability. Suppose first that condition (5.2) holds, and that the sequence tends monotonically to 0 at $\pm\infty$. Let N be a positive integer so that the sign of Δc is constant in the interval $[N,\infty)$ and also constant in the interval $(-\infty,N]$. This implies that $\Delta c \in \ell^1$. So $\|\Delta c\|_{1,2,2^m}$ is finite for all m, and in verifying inequality (1.3) it will be enough to show that $\sum_{m=M}^{\infty}\|\Delta c\|'_{1,2,2^m} < \infty$ for some integer M.

Choose M so that $2^M > 2N$. Then fix an integer $m \ge M$ and an integer $j \notin \{-1,0\}$. For each integer k in the interval $[(j-1/2)2^m, j2^m)$ the sign o Δc is constant on the interval $[k, k+2^{m+1})$, and the latter interval includes $[j2^m,(j+1)2^m)$. Hence

$$\sum_{n=j2^m}^{(j+1)2^m-1} |\Delta c(n)| \le \Big|\sum_{n=k}^{k+2^{m+1}-1} \Delta c(n)\Big| = |\Delta_{2^{m+1}} c(k)|$$

for all such integers k. Square this estimate, and add as k runs through most of an arithmetic progression of step-length 2^m to get

$$[\|\Delta c\|'_{1,2,2^m}]^2 \le \sum_{j=-\infty}^{\infty} |\Delta_{2^{m+1}} c(k_0 + j2^m)|^2 \tag{6.1}$$

for all integers k_0 in the interval $[2^{m-1},2^m)$. Now add as k_0 runs through that interval, and take a square-root to get

$$2^{(m-1)/2}\|\Delta c\|'_{1,2,2^m} \le \|\Delta_{2^{m+1}} c\|_2 .$$

Finally, divide by $2^{(m-1)/2}$ and sum on m to get

$$\sum_{m=M}^{\infty} \|\Delta c\|'_{1,2,2^m} \le 2 \sum_{m=M}^{\infty} 2^{-(m+1)/2}\|\Delta_{2^{m+1}} c\|_2 ,$$

which is finite.

At this point, we could appeal to Theorem 1 to deduce sufficient symmetry from regularity and the fact that condition (5.2) implies integrability when the coefficients tend to 0. We prefer, however, to work directly with condition (5.2) and the fact that sequences that tend monotonically to 0 at $\pm\infty$ have bounded variation. Given a positive integer m, let $\widetilde{c}(2^m)$ be the average of $c(n)$ over the interval $[2^m, 2^m+2^{m-1})$, and define $\widetilde{c}(-2^m)$ by averaging over the interval $[-2^m,-2^{m-1})$. As was pointed out in Section 1, condition (1.4) is equivalent, for sequences with bounded variation, to condition (1.5). Similarly, the latter condition is equivalent, for such sequences, to the requirement that

$$\sum_{m=1}^{\infty} |\widetilde{c}(2^m) - \widetilde{c}(-2^m)| < \infty \tag{6.2}$$

for sequences of bounded variation. Now

$$|\widetilde{c}(2^m) - \widetilde{c}(-2^m)| \le \frac{1}{2^{m-1}} \sum_{k=0}^{2^{m-1}-1} |c(-2^m+k) - c(-2^m+k+2^{m+1})| .$$

By the Schwarz inequality, the sum on the right is bounded above by $2^{(m-1)/2}\|\Delta_{2^{m+1}} c\|_2$. It follows that

$$\sum_{m=1}^{\infty} |\widetilde{c}(2^m) - \widetilde{c}(-2^m)| \le 2\sum_{m=1}^{\infty} 2^{-(m+1)/2}\|\Delta_{2^{m+1}} c\|_2 ,$$

which is finite. So condition (5.2) does indeed imply conditions (1.3) and (1.4) for sequences that tend monotonically to 0 at $\pm\infty$.

The converse implication holds for all sequences that tend to 0 at $\pm\infty$. We saw in the previous section how this follows from our first proof of Theorem 1. Proving the implication directly seems to require steps that resemble those in that proof. Suppose that conditions (1.3) and (1.4) hold, and again use the fact that

$$\Delta_{2^m} c(n) = \sum_{j=n}^{n+2^m-1} \Delta c(j) . \tag{6.3}$$

Split the sum on the right into a part, $\widetilde{s}_m(n)$ say, involving indices j with $-2^{m+1} \le j < 2^m$ and a part, $\widetilde{T}_m(n)$ say, involving all other values of j.

Cover the integers with disjoint translates of the interval $[0,2^m)$. Then $|\widetilde{T}_m(n)|$ is bounded above by the ℓ^1-norm of Δc over the translate containing n and possibly the next disjoint translate to the right, and $\widetilde{T}_m$ was defined so that the middle two translates do not have to be used here. Hence

$$\sum_{r=-\infty}^{\infty} |\widetilde{T}_m(k+r2^m)|^2 \le [2\|\Delta c\|'_{1,2,2^m}]^2$$

for all integers k. Add these estimates as k runs through the interval $[0, 2^m)$ and take a square root to get $\|\widetilde{T}_m\|_2 \le 2^{(m+2)/2}\|\Delta c\|'_{1,2,2^m}$. Therefore

$$\sum_{m=0}^{\infty} 2^{-m/2}\|\widetilde{T}_m\|_2 \le 2\|c\|_\Delta ,$$

which is finite.

The sum $\widetilde{s}_m(n)$ is empty unless n lies in the interval $(-2^{m+1}, 2^m)$. Suppose first that $n \in [0, 2^m)$, and split $\widetilde{s}_m(n)$ into subsums, $\widetilde{f}_{k,m}(n)$ say, in which the index j runs through the interval I_k that is equal to $[0, 1]$ if $k = ($ and to $(2^{k-1}, 2^k]$ otherwise. A particular subsum $\widetilde{f}_{k,m}(n)$ only occurs in formula (6.3) when $k \le m$; for the moment, define $\widetilde{f}_{k,m}(n)$ to be 0 when $k > m$ and when $n < 0$. The sum $\widetilde{f}_{k,m}(n)$ is empty when $n > 2^k$, and hence is also equal to 0 in that case. So, for fixed k, the sum $\widetilde{f}_{k,m}(n)$ vanishes unless $n \in [0, 2^k]$. In any case, $|\widetilde{f}_{k,m}(n)|$ is bounded above by the sum of the terms $|\Delta c(j)|$ as j runs through the interval I_k. Therefore

$$\Big[\sum_{n=-\infty}^{\infty} |\widetilde{f}_{k,m}(n)|^2\Big]^{1/2} \le (2^k+1)^{1/2} \sum_{j\in I_k} |\Delta c(j)| . \tag{6.4}$$

Since the quantity on the left is equal to 0 when $m < k$,

$$\sum_{m=0}^{\infty} 2^{-m/2}\Big[\sum_{n=-\infty}^{\infty} |\widetilde{f}_{k,m}(n)|^2\Big]^{1/2} \le \sum_{m=k}^{\infty} 2^{-m/2}(2^k+1)^{1/2} \sum_{j\in I_k} |\Delta c(j)| .$$

Reversing the order on the right yields the upper bound $8\sum_{j\in I_k}|\Delta c(j)$ for the contribution of the terms $\widetilde{f}_{k,m}(n)$, as m and n vary with $n \ge 0$ to the sum of the terms $2^{-m/2}\|\Delta_{2^m} c\|_2$. Adding these upper bounds as k varies shows that the total contribution of the terms $\widetilde{f}_{k,m}(n)$ is no more than $8\|\Delta c\|_1$.

The sum $\widetilde{s}_m(n)$ can be split in a similar way when $n \in (-2^{m+1}, -2^m]$ and a similar analysis shows that the total contribution of these terms to $\sum_{m=0}^{\infty} 2^{-m/2}\|\Delta_{2^m} c\|_2$ is no more than $8\|\Delta c\|_1$. Finally, suppose that $n \in (-2^m, 0)$. Let k be the largest integer for which the interval $[-2^k, 2^k]$ is included in $[n, n+2^m]$, and split $\Delta_{2^m} c(n)$ as

$$\Big(\sum_{j=n}^{-2^k-1} + \sum_{j=-2^k}^{2^k-1} + \sum_{j=2^k}^{n+2^m-1}\Big)\Delta c(j) . \tag{6.5}$$

The outer sums here can be further split and analysed much as above, and their contributions to $\sum_{m=0}^{\infty} 2^{-m/2}\|\Delta_{2^m} c\|_2$ are each bounded by $16\|\Delta c\|_1$

The middle sum in (6.5) is equal to $c(-2^k) - c(2^k)$. For fixed k the indices n for which this is the middle term in (6.5) are those for which at least one of the relations $n \in (-2^{k+1}, -2^k]$ and $n + 2^m \in [2^k, 2^{k+1})$ holds. There are at most 2^{k+1} such indices n. The contribution of these terms to $2^{-m/2}\|\Delta_{2^m}\|_2$ is bounded above by the product of $2^{(k+1-m)/2}$ and $|c(2^m) - c(-2^m)|$.

For fixed m, the indices k that can occur in the splitting (6.5) are those for which $2^k < 2^m$. Sum the upper bound above as k runs from 0 to $m-1$, and then add as m runs from 0 to ∞ to see that the contribution of these middle terms to $\sum_{m=0}^{\infty} 2^{-m/2}\|\Delta_{2^m} c\|_2$ is at most $4\sum_{m=0}^{\infty}|c(2^m) - c(-2^m)|$, which is finite.

This provides a second route from conditions (1.3) and (1.4) to the conclusion that the series must be integrable. Unlike the first proof, this method does not also lead directly to the fact that condition (1.3) and integrability imply condition (1.4). There is, however, a standard way to deal with that part of Theorem 1. It reduces to showing that if the coefficients (b_n) in a sine series satisfy the condition that

$$\sum_{m=0}^{\infty} \|\Delta b\|'_{1,2,2^m} \tag{6.6}$$

and if the sine series is integrable, then $\sum_{n=1}^{\infty} |b_n|/n < \infty$. Indeed, by the second method outlined above, condition (6.6) and the fact that $b_n \to 0$ as $n \to \infty$ imply that the cosine series with coefficients $(b_n)_{n=1}^{\infty}$ is integrable. If the sine series is also integrable, then so is the series $\sum_{n=1}^{\infty} b_n e^{int}$, and it then follows [37, Chapter VII, Theorem 8.7] that $\sum_{n=1}^{\infty} |b_n|/n < \infty$.

References

[1] B. Aubertin and J. J. F. Fournier, *An integrability theorem for Walsh series*, Boll. Un. Mat. Ital., to appear.

[2] A. E. Baernstein III and E. Sawyer, *Embedding and multiplier theorems for* $H^p(R^n)$, Mem. Amer. Math. Soc. 318 (1985).

[3] L. A. Balashov and S. A. Telyakovskiĭ, *Some properties of lacunary series and the integrability of trigonometric series*, Trudy Mat. Inst. Steklov. Akad. Nauk SSSR 143 (1977), 32–41 (in Russian); English transl.: Proc. Steklov Inst. Math. 1980 (1), 33–43.

[4] N. Bari, *A Treatise on Trigonometric Series*, 2 vols., translated by Margaret F. Mullins, MacMillan, New York, 1964.

[5] F. F. Bonsall, *Boundedness of Hankel matrices*, J. London Math. Soc. (2) 29 (1984), 289–300.

[6] W. O. Bray and V. Stanojević, *On the integrability of complex trigonometric series*, Proc. Amer. Math. Soc. 93 (1985), 51–58.

[7] M. Buntinas, *Some new multiplier theorems for Fourier series*, Proc. Amer. Math. Soc. 101 (1987), 497–502.

[8] M. Buntinas and N. Tanović-Miller, *New integrability and* L^1*-convergence classes for even trigonometric series II*, in: Approximation Theory, J. Szabados

and K. Tandori (eds.), Colloq. Math. Soc. János Bolyai 58, North-Holland, Ams terdam, 1991, 103–125.

[9] D. Ćeranić and N. Tanović-Miller, *An integrability and L^1-convergence clas for general trigonometric series*, to appear.

[10] C.-P. Chen, *L^1-convergence of Fourier series*, J. Austral. Math. Soc. Ser. A 4 (1986), 376–390.

[11] R. A. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in anal ysis*, Bull. Amer. Math. Soc. 83 (1977), 569–645.

[12] J. R. Dorronsoro, *Mean oscillation and Besov spaces*, Canad. Math. Bull. 2 (1985), 474–480.

[13] G. A. Fomin, *A class of trigonometric series*, Mat. Zametki 23 (1978), 213–22 (in Russian); English transl.: Math. Notes 23 (1978), 117–123.

[14] J. J. F. Fournier and W. M. Self, *Some sufficient conditions for uniform conver gence of Fourier series*, J. Math. Anal. Appl. 126 (1987), 355–374.

[15] J. J. F. Fournier and J. Stewart, *Amalgams of L^p and ℓ^q*, Bull. Amer. Math Soc. (N. S.) 13 (1985), 1–21.

[16] D. Grow and V. B. Stanojević, *Representations of Fourier coefficients in taube rian L^1-convergence classes*, J. Math. Anal. Appl. 160 (1991), 47–50.

[17] C. S. Herz, *Lipschitz spaces and Bernstein's theorem on absolutely convergen Fourier transforms*, J. Math. Mech. 18 (1968), 283–323.

[18] F. Holland, *Harmonic analysis on amalgams of L^p and ℓ^q*, J. London Math. Soc (2) 10 (1975), 295–305.

[19] F. Holland and D. Walsh, *Boundedness criteria for Hankel operators*, Proc. R Irish Acad. 84A (1984), 141–154.

[20] A. S. Kolmogorov, *Sur l'ordre de grandeur des coefficients de la série de Fourier-Lebesgue*, Bull. Internat. Acad. Polon. Sci. Lettres (A) Sci. Math. 1923, 83–86.

[21] F. Móricz, *On the integrability and L^1-convergence of complex trigonometric se ries*, Proc. Amer. Math. Soc. 113 (1991), 53–64.

[22] —, *On L^1-convergence of Walsh series. II*, Acta Math. Hungar. 58 (1991), 203–210

[23] J. Peetre, *New Thoughts on Besov Spaces*, Duke Univ. Math. Ser., Durham, N.C. 1976.

[24] M. Pepić and N. Tanović-Miller, to appear.

[25] M. Plancherel et G. Pólya, *Fonctions entières et intégrales de Fourier multiples Parties 1^e et 2^e*, Comment. Math. Helv. 9 (1936–1937), 224–248; 10 (1937–38) 110–163.

[26] F. Ricci and M. Taibleson, *Boundary values of harmonic functions in mixed norm spaces and their atomic structure*, Ann. Scuola Norm. Sup. Pisa (4) 10 (1983) 1–54.

[27] S. Sidon, *Hinreichende Bedingungen für den Fourier-Charakter einer trigonome trischen Reihe*, J. London Math. Soc. 14 (1939), 158–160.

[28] W. T. Sledd and D. A. Stegenga, *An H^1 multiplier theorem*, Ark. Mat. 19 (1981) 265–270.

[29] Č. V. Stanojević, *Classes of L^1-convergence of Fourier and Fourier–Stieltjes se ries*, Proc. Amer. Math. Soc. 82 (1981), 209–215.

[30] —, *Structure of Fourier coefficients and Fourier–Stieltjes coefficients of series with slowly varying convergence moduli*, Bull. Amer. Math. Soc. 19 (1988), 283–286.

[31] S. J. Szarek and T. Wolniewicz, *A proof of Fefferman's theorem on multipliers*, preprint 209, Institute of Mathematics, Polish Academy of Sciences, 1980.

[32] P. Szeptycki, *On functions and measures whose Fourier transforms are measures*, Math. Ann. 179 (1968), 31–41.

[33] N. Tanović-Miller, *On integrability and L^1-convergence of cosine series*, Boll. Un. Mat. Ital. (7) 4-B (1990), 499–516.

[34] S. A. Telyakovskiĭ, *Integrability conditions of trigonometric series and their applications to the study of linear methods of summing Fourier series*, Izv. Akad. Nauk SSSR Ser. Mat. 28 (1964), 1209–1236 (in Russian).

[35] —, *Concerning a sufficient condition of Sidon for the integrability of trigonometric series*, Mat. Zametki 14 (1973), 317–328 (in Russian); English transl.: Math. Notes 14 (1973), 742–748.

[36] W. H. Young, *On the Fourier series of bounded functions*, Proc. London Math. Soc. (2) 12 (1913), 41–70.

[37] A. Zygmund, *Trigonometric Series*, 2 volumes, Cambridge University Press, Cambridge 1959.

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Received October 29, 1992 (3018)
Revised version April 20, 1993